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© *R. Yu. Kazimirova, G. I. Ibragimov, R. M. Hasim*

MULTI-PURSUER PURSUIT DIFFERENTIAL GAME FOR AN INFINITE SYSTEM OF SECOND ORDER DIFFERENTIAL EQUATIONS

We study a pursuit differential game of many pursuers and one evader. The game is described by the infinite systems of m inertial equations. By definition, pursuit in the game is completed if the state and its derivative of one of the systems are equal to zero at some time. In the literature, such a condition of completion of pursuit is also called soft landing. We obtain a condition in terms of energies of players which is sufficient for completion of pursuit in the game. The pursuit strategies are also constructed.

Keywords: differential game, control, strategy, many pursuers, infinite system of differential equations, integral constraint.

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Introduction

Multi-player differential games is an important chapter of the differential game theory. Such differential games are studied mainly in finite dimensional spaces, as in [4, 5, 9, 11–13, 19, 21, 23, 24, 30].

One can consider differential games in infinite dimensional spaces as well. Such games can be obtained when we apply the decomposition method to investigating conflict-controlled systems governed by PDE's. This method was applied to study a number of controlled systems described by PDE by many researchers such as Butkovskiy [6], Chernous'ko [8], Avdonin and Ivanov [2], Satimov and Tukhtasinov [28], Philippe Martin et al. [20], Alimov and Albeverio [1], and Chaves-Silva et al. [7].

Also, some differential game problems of pursuit and evasion governed by PDE were analyzed using this method (see, for example, [3, 18, 27–29, 32, 33]). In result, an infinite system of linear differential equations is obtained. The simplicity of the equations in the system attracts the attention of many researchers. However, the main difficulty in analyzing the differential game problems arises due to the infinite number of equations in the system.

In the work of Satimov and Tukhtasinov [29], the differential games of one pursuer and one evader described by the following system

$$\ddot{z} + Az = -u + v, \quad Az = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial z}{\partial x_i}\right), \quad x \in \Omega, \quad a_{ij}(x) \in \bar{C}^1(\Omega), \tag{0.1}$$

were considered, where the operator is considered in the space $L_2(\Omega)$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise smooth boundary, the functions $a_{ij}(x)$ satisfy some conditions under which the operator A is elliptic. The domain of the operator A is the space $\mathring{C}^2(\Omega)$ of twice continuously differentiable finite functions. The operator A then has eigenvalues $\lambda_1, \lambda_2, \ldots,$ $0 < \lambda_1 \le \lambda_2 \le \ldots \to \infty$ and eigenfunctions $\varphi_1, \varphi_2, \ldots,$ that form a complete orthonormal system in $L_2(\Omega)$. Using the decomposition method the authors obtained a differential game problem described by the following infinite system of differential equations

$$\ddot{z}_k = -\lambda_k z_k - u_k + v_k, \; k = 1, 2, \ldots, \tag{0.2}$$

where u_k and v_k are control parameters of players, and $z_k, u_k, v_k \in \mathbb{R}$. In that work, for the differential games of one pursuer and one evader, sufficient conditions of terminating pursuit were obtained when the control functions of players satisfy integral and geometric constraints.

This approach leads us to suggest studying differential games governed by infinite system of differential equations (see, for example [14–16, 25, 26, 31]). In the work [17] for the following infinite system

$$\begin{aligned} \dot{x}_i &= -\alpha_i x_i - \beta_i y_i + u_{i1} - v_{i1}, \quad x_i(0) = x_{i0}, \\ \dot{y}_i &= \beta_i x_i - \alpha_i y_i + u_{i2} - v_{i2}, \qquad y_i(0) = y_{i0}, \end{aligned} \tag{0.3}$$

the problem of optimal pursuit was studied in Hilbert space l_2, where α_i, β_i are real numbers, and $\alpha_i \geq 0$.

In the present paper, we study a pursuit differential game of many pursuers and one evader. The game is described by the infinite systems of m inertial equations. By definition, pursuit in the game is completed if the state and its derivative of one of the systems are equal to zero at some time. In the literature, such a condition of completion of pursuit is called soft landing. We obtain a condition in terms of energies of players which is sufficient for completion of pursuit in the game. Also, we construct strategies for the pursuers that guarantee capturing the evader.

The paper is organized as follows. Section 1 is devoted to the statement of a problem. In Section 2, we study differential game of one pursuer and one evader. In Section 3, a pursuit differential game of many pursuers and one evader is studied.

§ 1. Statement of problem

We consider a differential game of m pursuers and one evader described by the infinite system of second order differential equations

$$\ddot{x}_{ik} = -\mu_{ik} x_{ik} - u_{ik} + v_k, \quad x_{ik}(t_0) = x_{ik}^0, \quad \dot{x}_{ik}(t_0) = x_{ik}^1, \quad i = 1, 2, \ldots, m; \; k = 1, 2, \ldots, \tag{1.1}$$

where t_0 is the initial time, x_{ik}, $x_{ik}^0, u_{ik}, v_k \in \mathbb{R}^1$, $x_i = (x_{i1}, x_{i2}, \ldots) \in l_2$ is the state variable of the ith system, μ_{ik} are given positive numbers. Also, system (1.1) can be written in the matrix form as follows:

$$\ddot{x}_i = A_i x_i - u_i + v, \quad x_i(t_0) = x_i^0, \quad \dot{x}_i(t_0) = x_i^1,$$

where $x_i^0 = (x_{i1}^0, x_{i2}^0, \ldots)$, $x_i^1 = (x_{i1}^1, x_{i2}^1, \ldots)$ are given initial states,

$$A_i = \begin{bmatrix} -\mu_{i1} & 0 & 0 & \ldots & 0 \\ 0 & -\mu_{i2} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}, \quad i = 1, 2, \ldots,$$

are infinite dimensional diagonal matrices, $u_i = (u_{i1}, u_{i2}, \ldots)$ and $v = (v_1, v_2, \ldots)$ are the control parameters of the ith pursuer, $i \in \{1, 2, \ldots, m\}$, and the evader, respectively. It is assumed that

$$\bar{x}_i^0 \triangleq (\sqrt{\mu_{i1}} x_{i1}^0, \sqrt{\mu_{i2}} x_{i2}^0, \ldots) \in l_2, \quad x_i^1 \in l_2, \quad \|\bar{x}_i^0\| + \|x_i^1\| \neq 0, \quad i = 1, 2, \ldots, m.$$

We denote by $B(r, a, b)$ the set of all functions $\varphi(t) = (\varphi_1(t), \varphi_2(t), \ldots)$, $a \leq t \leq b$, with the measurable coordinates $\varphi_k(t)$, $k = 1, 2, \ldots$, that satisfy the constraint

$$\int_a^b \sum_{k=1}^{\infty} \varphi_k^2(t)\, dt \leq r^2.$$

The control functions of the ith pursuer and the evader are defined as the functions $u_i(t) = (u_{i1}(t), u_{i2}(t), \ldots) \in B(\rho_i, t_0, \infty)$ and $v(t) = (v_1(t), v_2(t), \ldots) \in B(\sigma, t_0, \infty)$, respectively, where ρ_i, $i = 1, 2, \ldots, m$, and σ are given positive numbers.

Let $\rho = \left(\rho_1^2 + \ldots + \rho_m^2\right)^{1/2}$, $\sigma_i = \frac{\sigma}{\rho}\rho_i$, $i = 1, 2, \ldots, m$, and

$$\|v(t)\| = \left(\sum_{k=1}^{\infty} v_k^2(t)\right)^{1/2}, \qquad q(t) = \left(\sigma^2 - \int_{t_0}^{t} \|v(t)\|^2\, dt\right)^{1/2}.$$

It is clear that if $\rho > \sigma$, then $\rho_i > \sigma_i$ for all $i = 1, 2, \ldots, m$.

Definition 1.1. Let $v(t)$ be any control function of the evader. A function of the form

$$u_i(t, q, v) = \begin{cases} v - w_{i0}(t), & \theta_i \le t \le \theta_{i+1}, \\ 0, & t \notin [\theta_i, \theta_{i+1}], \end{cases} \qquad i = 1, 2, \ldots, m,$$

where θ_i is the time when $q(\theta_i) = \left(\sigma^2 - \sigma_1^2 - \ldots - \sigma_i^2\right)^{1/2}$, $w_{i0}(\cdot) \in B(\rho_i - \sigma_i, \theta_i, \theta_{i+1})$ is an arbitrary function, is called *a strategy of the ith pursuer*.

Definition 1.2. We say that *pursuit can be completed for the time T_0 in game* (1.1) *from the initial positions* $\{x_i^0, x_i^1\}$, $i = 1, \ldots, m$, if there exist strategies $u_i(t, q, v)$, $i = 1, \ldots, m$, of pursuers such that, for any control function $v(\cdot)$, the solutions of systems (1.1) with $u_i = u_i(t, q, v)$ and $v = v(t)$, $t_0 \le t \le T_0$, satisfy the conditions $x_{i_0}(\tau) = 0$, $\dot{x}_{i_0}(\tau) = 0$ at some time $t_0 \le \tau \le T_0$ and $1 \le i_0 \le m$.

The pursuers try to complete the pursuit as earlier as possible, and the purpose of the evader is opposite. In the literature, realization of the equations $x_{i_0}(\tau) = 0$ and $\dot{x}_{i_0}(\tau) = 0$ is called soft landing (see, for example, [10, 22]).

It is not difficult to verify that, for the given control functions $u_i(t)$ and $v(t)$, $t \ge t_0$, the solution of the ith system of (1.1) is $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots)$ with

$$x_{ik}(t) = x_{ik}^0 \cos\sqrt{\mu}_{ik}(t - t_0) + x_{ik}^1 \frac{\sin\sqrt{\mu_{ik}}(t - t_0)}{\sqrt{\mu_{ik}}} + \\ + \int_{t_0}^{t} \frac{\sin\sqrt{\mu_{ik}}(t - s)}{\sqrt{\mu_{ik}}} \left(-u_{ik}(s) + v_k(s)\right) ds, \quad (1.2)$$

where $i = 1, 2, \ldots, m$; $k = 1, 2, \ldots$, $t \ge t_0$. Using the techniques similar to [2, Chapter III, Sections 1 and 2] one can show that $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots) \in l_2$.

§ 2. The case of one pursuer and one evader

Let us first consider the case of one pursuer and one evader. We let $i = 1$ in system (1.1) and for simplicity of notations in this section we temporarily use notations x_k, u_k, x_k^0, x_k^1, μ_k instead of x_{1k}, u_{1k}, x_{1k}^0, x_{1k}^1, μ_{1k} respectively.

Thus, differential game of one pursuer and one evader is described by the equations

$$\ddot{x}_k = -\mu_k x_k - u_k + v_k, \quad x_k(t_0) = x_k^0, \quad \dot{x}_k(t_0) = x_k^1, \quad k = 1, 2, \ldots, \quad (2.1)$$

where x_k, $x_k^0, x_k^1, u_k, v_k \in \mathbb{R}^1$, $u = (u_1, u_2, \ldots)$ is pursuer's control parameter, $v = (v_1, v_2, \ldots)$ is evader's control parameter. We assume that

$$\bar{x}^0 = (\sqrt{\mu_1} x_1^0, \sqrt{\mu_2} x_2^0, \ldots) \in l_2, \quad x^1 = (x_1^1, x_2^1, \ldots) \in l_2, \quad \|\bar{x}^0\| + \|x^1\| \ne 0,$$

where

$$\|\bar{x}^0\| = \left(\sum_{k=1}^{\infty} \mu_k (x_k^0)^2\right)^{1/2}, \qquad \|x^1\| = \left(\sum_{k=1}^{\infty} (x_k^1)^2\right)^{1/2}.$$

The control functions of players are subject to the following integral constraints:

$$\int_{t_0}^{\infty} \|u(t)\|^2\, dt \le \rho_0^2, \qquad \int_{t_0}^{\infty} \|v(t)\|^2\, dt \le \sigma_0^2,$$

where ρ_0 and σ_0 are given positive numbers.

To prove the main result of the paper, which will be formulated in Section 3, we need the following statement.

Lemma 2.1. *If* $\mu_0 \triangleq \inf_{k \ge 1} \mu_k > 0$ *and* $\rho_0 > \sigma_0$*, then pursuit can be completed in the game of one pursuer and one evader* (2.1) *for the time*

$$T(t_0, \bar{x}^0, x^1) = t_0 + \frac{2}{(\rho_0 - \sigma_0)^2}\left(\|\bar{x}^0\|^2 + \|x^1\|^2\right) + \frac{1}{\sqrt{\mu_0}}.$$

P r o o f. By (1.2) the solution of (2.1) and its derivative can be written as follows:

$$x_k(t) = x_k^0 \cos\sqrt{\mu_k}(t - t_0) + x_k^1 \cdot \frac{\sin\sqrt{\mu_k}(t - t_0)}{\sqrt{\mu_k}} + \int_{t_0}^{t} \frac{\sin\sqrt{\mu_k}(t - s)}{\sqrt{\mu_k}}\big(-u_k(s) + v_k(s)\big)\, ds,$$

$$\dot{x}_k(t) = -\sqrt{\mu_k} x_k^0 \sin\sqrt{\mu_k}(t - t_0) + x_k^1 \cos\sqrt{\mu_k}(t - t_0) + \int_{t_0}^{t} \cos\sqrt{\mu_k}(t - s)\big(-u_k(s) + v_k(s)\big)\, ds. \tag{2.2}$$

Putting

$$u_k(s) = v_k(s) - \omega_k(s), \quad k = 1, 2, \ldots, \tag{2.3}$$

where $\omega_k(s)$, $k = 1, 2, \ldots$, are functions to be found, and

$$\lambda_k = \sqrt{\mu_k}, \quad \bar{\xi}_k(t) = \sqrt{\mu_k} \cdot x_k(t), \quad \bar{\eta}_k(t) = \dot{x}_k(t), \quad \bar{\xi}_{k0} = \sqrt{\mu_k} \cdot x_k^0, \quad \bar{\eta}_{k0} = x_k^1$$

in (2.2) we obtain, for $k = 1, 2, \ldots$,

$$\begin{aligned} \bar{\xi}_k(t) &= \bar{\xi}_{k0} \cos\lambda_k(t - t_0) + \bar{\eta}_{k0} \sin\lambda_k(t - t_0) + \int_{t_0}^{t} \sin\lambda_k(t - s)\omega_k(s)\, ds, \\ \bar{\eta}_k(t) &= -\bar{\xi}_{k0} \sin\lambda_k(t - t_0) + \bar{\eta}_{k0} \cos\lambda_k(t - t_0) + \int_{t_0}^{t} \cos\lambda_k(t - s)\omega_k(s)\, ds. \end{aligned} \tag{2.4}$$

We consider (2.4) under the following integral constraint:

$$\int_0^{\infty} \omega_k^2(s)\, ds \le \alpha_k^2, \qquad \alpha_k = \left(\frac{\bar{\xi}_{k0}^2 + \bar{\eta}_{k0}^2}{\|\bar{x}^0\|^2 + \|x^1\|^2}\right)^{1/2} (\rho_0 - \sigma_0). \tag{2.5}$$

Hence,

$$\begin{aligned}\sum_{k=1}^{\infty}\int_{0}^{\infty}\omega_k^2(s)\,ds \le \sum_{k=1}^{\infty}\alpha_k^2 &= \frac{(\rho_0-\sigma_0)^2}{\|\bar{x}^0\|^2+\|x^1\|^2}\left(\sum_{k=1}^{\infty}\bar{\xi}_{k0}^2+\sum_{k=1}^{\infty}\bar{\eta}_{k0}^2\right)\\ &= \frac{(\rho_0-\sigma_0)^2}{\|\bar{x}^0\|^2+\|x^1\|^2}\left(\sum_{k=1}^{\infty}\mu_k(x_k^0)^2+\sum_{k=1}^{\infty}(x_k^1)^2\right)\\ &= \frac{(\rho_0-\sigma_0)^2}{\|\bar{x}^0\|^2+\|x^1\|^2}\left(\|\bar{x}^0\|^2+\|x^1\|^2\right)=(\rho_0-\sigma_0)^2.\end{aligned} \tag{2.6}$$

To prove Lemma 2.1, we consider first a 2-system in (2.4) for the fixed k. For simplicity of notations, we drop the indices k and analyze the dynamics of the controlled object $(\bar{\xi},\bar{\eta})$ described by the following 2-system

$$\begin{aligned}\bar{\xi}(t) &= \bar{\xi}_0\cos\lambda(t-t_0)+\bar{\eta}_0\sin\lambda(t-t_0)+\int_{t_0}^{t}\sin\lambda(t-s)\omega(s)\,ds,\\ \bar{\eta}(t) &= -\bar{\xi}_0\sin\lambda(t-t_0)+\bar{\eta}_0\cos\lambda(t-t_0)+\int_{t_0}^{t}\cos\lambda(t-s)\omega(s)\,ds,\end{aligned} \tag{2.7}$$

where $\bar{\xi},\bar{\eta},\bar{\xi}_0,\bar{\eta}_0\in\mathbb{R}^1$, λ is a positive number, the scalar control function $\omega(t)$, $t\ge t_0$, satisfies the condition

$$\int_{t_0}^{\infty}\omega^2(s)\,ds\le\alpha^2,\quad \alpha>0.$$

The problem is to transfer the state $\big(\bar{\xi}(t),\bar{\eta}(t)\big)$ of system (2.7) from the initial state $\big(\bar{\xi}_0,\bar{\eta}_0\big)$ at the time $t=t_0$ to the origin of $\mathbb{R}^2$. In other words, we aim to obtain the equations

$$\begin{aligned}\bar{\xi}(\tau) &= \bar{\xi}_0\cos\lambda(\tau-t_0)+\bar{\eta}_0\sin\lambda(\tau-t_0)+\int_{t_0}^{\tau}\sin\lambda(\tau-s)\omega(s)\,ds=0,\\ \bar{\eta}(\tau) &= -\bar{\xi}_0\sin\lambda(\tau-t_0)+\bar{\eta}_0\cos\lambda(\tau-t_0)+\int_{t_0}^{\tau}\cos\lambda(\tau-s)\omega(s)\,ds=0\end{aligned} \tag{2.8}$$

at some $\tau>t_0$.

We prove the following statement.

Lemma 2.2. *The state of system* (2.7) *can be transferred from the initial state* $(\bar{\xi}_0,\bar{\eta}_0)$ *to the origin within the time*

$$T=t_0+\frac{2}{\alpha^2}\left(\bar{\xi}_0^2+\bar{\eta}_0^2\right)+\frac{1}{\lambda}.$$

P r o o f. The equation (2.8) can be written in the matrix form as follows:

$$\begin{bmatrix}\cos\lambda\tau & \sin\lambda\tau\\ -\sin\lambda\tau & \cos\lambda\tau\end{bmatrix}\begin{bmatrix}\bar{\xi}_0\cos\lambda t_0-\bar{\eta}_0\sin\lambda t_0\\ \bar{\xi}_0\sin\lambda t_0+\bar{\eta}_0\cos\lambda t_0\end{bmatrix}+\int_{t_0}^{\tau}\begin{bmatrix}\cos\lambda\tau & \sin\lambda\tau\\ -\sin\lambda\tau & \cos\lambda\tau\end{bmatrix}\begin{bmatrix}-\omega(s)\sin\lambda s\\ \omega(s)\cos\lambda s\end{bmatrix}ds=0.$$

We multiply this equation to the inverse of the matrix $\begin{bmatrix}\cos\lambda\tau & \sin\lambda\tau\\ -\sin\lambda\tau & \cos\lambda\tau\end{bmatrix}$ to obtain the system

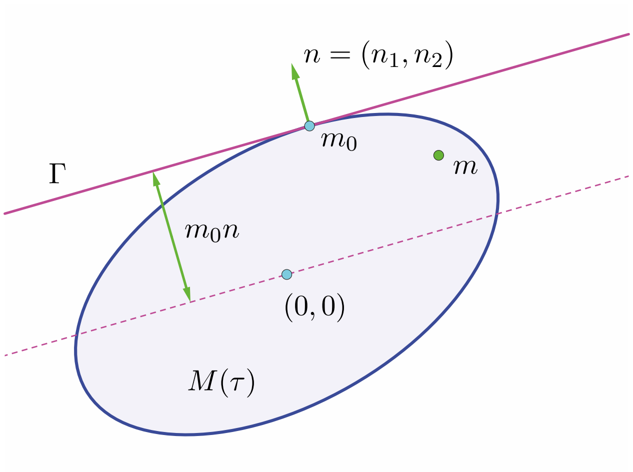

Fig. 1. The set $M(\tau)$, point m_0, and outer unit normal n

$$\begin{aligned} &\xi_0 - \int_{t_0}^{\tau} \omega(s) \sin \lambda s \, ds = 0, \\ &\eta_0 + \int_{t_0}^{\tau} \omega(s) \cos \lambda s \, ds = 0, \end{aligned} \tag{2.9}$$

where $\xi_0 = \bar{\xi}_0 \cos \lambda t_0 - \bar{\eta}_0 \sin \lambda t_0$, $\eta_0 = \bar{\xi}_0 \sin \lambda t_0 + \bar{\eta}_0 \cos \lambda t_0$.

To estimate the time τ from above, we consider the set (see Figure 1)

$$M(\tau) = \left\{ (\xi, \eta) \mid \xi = \int_{t_0}^{\tau} \omega(s) \sin \lambda s \, ds, \quad \eta = \int_{t_0}^{\tau} \omega(s) \cos \lambda s \, ds, \quad \omega \in B_\tau(\alpha) \right\},$$

where

$$B_\tau(\alpha) = \left\{ \omega(t), \ t_0 \le t \le \tau \mid \int_{t_0}^{\tau} \omega^2(t) \, dt \le \alpha^2, \quad \omega(\cdot) \in L_2[t_0, \tau] \right\}.$$

We only need to show that $(\xi_0, -\eta_0) \in M(\tau)$ to prove that (2.9) is satisfied at some $\tau > 0$ and $\omega(\cdot) \in B_\tau(\alpha)$.

For the set $M(\tau)$, we first prove the following statement.

Proposition 2.1. *The set $M(\tau)$ is convex and, for $\tau \le \tau_1$, $M(\tau) \subset M(\tau_1)$.*

P r o o f. To prove that $M(\tau)$ is convex, we take any two points $m_1, m_2 \in M(\tau)$. Then, there exist $\omega_1(\cdot), \omega_2(\cdot) \in B_\tau(\alpha)$ such that

$$m_1 = \int_{t_0}^{\tau} \omega_1(s)(\cos \lambda s, \sin \lambda s) \, ds, \qquad m_2 = \int_{t_0}^{\tau} \omega_2(s)(\cos \lambda s, \sin \lambda s) \, ds.$$

For any $0 \le \gamma \le 1$, we show that $\gamma m_1 + (1-\gamma) m_2 \in M(\tau)$. Indeed,

$$\gamma m_1 + (1-\gamma) m_2 = \int_{t_0}^{\tau} \big(\gamma \omega_1(s) + (1-\gamma)\omega_2(s)\big)(\cos \lambda s, \sin \lambda s) ds \in M(\tau)$$

since $\gamma\omega_1(\cdot)+(1-\gamma)\omega_2(\cdot)\in B_\tau(\alpha)$ because

$$\left(\int_{t_0}^{\tau}(\gamma\omega_1(s)+(1-\gamma)\omega_2(s))^2\,ds\right)^{1/2}\le\left(\int_{t_0}^{\tau}(\gamma\omega_1(s))^2\,ds\right)^{1/2}+\left(\int_{t_0}^{\tau}((1-\gamma)\omega_2(s))^2\,ds\right)^{1/2}$$
$$\le\gamma\alpha+(1-\gamma)\alpha=\alpha.$$

Next, to establish the inclusion $M(\tau)\subset M(\tau_1)$ for $\tau\le\tau_1$, we take any point $m\in M(\tau)$. Then, we have $m=\int_0^\tau(\sin\lambda s,\cos\lambda s)\omega(s)\,ds$ for some $\omega(\cdot)\in B_\tau(\alpha)$. Setting

$$\bar\omega(t)=\begin{cases}\omega(t), & t_0\le t\le\tau,\\ 0, & \tau<t\le\tau_1,\end{cases}$$

we obtain $\bar\omega(\cdot)\in B_\tau(\alpha)$ and $m=\int_{t_0}^{\tau_1}(\sin\lambda s,\cos\lambda s)\bar\omega(s)\,ds\in M(\tau_1)$, which is our claim. □

Let $m_0\in\partial M(\tau)$, where ∂M denotes the boundary of the set M, and $n=(n_1,n_2)$ be outer unit normal to the support straight line Γ to the set $M(\tau)$ at m_0 (see Figure 1). Then, for any point

$$m=\int_{t_0}^{\tau}(\sin\lambda s,\cos\lambda s)\omega(s)\,ds\in M(\tau),\quad \omega(\cdot)\in B_\tau(\alpha),$$

we have

$$nm\le nm_0, \tag{2.10}$$

where nm denotes the inner product of the vectors n and m. From this, for any $m\in M(\tau)$, by the Cauchy–Schwartz inequality we obtain

$$\begin{aligned}nm&=\int_{t_0}^{\tau}(n_1\sin\lambda s+n_2\cos\lambda s)\omega(s)\,ds\\&\le\left(\int_{t_0}^{\tau}(n_1\sin\lambda s+n_2\cos\lambda s)^2\,ds\right)^{1/2}\left(\int_{t_0}^{\tau}\omega^2(s)\,ds\right)^{1/2}\\&=\alpha F(n_1,n_2),\end{aligned}\tag{2.11}$$

where $F(n_1,n_2)=\left(\int_{t_0}^{\tau}(n_1\sin\lambda s+n_2\cos\lambda s)^2\,ds\right)^{1/2}$. Clearly, $F(n_1,n_2)\neq 0$, and the equality sign in inequality (2.11) holds true when

$$\omega(t)=\omega_0(t)=\frac{\alpha}{F(n_1,n_2)}(n_1\sin\lambda t+n_2\cos\lambda t),\quad t_0\le t\le\tau. \tag{2.12}$$

The inequality (2.10) is true, in particular, for the point m corresponding to the control $\omega_0(t)$ defined by (2.12): $\alpha F(n_1,n_2)\le nm_0$. Since (2.11) is true for any $m\in M(\tau)$, then for the point m_0 we have $nm_0\le\alpha F(n_1,n_2)$, and so, for the distance of the support line Γ from the origin, we have

$$nm_0=\alpha F(n_1,n_2). \tag{2.13}$$

Using the fact that the equality sign in (2.10) holds at $m = m_0$ and the equality sign in (2.10) holds for the only point corresponding to control (2.12) we conclude that the equation $nm = \alpha F(n_1, n_2)$ is true for the only point $m = m_0 \in M(\tau)$ corresponding to the control $\omega_0(t)$. Hence, the support line Γ beyond the point m_0 doesn't contain any other point of $M(\tau)$ (meaning that $M(\tau)$ is strictly convex).

Next, to estimate the minimum distance of the origin from the boundary $\partial M(\tau)$ of the set $M(\tau)$, we find the minimum of the function

$$F(n_1, n_2) = \left(n_1^2 \int_{t_0}^{\tau} \sin^2 \lambda s\, ds + 2n_1 n_2 \int_{t_0}^{\tau} \sin \lambda s \cos \lambda s\, ds + n_2^2 \int_{t_0}^{\tau} \cos^2 \lambda s\, ds \right)^{1/2}$$

subject to $n_1^2 + n_2^2 = 1$. To this end, we consider the following minimization problem

$$F^2(\tau) = a_1 n_1^2 + 2a_2 n_1 n_2 + a_3 n_2^2 \to \min, \quad n_1^2 + n_2^2 = 1, \tag{2.14}$$

where

$$a_1 = \int_{t_0}^{\tau} \sin^2 \lambda s\, ds, \quad a_2 = \int_{t_0}^{\tau} \sin \lambda s \cos \lambda s\, ds, \quad a_3 = \int_{t_0}^{\tau} \cos^2 \lambda s\, ds.$$

For the Lagrange function

$$H = a_1 n_1^2 + 2a_2 n_1 n_2 + a_3 n_2^2 + \mu(n_1^2 + n_2^2 - 1),$$

we write the following necessary conditions of minimum:

$$\frac{\partial H}{\partial n_1} = 2a_1 n_1 + 2a_2 n_2 + 2\mu n_1 = 0, \tag{2.15}$$

$$\frac{\partial H}{\partial n_2} = 2a_2 n_1 + 2a_3 n_2 + 2\mu n_2 = 0. \tag{2.16}$$

We multiply the equations (2.15) and (2.16) by n_1 and n_2, respectively, and add them up to obtain

$$a_1 n_1^2 + 2a_2 n_1 n_2 + a_3 n_2^2 = -\mu. \tag{2.17}$$

Also, (2.15) and (2.16) can be written in matrix form as follows:

$$\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = -\mu \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, \tag{2.18}$$

meaning that $-\mu$ is an eigenvalue of the matrix $A = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}$.

Since we are interested in finding the minimum of the function $F(n_1, n_2)$, combining (2.17) and (2.18) we conclude that $-\mu$ is the minimum value of $F(n_1, n_2)$.

It is easy to verify that the minimum eigenvalue of the matrix A is

$$\begin{aligned} -\mu &= \frac{1}{2}\left(a_1 + a_2 - \sqrt{(a_1 - a_3)^2 + 4a_2^2} \right) \\ &= \frac{1}{2}\left(\tau - t_0 - \left(\left(\int_{t_0}^{\tau} \cos 2\lambda s\, ds \right)^2 + \left(\int_{t_0}^{\tau} \sin 2\lambda s\, ds \right)^2 \right)^{1/2} \right). \end{aligned} \tag{2.19}$$

As

$$\left(\int_{t_0}^{\tau}\cos 2\lambda s\,ds\right)^2+\left(\int_{t_0}^{\tau}\sin 2\lambda s\,ds\right)^2=\frac{1}{4\lambda^2}\left[(\sin 2\lambda\tau-\sin 2\lambda t_0)^2+(\cos 2\lambda\tau-\cos 2\lambda t_0)^2\right]$$
$$=\frac{1}{2\lambda^2}\big(1-\cos 2(\tau-t_0)\big)$$
$$=\frac{1}{\lambda^2}\sin^2\lambda(\tau-t_0),$$

by (2.19) for the minimum of $F^2(n_1,n_2)$ we obtain the following estimate

$$-\mu=\frac{1}{2}\left(\tau-t_0-\frac{1}{\lambda}\big|\sin\lambda(\tau-t_0)\big|\right)\geq\frac{1}{2}\left(\tau-t_0-\frac{1}{\lambda}\right).$$

Thus, the minimum distance of the origin from the boundary $\partial M(\tau)$ of $M(\tau)$ is estimated from the below by $\alpha\sqrt{\frac{1}{2}\left(\tau-t_0-\frac{1}{\lambda}\right)}$ and so by (2.13) we have

$$|m_0|\geq nm_0=\alpha F(n_1,n_2)\geq\alpha\sqrt{\frac{1}{2}\left(\tau-t_0-\frac{1}{\lambda}\right)}. \tag{2.20}$$

So far we assumed m_0 to be a boundary point of $M(\tau)$. Since the right hand side of (2.20) is increasing in τ and approaches $+\infty$ as $\tau\to\infty$, therefore, if we assume m_0 to be any point in $\mathbb{R}^2$, then $m_0\in\partial M(\tau)$ at some τ that satisfy (2.20), that is $\tau\leq T_0$, where

$$T_0=t_0+\frac{2|m_0|^2}{\alpha^2}+\frac{1}{\lambda}. \tag{2.21}$$

This inequality shows that, for any $m_0\in\mathbb{R}^2$, at most at the time T_0 the point m_0 belongs to $\partial M(\tau)$. In particular, if $m_0=(\xi_0,-\eta_0)$, then $|m_0|^2=|\xi_0|^2+|\eta_0|^2$, and equations (2.21) can be written as follows:

$$\tau\leq t_0+\frac{2}{\alpha^2}(\xi_0^2+\eta_0^2)+\frac{1}{\lambda}. \tag{2.22}$$

Hence, the point $(\xi_0,-\eta_0)$ belongs to $M(\tau)$ at most at the time $T=t_0+\frac{2}{\alpha^2}(\xi_0^2+\eta_0^2)+\frac{1}{\lambda}$. Since

$$\xi_0^2+\eta_0^2=\left(\bar{\xi}_0\cos\lambda t_0+\bar{\eta}_0\sin\lambda t_0\right)^2+\left(\bar{\xi}_0\sin\lambda t_0+\bar{\eta}_0\cos\lambda t_0\right)^2=\bar{\xi}_0^2+\bar{\eta}_0^2,$$

therefore, equation (2.22) takes the form

$$\tau\leq t_0+\frac{2}{\alpha^2}(\bar{\xi}_0^2+\bar{\eta}_0^2)+\frac{1}{\lambda}.$$

The proof of Lemma 2.2 is complete. □

Next, we proceed to prove Lemma 2.1. By Lemma 2.2, there exist controls $\omega_k^0(t)$, $t_0\leq t\leq T_k$, such that the state of system (2.4) can be transferred to the origin within the time

$$T_k=t_0+\frac{2}{\alpha_k^2}\cdot(\bar{\xi}_{k0}^2+\bar{\eta}_{k0}^2)+\frac{1}{\lambda_k}.$$

By (2.5),

$$\alpha_k=\left(\frac{\bar{\xi}_{k0}^2+\bar{\eta}_{k0}^2}{\|\bar{x}^0\|^2+\|x^1\|^2}\right)^{1/2}(\rho_0-\sigma_0),$$

and so, in view of the inequality $\lambda_k \geq \sqrt{\mu_0}$, we have

$$T_k \leq t_0 + \frac{2}{(\rho_0 - \sigma_0)^2}\left(\|\bar{x}^0\|^2 + \|x^1\|^2\right) + \frac{1}{\sqrt{\mu_0}} = T_0.$$

Thus, for each k, system (2.4) can be transferred to the origin within the time T_0.

This means that, if the pursuer applies the strategy

$$u_k(t) = \begin{cases} v_k(t) - \omega_k^0(t), & t_0 \leq t \leq T_k, \\ v_k(t), & T_k < t \leq T_0, \\ 0, & t > T_0, \end{cases} \quad k = 1, 2, \ldots, \tag{2.23}$$

for which, by (2.6), we have

$$\int_{t_0}^{T_0} \|\bar{\omega}(s)\|^2\, ds = \sum_{k=1}^{\infty} \int_{t_0}^{T_k} \omega_k^2(s)\, ds \leq \sum_{k=1}^{\infty} \alpha_k^2 \leq (\rho_0 - \sigma_0)^2, \tag{2.24}$$

then pursuit is completed in game (2.1) by the time T_0. The strategy (2.23) can be represented as follows $u(t) = v(s) - \bar{\omega}(s)$, $t_0 \leq t \leq T_0$, where $\bar{\omega}(t) = (\bar{\omega}_1(t), \bar{\omega}_2(t), \ldots)$ with

$$\bar{\omega}_k(t) = \begin{cases} \omega_k^0(t), & t_0 \leq t \leq T_k, \\ 0, & T_k < t \leq T_0, \end{cases} \quad k = 1, 2, \ldots,$$

and it is admissible, since by (2.24),

$$\begin{aligned} \left(\int_{t_0}^{\infty} \|u(s)\|^2\, ds\right)^{1/2} &= \left(\int_{t_0}^{T_0} \|v(s) - \bar{\omega}(s)\|^2\, ds\right)^{1/2} \\ &\leq \left(\int_{t_0}^{T_0} \|v(s)\|^2\, ds\right)^{1/2} + \left(\int_{t_0}^{T_0} \|\bar{\omega}(s)\|^2\, ds\right)^{1/2} \\ &\leq \sigma_0 + \rho_0 - \sigma_0 = \rho_0. \end{aligned} \tag{2.25}$$

The proof of Lemma 2.1 is complete. □

§ 3. Pursuit game of many pursuers and one evader

In this section we formulate the main results of the paper. In Lemma 2.1, the condition $\mu_0 > 0$ is important, since, if $\mu_0 = 0$, then, by Theorem 3.1 of the work [16], evasion is possible from some initial states in infinite system (2.1) even though $\rho_0 > \sigma_0$.

Let $\mu_{i0} = \inf\limits_{k=1,2,\ldots} \mu_{ik}$. The following statement is the main result of the paper.

Theorem 3.1. *If $\mu_{i0} > 0$, $i = 1, 2, \ldots, m$, and $\rho_1^2 + \ldots + \rho_m^2 > \sigma^2$, then pursuit can be completed for a finite time in game (1.1) from any initial positions $\{x_i^0, x_i^1\}$, $i = 1, 2, \ldots, m$.*

P r o o f. To prove this theorem, we apply repeatedly Lemma 2.1. Let

$$\theta_1 = t_0 + \frac{2}{(\rho_1 - \sigma_1)^2}\left(\|\bar{x}_1^0\|^2 + \|x_1^1\|^2\right) + \frac{1}{\sqrt{\mu_{10}}}.$$

By Lemma 2.1, there exists a function $\omega_{10}(\cdot) \in B(\rho_1 - \sigma_1, t_0, \theta_1)$ such that if $v(\cdot) \in B(\sigma_1, t_0, \theta_1)$, that is

$$\int_{t_0}^{\theta_1} \|v(s)\|^2 \, ds \le \sigma_1^2, \tag{3.1}$$

then for the strategies

$$\begin{aligned} u_1(t, v(t)) &= \begin{cases} v(t) - w_{10}(t), & t_0 \le t \le \theta_1, \\ 0, & t \notin [t_0, \theta_1], \end{cases} \\ u_i(t, v(t)) &= 0, \quad t_0 \le t \le \theta_1, \quad i = 2, 3, \dots, m, \end{aligned} \tag{3.2}$$

of pursuers, pursuit is completed in game (1.1) by the pursuer x_1, that is, we'll have $x_1(\tau_1) = 0$, $\dot{x}_1(\tau_1) = 0$ at some time $\tau_1 \in [t_0, \theta_1]$, and we are done. The fact that the strategy $u_1(t, v(t))$ in (3.2) is admissible can be shown similarly to (2.25) using assumption (3.1). Hence, if the evader spends energy less than or equal to σ_1^2, then it will be captured by the first pursuer.

Let pursuit be not completed in the time interval $[t_0, \theta_1]$, that is either $x_1(t) \ne 0$ or $\dot{x}_1(t) \ne 0$ for all $t \in [t_0, \theta_1]$. We necessarily have then

$$\int_{t_0}^{\theta_1} \|v(s)\|^2 \, ds > \sigma_1^2.$$

In other words, the strategy of the first pursuer in (3.2) forces the evader to spend energy greater than σ_1^2.

Then, we consider system (1.1) for $i = 2$ with the initial position $x_2(\theta_1)$, $\dot{x}_2(\theta_1)$. Let

$$\theta_2 = \theta_1 + \frac{2}{(\rho_2 - \sigma_2)^2} \left(\|\bar{x}_2(\theta_1)\|^2 + \|\dot{x}_2(\theta_1)\|^2 \right) + \frac{1}{\sqrt{\mu_{20}}}.$$

We continue in this fashion. We define inductively the strategies of the pursuers. Let time θ_i, $1 \le i \le m - 1$, be defined. Then, we define time θ_{i+1} by the following equation:

$$\theta_{i+1} = \theta_i + \frac{2}{(\rho_{i+1} - \sigma_{i+1})^2} \left(\|\bar{x}_{i+1}(\theta_i)\|^2 + \|\dot{x}_{i+1}(\theta_i)\|^2 \right) + \frac{1}{\sqrt{\mu_{i+1,0}}}.$$

By Lemma 2.1, there exists a function $\omega_{i+1,0}(\cdot) \in B(\rho_{i+1} - \sigma_{i+1}, \theta_i, \theta_{i+1})$ such that if $v(\cdot) \in$ $\in B(\sigma_{i+1}, \theta_i, \theta_{i+1})$, then for the strategies

$$\begin{aligned} u_{i+1}(t, v(t)) &= \begin{cases} v(t) - w_{i+1,0}(t), & \theta_i \le t \le \theta_{i+1}, \\ 0, & t \notin [\theta_i, \theta_{i+1}], \end{cases} \\ u_i(t, v(t)) &= 0, \quad \theta_i \le t \le \theta_{i+1}, \quad i \in \{1, \dots, m\} \setminus \{i+1\}, \end{aligned} \tag{3.3}$$

of pursuers, the strategy $u_{i+1}(t, v(t))$ of the pursuer x_{i+1} is admissible and pursuit is completed in game (1.1) by the pursuer x_{i+1}, that is, we'll have $x_{i+1}(\tau_{i+1}) = 0$, $\dot{x}_{i+1}(\tau_{i+1}) = 0$ at some time $\tau_{i+1} \in [\theta_i, \theta_{i+1}]$ and we are done. Hence, if the evader spends energy less than or equal to σ_{i+1}^2 on the time interval $[\theta_i, \theta_{i+1}]$, then it will be captured by the pursuer x_{i+1}.

Let $x_{i+1}(t) \ne 0$ or $\dot{x}_{i+1}(t) \ne 0$ for all $t \in [\theta_i, \theta_{i+1}]$. We necessarily have then

$$\int_{\theta_i}^{\theta_{i+1}} \|v(s)\|^2 \, ds > \sigma_{i+1}^2, \quad i \in \{0, 1, \dots, m-1\}, \quad \theta_0 = t_0. \tag{3.4}$$

In other words, strategies (3.3) of pursuers force the evader to spend energy greater than σ_{i+1}^2.

Thus, to prove that pursuit can be completed in game (1.1), it suffices to establish that inequalities (3.4) fail to hold at least for one $i \in \{0, 1, \ldots, m-1\}$. We claim that inequalities (3.4) cannot be satisfied for all $i = 0, 1, \ldots, m-1$.

Assume the contrary, let inequalities (3.4) be satisfied for all $i = 0, 1, \ldots, m-1$. We then obtain from these inequalities the following:

$$\sigma^2 \geq \int\limits_{t_0}^{\theta_m} \|v(s)\|^2\,ds = \sum_{i=0}^{m-1} \int\limits_{\theta_i}^{\theta_{i+1}} \|v(s)\|^2\,ds > \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_m^2 = \sigma^2,$$

a contradiction. Hence, there is $k \in \{0, 1, 2, \ldots, m-1\}$ such that

$$\int\limits_{\theta_k}^{\theta_{k+1}} \|v(s)\|^2\,ds \leq \sigma_{k+1}^2$$

is true. Then, pursuit is completed in game (1.1) by the pursuer x_{k+1}, that is, we'll have $x_{k+1}(\tau_{k+1}) = 0$, $\dot{x}_{k+1}(\tau_{k+1}) = 0$ at some time $\tau_{k+1} \in [\theta_k, \theta_{k+1}]$. The proof of the theorem is complete. □

§ 4. Conclusions

We have studied a pursuit differential game of many pursuers and one evader. The game is described by the infinite systems of m inertial equations. We have obtained a condition in terms of energies of players which is sufficient for completion of pursuit in the game.

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Ruzakhon Yunus kizi Kazimirova, PhD student, Department of Mathematics and Statistics, Universiti Putra Malaysia, Serdang, Selangor, 43400, Malaysia;
Department of Mathematics, Andijan State University, Andijan, 170100, Uzbekistan.
E-mail: kazimirovarozaxon@gmail.com

Gafurjan Ismailovich Ibragimov, Doctor of Physics and Mathematics, Professor, Department of Dynamical Systems and Their Applications, V. I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, University street, 9, Tashkent, 100174, Uzbekistan;
Department of Applied Mathematics, Tashkent State University of Economics, Islam Karimov street, 49, Tashkent, 100066, Uzbekistan.
ORCID: https://orcid.org/0000-0002-4282-7482
E-mail: gofurjon.ibragimov@tsue.uz

Risman Mat Hasim, PhD, Senior Teacher, Department of Mathematics and Statistics, University Putra Malaysia, Serdang, Selangor, 43400, Malaysia.
ORCID: https://orcid.org/0000-0002-5358-587X
E-mail: risman@upm.edu.my

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Р. Ю. Казимирова, Г. И. Ибрагимов, Р. М. Хасим

Дифференциальная игра преследования со многими преследователями для бесконечной системы дифференциальных уравнений второго порядка

Ключевые слова: дифференциальная игра, управление, стратегия, много преследователей, бесконечная система дифференциальных уравнений, интегральное ограничение.

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Изучается дифференциальная игра преследования со многими преследователями и одним убегающим. Игра описывается бесконечной системой m инерционных уравнений. По определению преследование завершается, если состояние одной из систем и его производная равны нулю в некоторый момент времени. В литературе такое условие завершения игры называется мягкой посадкой. В терминах энергий игроков получено условие, которое является достаточным для завершения преследования в игре. Также построены стратегии преследующих, гарантирующие завершение преследования в игре.

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Казимирова Рузахон Юнус кизи, Университет Путра Малайзии, 43400, Малайзия, Селангор, г. Серданг;
математический факультет, Андижанский государственный университет, 170100, Узбекистан, г. Андижан.
E-mail: kazimirovarozaxon@gmail.com

Ибрагимов Гафуржан Исмаилович, д. ф.-м. н., профессор, лаборатория динамических систем и их приложений, Институт математики имени В. И. Романовского АН Узбекистана, 100174, Узбекистан, г. Ташкент, ул. Университетская, 9;
кафедра прикладной математики, Ташкентский государственный экономический университет, 100066, Узбекистан, г. Ташкент, ул. Ислама Каримова, 49.
ORCID: https://orcid.org/0000-0002-4282-7482
E-mail: gofurjon.ibragimov@tsue.uz

Хасим Рисман Мат, к. ф.-м. н., Университет Путра Малайзии, 43400, Малайзия, Селангор, г. Серданг.
ORCID: https://orcid.org/0000-0002-5358-587X
E-mail: risman@upm.edu.my

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