MSC2010: 93B60, 93B55, 93B52, 93D20, 93C15, 93C05, 34H15, 93C55

© ***V. A. Zaitsev, I. G. Kim***

ON FINITE SPECTRUM ASSIGNMENT PROBLEM IN BILINEAR SYSTEMS WITH STATE DELAY

We consider a bilinear control system defined by a linear time-invariant system of differential equations with delay in the state variable. We study an arbitrary finite spectrum assignment problem by stationary control. One needs to construct constant control vector such that the characteristic quasi-polynomial of the closed-loop system becomes a polynomial with arbitrary preassigned coefficients. We obtain conditions on coefficients of the system under which the criterion was found for solvability of this finite spectrum assignment problem. This criterion is expressed in terms of rank conditions for matrices of the special form. Interconnection of these rank conditions with the property of consistency for truncated system without delay is shown. Corollaries on stabilization of a bilinear system with delay are obtained. The results extend the previously obtained results on spectrum assignment for linear systems with static output feedback with delay and for bilinear systems without delay. The results obtained are transferred to discrete-time bilinear systems with delay. An illustrative example is considered.

Keywords: linear delay systems, spectrum assignment, stabilization, bilinear system.

DOI: 10.20537/vm190102

Introduction

Stabilization problems for bilinear systems with delays were studied in many papers; see, e.g., [1–5] for continuous-time systems, [6–8] for discrete-time systems. In the papers [1–3], on the basis of the Lyapunov–Krasovsky method, the conditions are given for stabilization of the systems by means of state feedback. These conditions are expressed in terms of solutions for algebraic Riccati equations [1], of linear matrix inequalities [2, 3]. For obtaining sufficient conditions for global asymptotic stabilization by static state feedback [4, 5] and by dynamic output feedback [4], the LaSalle invariance principle is applied. The problem of stabilization by means of output feedback for discrete-time systems with delay is considered in [7, 8]. Conditions for stabilization are presented and the procedure for constructing a stabilizing regulator is given. In the present paper, we obtain conditions for arbitrary finite spectrum assignability for time-delay bilinear systems by stationary feedback and, as a consequence, stabilization conditions both for continuous-time and discrete-time systems.

§ 1. Continuous-time systems with delay

Let $\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{R}$; $\mathbb{K}^n=\{x=\operatorname{col}(x_1,\dots,x_n)\colon x_i\in\mathbb{K}\}$ is the linear space of vectors over $\mathbb{K}$; $M_{m,n}(\mathbb{K})$ is the space of $m\times n$-matrices over $\mathbb{K}$; $M_n(\mathbb{K}):=M_{n,n}(\mathbb{K})$; $I\in M_n(\mathbb{K})$ is the identity matrix; T is the transposition of a vector or a matrix; $*$ is the Hermitian conjugation, i.e., $A^*=\overline{A}^T$; $\chi(H;\lambda)$ and $\operatorname{Sp}H$ are the characteristic polynomial and the trace of a matrix $H\in M_n(\mathbb{K})$, respectively.

Consider a bilinear time-invariant differential system with delay in the state variable of the following form:

$$\dot x(t)=A_0x(t)+B_0x(t-h)+\Big(\sum_{j=1}^{r}u_jA_j\Big)x(t)+\Big(\sum_{\ell=1}^{s}v_\ell B_\ell\Big)x(t-h),\quad t>0, \tag{1.1}$$

with initial conditions $x(\tau) = \mu(\tau)$, $\tau \in [-h, 0]$; here A_j, $B_\ell \in M_n(\mathbb{K})$, $j = \overline{0, r}$, $\ell = \overline{0, s}$; $h > 0$ is a constant delay, $\mu\colon [-h, 0] \to \mathbb{K}^n$ is a continuous function; $x \in \mathbb{K}^n$ is a state vector, $u = \operatorname{col}(u_1, \ldots, u_r) \in \mathbb{K}^r$, $v = \operatorname{col}(v_1, \ldots, v_s) \in \mathbb{K}^s$ are control vectors.

In [9] the following linear time-invariant differential control system with delay was considered:

$$\dot{x}(t) = Ax(t) + Px(t-h) + Hw(t), \quad t > 0, \tag{1.2}$$
$$y(t) = C^* x(t), \tag{1.3}$$

where $A, P \in M_n(\mathbb{K})$, $H \in M_{n,m}(\mathbb{K})$, $C \in M_{n,k}(\mathbb{K})$, $h > 0$ is a constant delay, $x \in \mathbb{K}^n$ is a state vector, $w \in \mathbb{K}^m$ and $y \in \mathbb{K}^k$ are input and output vectors, respectively. For the system (1.2), (1.3) in [9] the controller is constructed as linear static output feedback with delay

$$w(t) = Q_0 y(t) + Q_1 y(t-h), \quad t > 0, \tag{1.4}$$

where $Q_0, Q_1 \in M_{m,k}(\mathbb{K})$ are constant. The corresponding closed-loop system (1.2), (1.3), (1.4) has the form

$$\dot{x}(t) = (A + HQ_0C^*)x(t) + (P + HQ_1C^*)x(t-h). \tag{1.5}$$

In [9, § 1] sufficient conditions are obtained for assigning an arbitrary finite spectrum for the system (1.5). The system (1.5) can be considered as a particular case of the system (1.1). In fact, every system (1.5), where $H = [h_1, \ldots, h_m]$, $C = [c_1, \ldots, c_k]$, $h_i, c_j \in \mathbb{K}^n$, $Q_0 = \{\alpha_{ij}\}$, $Q_1 = \{\beta_{ij}\}$, $\alpha_{ij}, \beta_{ij} \in \mathbb{K}$, $i = \overline{1, m}$, $j = \overline{1, k}$, can be rewritten in the form (1.1), where $r = s = mk$, $A_0 = A$, $B_0 = P$, $A_1 = B_1 = h_1 c_1^*$, $A_2 = B_2 = h_1 c_2^*$, $\ldots$, $A_k = B_k = h_1 c_k^*$, $A_{k+1} = B_{k+1} = h_2 c_1^*$, $\ldots$, $A_{2k} = B_{2k} = h_2 c_k^*$, $\ldots$, $A_r = h_m c_k^*$, $u = \operatorname{col}(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1k}, \alpha_{21}, \ldots, \alpha_{2k}, \ldots, \alpha_{m1}, \ldots, \alpha_{mk})$, $v = \operatorname{col}(\beta_{11}, \beta_{12}, \ldots, \beta_{1k}, \beta_{21}, \ldots, \beta_{2k}, \ldots, \beta_{m1}, \ldots, \beta_{mk})$.

In the present paper, we obtain sufficient conditions for assigning an arbitrary finite spectrum for the system (1.1). These results extend the results [9, § 1] from the system (1.5) to the system (1.1).

Denote by

$$\varphi(\lambda, e^{-\lambda h}) = \det\Big[\lambda I - \Big(A_0 + \sum_{j=1}^{r} u_j A_j\Big) - e^{-\lambda h}\Big(B_0 + \sum_{\ell=1}^{s} v_\ell B_\ell\Big)\Big]$$

the characteristic function of the system (1.1). This function is quasi-polynomial. The characteristic equation $\varphi(\lambda, e^{-\lambda h}) = 0$ of the system (1.1) has the form

$$\lambda^n + \sum_{i=1}^{n} \sum_{k=0}^{i} \delta_{ik} \lambda^{n-i} e^{-\lambda hk} = 0. \tag{1.6}$$

Here δ_{ik} depend on A_j, B_ℓ, u_j, v_ℓ. The set $\sigma = \{\lambda \in \mathbb{C} : \varphi(\lambda, e^{-\lambda h}) = 0\}$ of the roots of (1.6) is called the spectrum of the system (1.1). If $\mathbb{K} = \mathbb{R}$, then the spectrum σ is symmetric with respect to the real axis. In general, the spectrum σ of a system with delay (1.1) is countable. If $\delta_{ik} = 0$ for all $i = \overline{1, n}$, $k = \overline{1, i}$ in the equation (1.6), then the characteristic quasi-polynomial is polynomial and the spectrum σ is finite. Consider the problem of assigning an arbitrary finite spectrum σ for the system (1.1) by constant control.

Definition 1. We say that the system (1.1) is *arbitrary finite spectrum assignable by constant control* if for any $\gamma_i \in \mathbb{K}$, $i = \overline{1, n}$, there exist $u \in \mathbb{K}^r$, $v \in \mathbb{K}^s$ such that:

$$\varphi(\lambda, e^{-\lambda h}) = \lambda^n + \gamma_1 \lambda^{n-1} + \ldots + \gamma_n.$$

Suppose that the coefficients of the system (1.1) have the following special form: the matrix A_0 has the lower Hessenberg form with non-zero superdiagonal entries; for some $p \in \{1, \ldots, n\}$, the first $p-1$ rows and the last $n-p$ columns of A_j, $j=\overline{1,r}$, are equal to zero, i.e.,

$$A_0=\begin{pmatrix} a_{11} & a_{12} & 0 & \ldots & 0 \\ a_{21} & a_{22} & a_{23} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n-1,1} & a_{n-1,2} & \ldots & \ldots & a_{n-1,n} \\ a_{n1} & a_{n2} & \ldots & \ldots & a_{nn} \end{pmatrix}, \qquad a_{i,i+1}\neq 0, \quad i=\overline{1,n-1}; \tag{1.7}$$

$$A_j=\begin{pmatrix} 0 & 0 \\ \widehat{A}_j & 0 \end{pmatrix}, \quad \widehat{A}_j\in M_{n-p+1,p}(\mathbb{K}), \quad j=\overline{1,r}. \tag{1.8}$$

For that system without delay (i.e., for the case $B_\ell=0$, $\ell=\overline{0,s}$) it was proved in [10] (see also [11]) that the system is arbitrary finite spectrum assignable by constant control iff the rank of the matrix $\varGamma=\{\mathrm{Sp}\,(A_jA_0^{i-1})_{i,j=1}^{n,r}\}$ is equal to n. Here we extend this result to systems with delay. Suppose that the matrices B_ℓ, $\ell=\overline{0,s}$, of the system (1.1) have the special form as well: the first $p-1$ rows and the last $n-p$ columns of B_ℓ, $\ell=\overline{0,s}$, are equal to zero, i.e.,

$$B_\ell=\begin{pmatrix} 0 & 0 \\ \widehat{B}_\ell & 0 \end{pmatrix}, \quad \widehat{B}_\ell\in M_{n-p+1,p}(\mathbb{K}), \quad \ell=\overline{0,s}, \qquad p\in\{1,\ldots,n\}. \tag{1.9}$$

The number p in (1.9) is the same as in (1.8).

Let $\chi(A_0;\lambda)=\lambda^n+\alpha_1\lambda^{n-1}+\ldots+\alpha_n$. Set $\alpha_0:=1$. From the matrix A_0, we construct the matrices

$$F_\nu=\alpha_0A_0^\nu+\alpha_1A_0^{\nu-1}+\ldots+\alpha_\nu I, \quad \nu=\overline{0,n-1}. \tag{1.10}$$

Further, we will use the following lemma (see [12, Lemma 1]).

Lemma 1. *Suppose a matrix A_0 has the form (1.7) and a matrix $D\in M_n(\mathbb{K})$ has the following form for some $p\in\{1,\ldots,n\}$:*

$$D=\begin{pmatrix} 0 & 0 \\ D_1 & 0 \end{pmatrix}, \quad D_1\in M_{n-p+1,p}(\mathbb{K}). \tag{1.11}$$

Let $\chi(A_0+D;\lambda)=\lambda^n+\gamma_1\lambda^{n-1}+\ldots+\gamma_n$. Then $\gamma_i=\alpha_i-\mathrm{Sp}\,(DF_{i-1})$ for all $i=1,\ldots,n$.

From the system (1.1) we construct the matrices $\varGamma_0\in M_{n,r}(\mathbb{K})$, $\varGamma_1\in M_{n,s}(\mathbb{K})$, $\varLambda_1\in M_{n,1}(\mathbb{K})$:

$$\varGamma_0=\begin{pmatrix} \mathrm{Sp}\,(A_1) & \mathrm{Sp}\,(A_2) & \ldots & \mathrm{Sp}\,(A_r) \\ \mathrm{Sp}\,(A_1A_0) & \mathrm{Sp}\,(A_2A_0) & \ldots & \mathrm{Sp}\,(A_rA_0) \\ \ldots & \ldots & \ldots & \ldots \\ \mathrm{Sp}\,(A_1A_0^{n-1}) & \mathrm{Sp}\,(A_2A_0^{n-1}) & \ldots & \mathrm{Sp}\,(A_rA_0^{n-1}) \end{pmatrix}, \tag{1.12}$$

$$\varGamma_1=\begin{pmatrix} \mathrm{Sp}\,(B_1) & \mathrm{Sp}\,(B_2) & \ldots & \mathrm{Sp}\,(B_s) \\ \mathrm{Sp}\,(B_1A_0) & \mathrm{Sp}\,(B_2A_0) & \ldots & \mathrm{Sp}\,(B_sA_0) \\ \ldots & \ldots & \ldots & \ldots \\ \mathrm{Sp}\,(B_1A_0^{n-1}) & \mathrm{Sp}\,(B_2A_0^{n-1}) & \ldots & \mathrm{Sp}\,(B_sA_0^{n-1}) \end{pmatrix}, \quad \varLambda_1=\begin{pmatrix} \mathrm{Sp}\,(B_0) \\ \mathrm{Sp}\,(B_0A_0) \\ \ldots \\ \mathrm{Sp}\,(B_0A_0^{n-1}) \end{pmatrix}; \tag{1.13}$$

and construct the matrix $\varDelta_1=[\varGamma_1,\varLambda_1]\in M_{n,s+1}(\mathbb{K})$.

Theorem 1. *Suppose that the matrices of the system (1.1) have the special form (1.7), (1.8), (1.9). Then the system (1.1) is arbitrary finite spectrum assignable by constant control iff the following conditions hold:*

$$\mathrm{rank}\,\varGamma_0=n, \tag{1.14}$$

$$\mathrm{rank}\,\varGamma_1=\mathrm{rank}\,\varDelta_1. \tag{1.15}$$

P r o o f. Suppose the matrices of the system (1.1) have the form (1.7), (1.8), (1.9). Consider the problem of assigning an arbitrary finite spectrum. Let a polynomial

$$q(\lambda) = \lambda^n + \gamma_1 \lambda^{n-1} + \ldots + \gamma_n \tag{1.16}$$

with numbers $\gamma_i \in \mathbb{K}$ be given. One needs to find $u \in \mathbb{K}^r$, $v \in \mathbb{K}^s$ such that the characteristic quasi-polynomial $\varphi(\lambda, e^{-\lambda h})$ of the system (1.1) satisfies the equality

$$\varphi(\lambda, e^{-\lambda h}) = q(\lambda). \tag{1.17}$$

Denote

$$D = \sum_{j=1}^{r} u_j A_j + e^{-\lambda h}\Big(B_0 + \sum_{\ell=1}^{s} v_\ell B_\ell\Big). \tag{1.18}$$

We have

$$\varphi(\lambda, e^{-\lambda h}) = \det\big(\lambda I - (A_0 + D)\big) = \chi(A_0 + D; \lambda). \tag{1.19}$$

It follows from conditions (1.8), (1.9) that the matrix (1.18) has the form (1.11). Taking into account (1.19), (1.17), (1.16), condition (1.7) and applying Lemma 1, we obtain that the system (1.1) is arbitrary finite spectrum assignable by constant control iff there exist $u \in \mathbb{K}^r$, $v \in \mathbb{K}^s$ such that for all $i = \overline{1, n}$ the following equalities hold:

$$\gamma_i = \alpha_i - \operatorname{Sp}\Big(\big(\sum_{j=1}^{r} u_j A_j\big) F_{i-1}\Big) - e^{-\lambda h} \operatorname{Sp}\Big(\big(B_0 + \sum_{\ell=1}^{s} v_\ell B_\ell\big) F_{i-1}\Big). \tag{1.20}$$

Equalities (1.20) hold iff

$$\gamma_i = \alpha_i - \operatorname{Sp}\Big(\big(\sum_{j=1}^{r} u_j A_j\big) F_{i-1}\Big), \qquad \operatorname{Sp}\Big(\big(B_0 + \sum_{\ell=1}^{s} v_\ell B_\ell\big) F_{i-1}\Big) = 0, \qquad i = 1, \ldots, n. \tag{1.21}$$

Taking into account the definition (1.10) of the matrices F_ν (and using denotation $A_0^0 := I$), we have

$$\operatorname{Sp}\Big(\big(\sum_{j=1}^{r} u_j A_j\big) F_{i-1}\Big) = \sum_{\nu=0}^{i-1} \alpha_{i-1-\nu}\Big(\sum_{j=1}^{r} u_j \operatorname{Sp}(A_j A_0^\nu)\Big), \quad i = 1, \ldots, n,$$

$$\operatorname{Sp}\Big(\big(B_0 + \sum_{\ell=1}^{s} v_\ell B_\ell\big) F_{i-1}\Big) = \sum_{\nu=0}^{i-1} \alpha_{i-1-\nu}\Big(\operatorname{Sp}(B_0 A_0^\nu) + \sum_{\ell=1}^{s} v_\ell \operatorname{Sp}(B_\ell A_0^\nu)\Big), \quad i = 1, \ldots, n.$$

Therefore the equalities (1.21) are equivalent to two systems of linear equations

$$\sum_{\nu=0}^{i-1} \alpha_{i-1-\nu}\Big(\sum_{j=1}^{r} u_j \operatorname{Sp}(A_j A_0^\nu)\Big) = \alpha_i - \gamma_i, \qquad i = 1, \ldots, n, \tag{1.22}$$

$$\sum_{\nu=0}^{i-1} \alpha_{i-1-\nu}\Big(\sum_{\ell=1}^{s} v_\ell \operatorname{Sp}(B_\ell A_0^\nu)\Big) = -\sum_{\nu=0}^{i-1} \alpha_{i-1-\nu} \operatorname{Sp}(B_0 A_0^\nu), \qquad i = 1, \ldots, n, \tag{1.23}$$

with r unknown variables $u_1, \ldots, u_r$ and with s unknown variables $v_1, \ldots, v_s$. Let us rewrite (1.22), (1.23) in the vector form. Let us construct the matrices

$$G := \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ \alpha_1 & 1 & 0 & \ldots & 0 \\ \alpha_2 & \alpha_1 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} & \ldots & 1 \end{pmatrix}, \tag{1.24}$$

and (1.12), (1.13). Denote $w_0 := \mathrm{col}\,(\alpha_1 - \gamma_1, \dots, \alpha_n - \gamma_n) \in \mathbb{K}^n$. Then one can rewrite systems (1.22), (1.23) in the vector form

$$G\varGamma_0 u = w_0, \tag{1.25}$$
$$G\varGamma_1 v = -G\varLambda_1. \tag{1.26}$$

Taking into account that $\det G = 1 \neq 0$, we see that the system (1.25) is resolvable with respect to u (over $\mathbb{K}$) for any pregiven $\gamma_i \in \mathbb{K}$, $i = \overline{1,n}$, iff condition (1.14) holds, and the system (1.26) is resolvable with respect to v (over $\mathbb{K}$) iff condition (1.15) holds. Finding u, v from (1.25), (1.26), we assign a desirable polynomial (1.16) as the characteristic function for the system (1.1). □

Remark 1. Suppose that the system (1.1) has the form (1.5). Suppose that for this system (1.5) the sufficient conditions of [9, § 1] hold for assigning an arbitrary finite spectrum, i.e., the matrices of the system have the special form and the matrices

$$C^*H, \quad C^*A_0H, \quad \dots, \quad C^*A_0^{n-1}H$$

are linearly independent. One can check that, in this case, the coefficients of the system (1.5) (which is considered as the system (1.1)) have the form (1.7), (1.8), (1.9) and conditions (1.14), (1.15) hold. Thus, Theorem 1 extends the results of [9, § 1] from systems (1.5) to systems (1.1). □

Remark 2. Suppose that the system (1.1) does not have delay, i.e., $B_\ell = 0$, $\ell = \overline{0,r}$. Then condition (1.15) holds. In that case, Theorem 1 coincides with [10, Theorem 2]. Thus, Theorem 1 extends the results of [10] from bilinear systems without delay to bilinear systems (1.1) with delay. □

Consider a problem of stabilization for the system (1.1) by constant control: one needs to construct $u \in \mathbb{K}^r$, $v \in \mathbb{K}^s$ such that the system (1.1) is asymptotically stable. The system (1.1) is asymptotically stable if the spectrum σ lies in the left half-plane $\omega = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\}$. If the system (1.1) is arbitrary finite spectrum assignable by constant control, then choosing the polynomial (1.16) in such a way that its roots belong to ω, one can obtain asymptotical stability for the system (1.1). Thus, the following obvious corollary follows from Theorem 1.

Corollary 1. *Suppose that the matrices of the system* (1.1) *have the special form* (1.7), (1.8), (1.9). *Suppose conditions* (1.14), (1.15) *hold. Then the system* (1.1) *is asymptotically stabilizable by constant control.*

For the system (1.1), let us construct the “truncated system” (without delay) assuming $B_\ell = 0$, $\ell = \overline{0,s}$:

$$\dot{x}(t) = \Bigl(A_0 + \sum_{j=1}^{r} u_j A_j\Bigr) x(t). \tag{1.27}$$

Denote by $X(t,s)$ the transition matrix of the free system $\dot{x}(t) = A_0 x(t)$. Hence, $X(t,s) = e^{(t-s)A_0}$.

Definition 2. The system (1.27) is said to be *consistent on* $[t_0, t_1]$ if for any $H \in M_n(\mathbb{K})$ there exists a piecewise continuous control function $\widehat{u} : [t_0, t_1] \to \mathbb{K}^r$ such that the solution of the $n \times n$-matrix initial value problem

$$\dot{Z}(t) = A_0 Z(t) + \sum_{j=1}^{r} \bigl(\widehat{u}_j(t) A_j\bigr) X(t, t_0), \qquad Z(t_0) = 0,$$

satisfies condition $Z(t_1) = H$.

The property of consistency was introduced in [13] for continuous-time systems (1.27), which are not necessarily time-invariant. Time-invariant consistent systems (1.27) with continuous time were investigated in [14, 15]. It was proved in [14, Assertion 5] that, for system (1.27) with a cyclic matrix A_0 (in particular, with A_0 of the form (1.7)), the property of consistency is sufficient for condition (1.14) to be fulfilled. Thus, the following theorem holds.

Theorem 2. *Suppose that the matrices of the system* (1.1) *have the special form* (1.7), (1.8), (1.9). *Suppose that the truncated system* (1.27) *is consistent and condition* (1.15) *holds. Then the system* (1.1) *is arbitrary finite spectrum assignable by constant control.*

Remark 3. Suppose that system (1.1) does not have delay, i.e., $B_\ell = 0$, $\ell = \overline{0,r}$. Then condition (1.15) holds. In that case Theorem 2 coincides with the assertion ($1 \Longrightarrow 3$) of Theorem 2 in [14]. Thus, Theorem 1 together with Theorem 2 extends Theorem 2 of [14] from bilinear systems without delay (1.27) to bilinear systems (1.1) with delay. □

§ 2. Discrete-time systems with delay

Consider a bilinear time-invariant discrete-time system with delay in the state variable of the following form:

$$x(t+1) = A_0x(t) + B_0x(t-h) + \Big(\sum_{j=1}^{r} u_jA_j\Big)x(t) + \Big(\sum_{\ell=1}^{s} v_\ell B_\ell\Big)x(t-h), \tag{2.1}$$

$t = 0, 1, 2, \ldots$, with initial conditions $x(\tau) = \mu(\tau)$, $\tau = -h, \ldots, 0$; here $A_j, B_\ell \in M_n(\mathbb{K})$, $j = \overline{0,r}$, $\ell = \overline{0,s}$; $h > 0$ is an integer constant delay; $x \in \mathbb{K}^n$ is a state vector, $u = \operatorname{col}(u_1, \ldots, u_r) \in \mathbb{K}^r$, $v = \operatorname{col}(v_1, \ldots, v_s) \in \mathbb{K}^s$ are control vectors.

Denote by

$$\psi(\lambda) = \det\Big[\lambda I - \Big(A_0 + \sum_{j=1}^{r} u_jA_j\Big) - \lambda^{-h}\Big(B_0 + \sum_{\ell=1}^{s} v_\ell B_\ell\Big)\Big]$$

the characteristic function of the system (2.1). This function is rational. The characteristic equation $\psi(\lambda) = 0$ of the system (2.1) has the form

$$\lambda^n + \sum_{i=1}^{n}\sum_{k=0}^{i} \delta_{ik}\lambda^{n-i-hk} = 0. \tag{2.2}$$

The set $\rho = \{\lambda \in \mathbb{C} : \psi(\lambda) = 0\}$ of the roots of (2.2) is called the spectrum of the system (2.1). If $\mathbb{K} = \mathbb{R}$, then the spectrum ρ is symmetric with respect to the real axis. The spectrum ρ of a discrete-time system with delay (2.1) consists of a finite amount $N \geqslant n$ of numbers $\lambda_m \in \mathbb{C}$, $m = \overline{1,N}$, in general. Iff $\delta_{ik} = 0$ for all $i = \overline{1,n}$, $k = \overline{1,i}$ in the equation (2.2), then the spectrum ρ consists of exactly n points (with accounting the multiplicity). Consider the problem of assigning an arbitrary n-point spectrum ρ for the system (2.1) by constant control.

Definition 3. We say that the system (2.1) is *arbitrary n-point spectrum assignable by constant control* if for any $\gamma_i \in \mathbb{K}$, $i = \overline{1,n}$, there exist $u \in \mathbb{K}^r$, $v \in \mathbb{K}^s$ such that:

$$\psi(\lambda) = \lambda^n + \gamma_1\lambda^{n-1} + \ldots + \gamma_n.$$

Theorem 3. *Suppose that the matrices of the system* (2.1) *have the special form* (1.7), (1.8), (1.9). *Then the system* (2.1) *is arbitrary n-point spectrum assignable by constant control iff conditions* (1.14), (1.15) *hold.*

The proof of Theorem 3 is identical to the proof of Theorem 1.

Corollary 2. *Suppose that the matrices of the system* (2.1) *have the special form* (1.7), (1.8), (1.9). *Suppose conditions* (1.14), (1.15) *hold. Then the system* (2.1) *is asymptotically stabilizable by constant control.*

For the system (2.1), consider the truncated system

$$x(t+1) = \Big(A_0 + \sum_{j=1}^{r} u_j A_j\Big) x(t). \tag{2.3}$$

Denote by $X(t,s)$ the transition matrix of the free system $x(t+1) = A_0 x(t)$. Hence, $X(t,s) = A_0^{t-s}$, $t \geqslant s$.

Definition 4. The system (2.3) is said to be *consistent on* $[t_0, t_1) \subset \mathbb{Z}$ [16] if, for any matrix $H \in M_n(\mathbb{K})$, there exists a $\widehat{u}(t) = \operatorname{col}\big(\widehat{u}_1(t), \ldots, \widehat{u}_r(t)\big)$, $t = t_0, \ldots, t_1 - 1$, such that the solution of the $n \times n$-matrix initial value problem

$$Z(t+1) = A_0 Z(t) + \sum_{j=1}^{r} \big(\widehat{u}_j(t) A_j\big) X(t, t_0), \qquad Z(t_0) = 0,$$

satisfies condition $Z(t_1) = H$.

The property of consistency was introduced in [16] for discrete-time systems (2.3), which are not necessarily time-invariant. Consistent systems (2.3) with discrete time were investigated in [16, 17]. It was proved in [17, Assertion 3] that, for time-invariant system (2.3) with a cyclic matrix A_0 (in particular, with A_0 of the form (1.7)), the property of consistency is sufficient for condition (1.14) to be fulfilled. Thus, the following theorem holds.

Theorem 4. *Suppose that the matrices of the system* (2.1) *have the special form* (1.7), (1.8), (1.9). *Suppose that the truncated system* (2.3) *is consistent, and condition* (1.15) *holds. Then the system* (2.1) *is arbitrary n-point spectrum assignable by constant control.*

Remark 4. Suppose that system (2.1) does not have delay, i.e., $B_\ell = 0$, $\ell = \overline{0, r}$. Then condition (1.15) holds. In that case, Theorem 4 coincides with the assertion ($1 \Longrightarrow 3$) of Theorem 6 in [17]. Thus, Theorem 3 together with Theorem 4 extends Theorem 6 of [17] from bilinear systems without delay (2.3) to bilinear systems (2.1) with delay. □

Remark 5. The condition $r \geqslant n$ is obviously necessary both for condition (1.14) and for the property of consistency of the truncated system (see [14, Corollary 5] for continuous-time systems and [17, Corollary 7] for discrete-time systems). Nevertheless, there is no necessary estimation to s for condition (1.15) to be fulfilled.

§ 3. Example

Consider an example illustrating Theorem 1. Suppose $\mathbb{K} = \mathbb{C}$, $n = 3$, $r = 3$, $s = 2$, $p = 2$ and matrices of the system (1.1) have the following form:

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ 0 & i & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i & 0 \\ i & 0 & 0 \end{pmatrix},$$
$$B_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ i & 1 & 0 \\ 0 & -i & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ i & -1 & 0 \end{pmatrix}. \tag{3.1}$$

Matrices (3.1) of the system (1.1) have the special form (1.7), (1.8), (1.9). We have $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = -1$. Let's calculate the matrices (1.24), (1.12), (1.13):

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varGamma_0 = \begin{pmatrix} 0 & 0 & -i \\ -1 & 2i & 0 \\ 1 & 0 & i \end{pmatrix}, \quad \varGamma_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & i \end{pmatrix}, \quad \varLambda_1 = \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}. \tag{3.2}$$

Obviously, conditions (1.14), (1.15) hold. Hence, by Theorem 1, the system (1.1) with the matrices (3.1) is arbitrary finite spectrum assignable by constant control. Let us construct that control $u \in \mathbb{K}^3$, $v \in \mathbb{K}^2$. Suppose, for example, that:

$$q(\lambda) = (\lambda + 1)^3.$$

We have $\gamma_1 = 3$, $\gamma_2 = 3$, $\gamma_3 = 1$. Hence,

$$w_0 = \operatorname{col}(\alpha_1 - \gamma_1, \alpha_2 - \gamma_2, \alpha_3 - \gamma_3) = \operatorname{col}(-3, -3, -2). \tag{3.3}$$

Resolving the systems (1.25), (1.26) with coefficients (3.2), (3.3), we obtain

$$u = \operatorname{col}(-5, 4i, -3i), \quad v = \operatorname{col}(0, i). \tag{3.4}$$

The system (1.1) with the matrices (3.1) and with the control (3.4) takes the form

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -3 & 1 \\ -1 & -4 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ 0 & -i & 0 \end{pmatrix} x(t-h). \tag{3.5}$$

Calculating the characteristic function for the system (3.5), we obtain that

$$\varphi(\lambda) = (\lambda + 1)^3.$$

In particular the system (3.5) is asymptotically stable. □

Funding. The research was supported by the Russian Foundation for Basic Research (project no. 18–51–41005) and by the Ministry of Education and Science of the Russian Federation in the framework of the basic part (project no. 1.5211.2017/8.9).

REFERENCES

1. Niculescu S.-I., Tarbouriech S., Dion J.-M., Dugard L. Stability criteria for bilinear systems with delayed state and saturating actuators, *Proceedings of 1995 34th IEEE Conference on Decision and Control*, 1995, pp. 2064–2069. https://doi.org/10.1109/CDC.1995.480652
2. Yoneyama J. Stabilization of Takagi-Sugeno fuzzy bilinear time-delay systems, *2010 IEEE International Symposium on Intelligent Control*, 2010, pp. 8–10. https://doi.org/10.1109/ISIC.2010.5612886
3. Liu P.-L. Stabilization criteria for bilinear systems with time-varying delay, *Universal Journal of Electrical and Electronic Engineering*, 2014, vol. 2, pp. 52–58. https://doi.org/10.13189/ujeee.2014.020202
4. Ho D.W.C., Lu G., Zheng Y. Global stabilisation for bilinear systems with time delay, *IEE Proceedings – Control Theory and Applications*, 2002, vol. 149, issue 1, pp. 89–94. https://doi.org/10.1049/ip-cta:20020114
5. Lu G., Ho D.W.C. Solution existence and stabilization for bilinear descriptor systems with time-delay, *2006 9th International Conference on Control, Automation, Robotics and Vision*, 2006. https://doi.org/10.1109/ICARCV.2006.345310
6. Liu C.-H., Hsiao M.-Y., Tsai S.-H. Stabilization for a class of time-delay discrete bilinear system, *Information Technology Journal*, 2011, vol. 10, issue 2, pp. 457–460. https://doi.org/10.3923/itj.2011.457.460

7. Yang X., Stepanenko Yu. Stabilization of discrete bilinear systems with time delayed feedback, *Proceeding of the American Control Conference*, 1993, pp. 1051–1055.
https://doi.org/10.23919/ACC.1993.4793026
8. Yang X., Chen L.-K. Stability of discrete bilinear systems with time-delayed feedback functions, *IEEE Transactions on Automatic Control*, 1993, vol. 38, issue 1, pp. 158–163.
https://doi.org/10.1109/9.186331
9. Zaitsev V.A., Kim I.G. Finite spectrum assignment problem in linear systems with state delay by static output feedback, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, 2016, vol. 26, issue 4, pp. 463–473 (in Russian). https://doi.org/10.20537/vm160402
10. Zaitsev V.A. Necessary and sufficient conditions in a spectrum control problem, *Differential Equations*, 2010, vol. 46, issue 12, pp. 1789–1793. https://doi.org/10.1134/S0012266110120128
11. Zaitsev V.A. Control of spectrum in bilinear systems, *Differential Equations*, 2010, vol. 46, issue 7, pp. 1071–1075. https://doi.org/10.1134/S0012266110070153
12. Kim I.G., Zaitsev V.A. Spectrum assignment by static output feedback for linear systems with time delays in states, *2018 14th International Conference "Stability and Oscillations of Nonlinear Control Systems" (Pyatnitskiy's Conference) (STAB)*, IEEE.
https://doi.org/10.1109/stab.2018.8408365
13. Zaitsev V.A., Tonkov E.L. Attainability, compatibility and V.M. Millionshchikov's method of rotations, *Russian Mathematics*, vol. 43, no. 2, pp. 42–52. https://zbmath.org/?q=an:1049.93504
14. Zaitsev V.A. Consistent systems and pole assignment: I, *Differential Equations*, 2012, vol. 48, issue 1, pp. 120–135. https://doi.org/10.1134/S001226611110120
15. Zaitsev V.A. Consistent systems and pole assignment: II, *Differential Equations*, 2012, vol. 48, issue 6, pp. 857–866. https://doi.org/10.1134/S0012266112060092
16. Zaitsev V.A. Consistency and eigenvalue assignment for discrete-time bilinear systems: I, *Differential Equations*, 2014, vol. 50, issue 11, pp. 1495–1507.
https://doi.org/10.1134/S001226611411007X
17. Zaitsev V.A. Consistency and eigenvalue assignment for discrete-time bilinear systems: II, *Differential Equations*, 2015, vol. 51, issue 4, pp. 510–522. https://doi.org/10.1134/S0012266115040084

Received 15.01.2019

Zaitsev Vasilii Aleksandrovich, Doctor of Physics and Mathematics, Professor, Department of Differential Equations, Udmurt State University, ul. Universitetskaya, 1, Izhevsk, 426034, Russia.
E-mail: verba@udm.ru

Kim Inna Geral'dovna, Senior Lecturer, Department of Mathematical Analysis, Udmurt State University, ul. Universitetskaya, 1, Izhevsk, 426034, Russia.
E-mail: kimingeral@gmail.com

В. А. Зайцев, И. Г. Ким

Задача назначения конечного спектра в билинейных системах с запаздыванием в состоянии

Цитата: Вестник Удмуртского университета. Математика. Механика. Компьютерные науки. 2019. Т. 29. Вып. 1. С. 19–28.

Ключевые слова: линейные системы с запаздыванием, управление спектром, стабилизация, билинейная система.

УДК 517.929, 517.977

DOI: 10.20537/vm190102

Рассматривается билинейная управляемая система, заданная линейной стационарной системой дифференциальных уравнений с запаздыванием в состоянии. Исследуется задача назначения произвольного конечного спектра посредством стационарного управления. Требуется построить постоянный вектор управления таким образом, чтобы характеристический квазиполином замкнутой системы обращался в полином с произвольными наперед заданными коэффициентами. Получены условия на коэффициенты системы, при которых найден критерий разрешимости данной задачи назначения конечного спектра. Критерий выражен в терминах ранговых условий для матриц специального вида. Показана взаимосвязь этих ранговых условий со свойством согласованности усеченной системы без запаздывания. Получены следствия о стабилизации билинейной системы с запаздыванием. Результаты обобщают полученные ранее результаты о назначении спектра для линейных систем со статической обратной связью по выходу с запаздыванием и для билинейных систем без запаздывания. Полученные результаты переносятся на билинейные системы с запаздыванием с дискретным временем. Рассмотрен иллюстрирующий пример.

Финансирование. Работа выполнена при поддержке Российского фонда фундаментальных исследований (проект № 18–51–41005) и Министерства науки и высшего образования РФ в рамках базовой части госзадания в сфере науки (проект № 1.5211.2017/8.9).

Поступила в редакцию 15.01.2019

Зайцев Василий Александрович, д. ф.-м. н., профессор, кафедра дифференциальных уравнений, Удмуртский государственный университет, 426034, Россия, г. Ижевск, ул. Университетская, 1.
E-mail: verba@udm.ru

Ким Инна Геральдовна, старший преподаватель, кафедра математического анализа, Удмуртский государственный университет, 426034, Россия, г. Ижевск, ул. Университетская, 1.
E-mail: kimingeral@gmail.com