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PSEUDOSPECTRAL METHOD FOR SECOND-ORDER AUTONOMOUS NONLINEAR DIFFERENTIAL EQUATIONS

Autonomous nonlinear differential equations constituted a system of ordinary differential equations, which often applied in different areas of mechanics, quantum physics, chemical engineering science, physical science, and applied mathematics. It is assumed that the second-order autonomous nonlinear differential equations have the types $u''(x) - u'(x) = f[u(x)]$ and $u''(x) + f[u(x)]u'(x) + u(x) = 0$ on the range $[-1, 1]$ with the boundary values $u[-1]$ and $u[1]$ provided. We use the pseudospectral method based on the Chebyshev differentiation matrix with Chebyshev–Gauss–Lobatto points to solve these problems. Moreover, we build two new iterative procedures to find the approximate solutions. In this paper, we use the programming language Mathematica version 10.4 to represent the algorithms, numerical results and figures. In the numerical results, we apply the well-known Van der Pol oscillator equation and gave good results. Therefore, they will be able to be applied to other nonlinear systems such as the Rayleigh equations, the Lienard equations, and the Emden–Fowler equations.

Keywords: pseudospectral method, Chebyshev differentiation matrix, Chebyshev polynomial, autonomous equations, nonlinear differential equations, Van der Pol oscillator.

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Introduction

It is well-known that the autonomous nonlinear differential equations constitute a system of the ODEs, which often arise in different areas of mechanics, quantum physics, chemical engineering science, analytical chemistry and their applications in engineering, physical science, and applied mathematics [1–8]. For instance, the Val de Pol equations have been used in physical and biological sciences and in [3,4]; the autonomous equations have been done in the nonlinear oscillations, in the physical systems as the Duffing oscillator, the pendulum, the nonlinear dynamics, the deterministic chaos and the nonlinear electronic circuits [5–7].

Hence, we need to find analytical methods to determine solutions for these problems, which is very important. Special numerical methods compute the approximate solutions.

We have the general form of the autonomous nonlinear second-order differential equations

$$\frac{d^2}{dx^2}u = f[u, u'].$$

In this paper, we consider two forms of the autonomous nonlinear problems. The first form is

$$\frac{d^2}{dx^2}u(x) - \frac{d}{dx}u(x) = g[u(x)], \quad x \in [-1, 1], \quad u[-1] = a, \ u[1] = b, \tag{0.1}$$

and the second form is

$$\frac{d^2}{dx^2}u(x) + h[u(x)]\frac{d}{dx}u(x) + u(x) = 0, \quad x \in [-1, 1], \quad u[-1] = c, \ u[1] = d, \tag{0.2}$$

where g and h are the differentiable functions of $u(x)$; a, b, c and d are known boundary values.

Several methods have been studied to determine solutions for the autonomous problems. The popular method is the analysis method to reduce the autonomous equations to the Abel equations of the first kind [1, 8, 9]. The other methods used are as follows: the functional parameter methods combined with mechanical quadratures, Newton's and gradient methods construct numerical procedures to determine approximate solutions in nonlinear systems [10]; the measure theory to solve a wide range of second-order boundary value ODEs, in which the author computed approximate solutions by a finite combination of atomic measures and the problem converted approximately to a finite-dimensional linear programming problem [11]; the natural decomposition method based on the natural transform method and the Adomian decomposition method determine exact solutions for the nonlinear ODEs [12]; the Taylor-type iterative methods compute the transformed function to solve strongly nonlinear differential equations [13]; the method has been used neural networks for the numerical solutions of nonlinear differential equations [14]; the feed-forward neural network determined the approximate solutions of the nonlinear ODEs without the need for training [15]; the exponential function method determined the solutions for nonlinear ODEs with constant coefficients in a semi-infinite domain [16]; the collocation method is based on the rational Chebyshev functions to solve the nonlinear ODEs [17]; the collocation method via the Jacobi polynomials solved the nonlinear ODEs [18]; the multistep methods, the Runge–Kutta methods and the predictor-corrector methods solved the nonlinear autonomous ODEs [19]; the nonlinear modal superposition method has been used the power series expansions and the mathematical transformation from the physics system coordinate to the modal coordinate for the weakly nonlinear autonomous systems [20], and others.

In this paper, we study the pseudospectral method based on the Chebyshev differentiation matrix to solve problems (0.1) and (0.2). The first time the collocation approach was used for partial differential equations with periodic solutions by Kreiss H.-O. and Oliger J. [21]. They refer to the pseudospectral method by Orszag S.A. [22]. Due to their universality, high efficiency, accuracy, the pseudospectral methods were expanded, developed in different forms such as the Fourier pseudospectral method, the Laguerre pseudospectral method and the Chebyshev pseudospectral method [23,24], ... In fact, the pseudospectral method can be applied for numerical solving different problems [25]. For example: the pseudospectral fictitious point method was used for solving the high order initial-boundary value problems [26]; the pseudospectral method was used for solving the nonlinear Pendulum equations and the Duffing oscillator [27,28], for solving third-order differential equations [29]; the Chebyshev pseudospectral method was used for solving the class of van der Waals flows with non-convex flux functions [30] etc.

§ 1. Chebyshev differentiation matrix

A grid function $v(x)$ is defined on the Chebyshev–Gauss–Lobatto points (nodes)

$$x = \{x_0, x_1, \dots, x_n\}$$

such that $x_k = \cos(k\pi/n)$, $k = \overline{0,n}$. They are the extrema of the n-th order in the Chebyshev polynomial $T_n(x) = \cos(n\cos^{-1}x)$. The function $v(x)$ is interpolated by constructing the n-th order interpolation polynomial $g_j(x)$ such that $g_j(x_k) = \delta_{j,k}$ [23,30–33]

$$p(x) = \sum_{j=0}^{n} p_j g_j(x),$$

where $p(x)$ is the unique polynomial of degree n and $p_j = v(x_j)$, $j = \overline{0,n}$. The following can be shown:

$$g_j(x) = \frac{(-1)^{j+1}(1-x^2)T_n'(x)}{c_j n^2 (x - x_j)}, \quad j = \overline{0,n},$$

where

$$c_j = \begin{cases} 2, & j = 0 \text{ or } n, \\ 1, & \text{otherwise}. \end{cases} \tag{1.1}$$

As we know the values of $p(x)$ at $n+1$ points, we would like to find approximately the values of the derivative of $p(x)$ at those points $p'(x) = \dfrac{d}{dx}p(x)$. We can write the same in the matrix form:

$$p' = D_c p,$$

where $D_c = \left\{d^{(1)}_{i,j}\right\}$ is an $(n+1)\times(n+1)$ differentiation matrix (or derivative matrix).

Evidently, the derivative of $p(x_j)$ becomes

$$p'(x_j) = \sum_{k=0}^{n} (D_c)_{j,k}\, p(x_k), \quad j = \overline{0, n}.$$

We have the entries $d^{(1)}_{i,j} = g'_i(x_j)$ which are

$$d^{(1)}_{0,0} = \frac{2n^2+1}{6} = -d^{(1)}_{n,n}, \qquad d^{(1)}_{i,i} = -\frac{x_i}{2(1-x_i^2)}, \quad i = \overline{1, n-1},$$
$$d^{(1)}_{i,j} = \frac{c_i}{c_j}\frac{(-1)^{i+j}}{x_i - x_j}, \quad i \neq j, \quad i,j = \overline{1, n-1},$$

where c_k is determined by the formula (1.1).

Similarly, $p'(x)$ is a polynomial of degree $n-1$; there exists the second differentiation matrix D_c^2,

$$p'' = D_c^2 p,$$

and

$$p''(x_j) = \sum_{k=0}^{n} \left(D_c^2\right)_{j,k} p(x_k), \quad j = \overline{0, n}.$$

§ 2. Pseudospectral method using CDM

Suppose that

$$\frac{d^2}{dx^2}u(x) = t(x), \quad u(-1) = \alpha, \quad u(1) = \beta, \tag{2.1}$$

and the collocation points $\{x_i\}$ such that $1 > x_0 > x_1 > \ldots > x_n = -1$.

We know that

$$\frac{d^2}{dx^2}u_n(x_i) = \sum_{k=0}^{n} (D_C^2)_{i,k} u_n(x_k).$$

Therefore, equation (2.1) becomes

$$\sum_{k=0}^{n} (D_c^2)_{i,k} u_n(x_k) = t(x_i), \quad i = \overline{1,\ n-1}, \quad u_n(x_n) = \alpha, \quad u_n(x_0) = \beta.$$

Alternately, we partition the matrix D_c into matrices [23, 31]:

$$e_0^{(1)} = \begin{pmatrix} d^{(1)}_{1,0} \\ d^{(1)}_{2,0} \\ \vdots \\ d^{(1)}_{n-1,0} \end{pmatrix}, \quad E^{(1)} = \begin{pmatrix} d^{(1)}_{1,1} & d^{(1)}_{1,2} & \cdots & d^{(1)}_{1,n-1} \\ d^{(1)}_{2,1} & d^{(1)}_{2,2} & \cdots & d^{(1)}_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ d^{(1)}_{n-1,1} & d^{(1)}_{n-1,2} & \cdots & d^{(1)}_{n-1,n-1} \end{pmatrix}, \quad e_n^{(1)} = \begin{pmatrix} d^{(1)}_{1,n} \\ d^{(1)}_{2,n} \\ \vdots \\ d^{(1)}_{n-1,n} \end{pmatrix}.$$

Or we can rewrite the same short form

$$e_0^{(1)} = \{d_{i,0}^{(1)}\}, \quad E^{(1)} = \{d_{i,j}^{(1)}\}, \quad e_n^{(1)} = \{d_{i,n}^{(1)}\}, \quad i,j = \overline{1, n-1}.$$

Similarly, we partition the matrix D_c^2 into matrices: $e_0^{(2)}$, $E^{(2)}$, and $e_n^{(2)}$. So the equation (2.1) can be written then in the matrix form

$$\beta e_0^{(2)} + E^{(2)} u + \alpha e_n^{(2)} = t,$$

where u and t denote the vectors

$$u = \begin{pmatrix} u_n(x_1) \\ \vdots \\ u_n(x_{n-1}) \end{pmatrix}, \quad t = \begin{pmatrix} t(x_1) \\ \vdots \\ t(x_{n-1}) \end{pmatrix}.$$

§ 3. Applications

We apply the PSM using CDM to the equation (0.1). Therefore, we can rewrite the equation (0.1) in the following matrix form:

$$\left(E^{(2)} - E^{(1)}\right) u + b\left(e_0^{(2)} - e_0^{(1)}\right) + a\left(e_n^{(2)} - e_n^{(1)}\right) = G, \tag{3.1}$$

here $G(u)$ denotes the vector with elements $\{g\,[u_n(x_i)]\}$, $i = \overline{1, n-1}$.

To find the solutions $u_n(x_i)$ of the equation (3.1), we might be able to approach it with an iterative procedure as follows:

Procedure 1;
Begin

$T := E^{(2)} - E^{(1)}$;
$u^{(old)} := I^T$;
$\varepsilon := 1$;
$er := 10^{-8}$;
While $\varepsilon > er$ **do**
Begin

$u^{(new)} := T^{-1}\left[G(u^{(old)}) - b\left(e_0^{(2)} - e_0^{(1)}\right) - a\left(e_n^{(2)} - e_0^{(1)}\right)\right]$;
$\varepsilon := \left|Min\left\{u_1^{(new)} - u_1^{(old)}, u_2^{(new)} - u_2^{(old)}, \ldots, u_{n-1}^{(new)} - u_{n-1}^{(old)}\right\}\right|$;
$u^{(old)} := u^{(new)}$;

End;
$u^{(old)}$;

End;

here I is the unit vector and er is the error that might change.

Similarly, we can rewrite the equation (0.2) in the matrix form as follows:

$$\left(E^{(2)} + HE^{(1)} + J\right) u + d\left(e_0^{(2)} + He_0^{(1)}\right) + c\left(e_n^{(2)} + He_n^{(1)}\right) = 0, \tag{3.2}$$

where H denotes the diagonal matrix with elements $h[u(x_i)]$, $i = \overline{1, n-1}$; J is a unit matrix of order $n-1$.

To find the solutions $u_n(x_i)$ of the equation (3.2), we might be able to approach it with an iterative procedure as follows:

Procedure 2;
Begin

$u^{(old)} := I^T$;
$\varepsilon := 1$;
$er := 10^{-8}$;
While $\varepsilon > er$ **do**
Begin
$H := H\left(u^{(old)}\right)$;
$T := E^{(2)} + HE^{(1)} + J$;
$u^{(new)} := T^{-1}\left[-d\left(e_0^{(2)} + He_0^{(1)}\right) - c\left(e_n^{(2)} + He_0^{(1)}\right)\right]$;
$\varepsilon := \left|Min\left\{u_1^{(new)} - u_1^{(old)}, u_2^{(new)} - u_2^{(old)}, \ldots, u_{n-1}^{(new)} - u_{n-1}^{(old)}\right\}\right|$;
$u^{(old)} := u^{(new)}$;
End;
$u^{(old)}$;
End;

here I is the unit vector, er is the error that might change, and J is a unit matrix of order $n-1$.

§ 4. Numerical results

In this section, we use the programming language Mathematica 10.4 to represent the algorithms. Furthermore, we have used the function NDSolve to compute numerical results at the column NDSolve in each example for comparison [34].

Example 1. Consider the Van der Pol oscillator equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) - \sigma\left(1 - u^2(x)\right)\frac{\mathrm{d}}{\mathrm{d}x}u(x) + u(x) = 0, \quad x \in [-1, 1], \quad u[-1] = c,\ u[1] = d, \tag{4.1}$$

where $\sigma = \text{const} > 0$ [3, 4].

From section 3, we can rewrite the equation (4.1) in the matrix form

$$\left(E^{(2)} - HE^{(1)} + J\right)u + d\left(e_0^{(2)} - He_0^{(1)}\right) + c\left(e_n^{(2)} - He_n^{(1)}\right) = 0,$$

where $H = \sigma(1 - u^2(x_i))$, and J denotes the unit matrix.

Hence, the formula to loop in the Procedure 2 is

$$\left(E^{(2)} - H(u^{(old)})E^{(1)} + J\right)u^{(new)} = d\left(H(u^{(old)})e_0^{(1)} - e_0^{(2)}\right) + c\left(H(u^{(old)})e_n^{(1)} - e_n^{(2)}\right) = 0.$$

With $n = 80$, the boundary conditions $c = 0.1$, $d = 0.5$ and the error $\varepsilon = 10^{-8}$, we have Table 1, which shows the numerical results for two cases $\sigma = 0.01$ and $\sigma = 100$, where the $u_n(x_i)$ columns are the numerical results of the method, and the NDSolve columns are the numerical results computed by Mathematica 10.4 corresponding to each point x_i. Moreover, Figure 1 illustrates the graphics of the Van der Pol oscillator equations with $\sigma = \{0.01, 1, 10, 100\}$ in two cases: $c = 0.1$, $d = 0.5$ (Fig. 1, a) and $c = 0.1$, $d = 0.1$ (Fig. 1, b); in Figure 1, the dots are the results of the PSM and the lines are the graphics computed by the Mathematica 10.4.

Besides, using the numerical results just obtained, we also evaluate the highest differences between two the columns $u_n(x_i)$ and NDSolve and they have been presented in Table 2.

Therefore, we see that the highest differences between the two columns $u_n(x_i)$ and NDSolve in these cases are very small (10^{-8}).

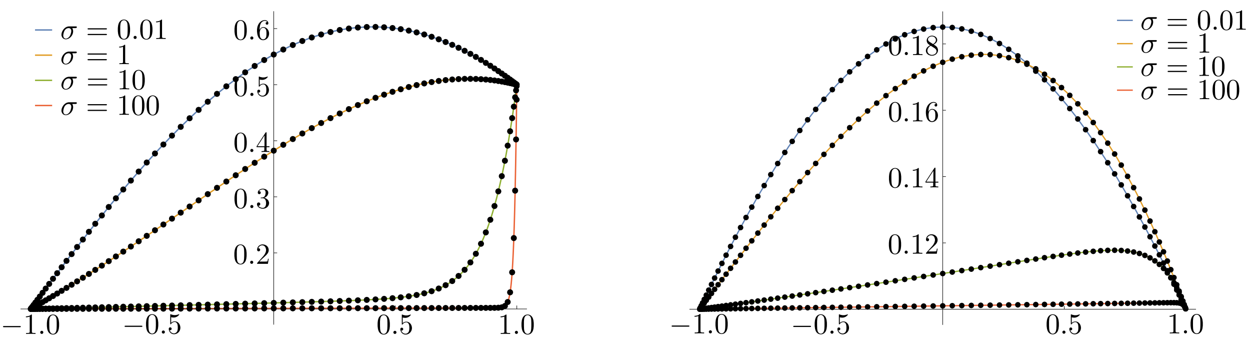
Example 2. Consider the following autonomous differential equations:

$$u''(x) - u'(x) = s_1u^m + s_2u^k + s_3u^t, \quad x \in [-1, 1], \quad u[-1] = a,\ u[1] = b, \tag{4.2}$$

where $m, k, t \in \mathbb{Q}$, $s_i \in \mathbb{R}$, $i = 1, 2, 3$, and the boundary values a and b are given.

Table 1. Numerical results of Van der Pol oscillator equations

		The case $\sigma = 0.01$		The case $\sigma = 100$	
i	x_i	$u_n(x_i)$	NDSolve	$u_n(x_i)$	NDSolve
1	0.99922904	0.50026011	0.50026012	0.47329497	0.47329499
6	0.97236992	0.50913422	0.50913420	0.12968218	0.12968218
12	0.89100652	0.53373965	0.53373962	0.10193792	0.10193791
18	0.76040597	0.56576495	0.56576492	0.10179464	0.10179462
24	0.58778525	0.59320554	0.59320550	0.10161722	0.10161720
30	0.38268343	0.60283001	0.60282997	0.10140683	0.10140681
35	0.19509032	0.58949973	0.58949970	0.10121473	0.10121477
40	0	0.55374163	0.55374160	0.10101546	0.10101544
46	−0.23344536	0.48360122	0.48360120	0.10077747	0.10077746
52	−0.45399050	0.39328998	0.39328997	0.10055316	0.10055315
58	−0.64944805	0.29732058	0.29732060	0.10035479	0.10035478
64	−0.80901699	0.21052966	0.21052967	0.10019314	0.10019313
70	−0.92387953	0.14469415	0.14469416	0.10007694	0.10007694
75	−0.98078528	0.11134367	0.11134365	0.10001941	0.10001942
79	−0.99922904	0.10045585	0.10045585	0.10000078	0.10000078



(a) The case $c = 0.1$ and $d = 0.5$

(b) The case $c = d = 0.1$

Fig 1. Graphics of the Van der Pol oscillator equations, here dots are the numerical results of the PSM and the lines are graphics computed by the Mathematica 10.4

Similarly, from section 3, we can rewrite the equation (4.2) in the matrix form

$$\left(E^{(2)} - E^{(1)}\right) u + b\left(e_0^{(2)} - e_0^{(1)}\right) + a\left(e_n^{(2)} - e_n^{(1)}\right) = S,$$

here $S = s_1 u^m(x_i) + s_2 u^k(x_i) + s_3 u^t(x_i)$.

Hence, the formula to loop in Procedure 1 is

$$\left(E^{(2)} - E^{(1)}\right) u^{(new)} = S(u^{(old)}) - b\left(e_0^{(2)} - e_0^{(1)}\right) - a\left(e_n^{(2)} - e_n^{(1)}\right).$$

We consider this equation in the four cases (these are the problems 2.2.1–3, 2.2.1–6, 2.2.1–7, 2.2.1–22 in the book [35]):

- the first case: $s_1 = -288$, $s_2 = s_3 = 0$, $m = -2$, $a = 8$, and $b = 10$;
- the second case: $s_1 = -\dfrac{1}{3}5^{3/2}$, $s_2 = s_3 = 0$, $m = -\dfrac{1}{2}$, and $a = b = 3$;
- the third case: $s_1 = -\dfrac{2}{9}$, $s_2 = \dfrac{19}{9}0.2^{3/2}$, $s_3 = 0$, $m = 1$, $k = -\dfrac{1}{2}$, $a = 6$, and $b = 5$;

Table 2. The highest differences between two columns $u_n(x_i)$ and NDSolve of the Van der Pol oscillator equations

σ	The case $c = 0.1, d = 0.5$	The case $c = 0.1, d = 0.5$
0.01	3.2579×10^{-8}	2.95516×10^{-8}
1	2.06538×10^{-8}	2.44241×10^{-8}
10	1.89381×10^{-8}	1.53047×10^{-8}
100	3.93109×10^{-8}	6.0991×10^{-10}

- the fourth case: $s_1 = 1$, $s_2 = 2$, $s_3 = -4$, $m = 1$, $k = -1$, $t = -3$, $a = 4$, and $b = 3$.

We choose $\varepsilon = 10^{-8}$; in the first two cases, we have the numerical solutions $u_n(x_i)$ of the equation (4.2) with $n = 64$ in Table 3; and Table 4 presents the numerical solutions in the last two cases with $n = 128$.

Table 3. Numerical results of the equations (4.2) in the first two cases

		The first case		The second case	
i	x_i	$u_n(x_i)$	NDSolve	$u_n(x_i)$	NDSolve
1	0.99879546	10.00185524	10.00185522	3.00308323	3.00308315
5	0.97003125	10.04427502	10.04427497	3.07470296	3.07470285
10	0.88192126	10.15273919	10.15273913	3.27119328	3.27119308
15	0.74095113	10.26524485	10.26524475	3.52052609	3.52052577
20	0.55557023	10.31499585	10.31499573	3.74408404	3.74408360
25	0.33688985	10.25476795	10.25476810	3.88284876	3.88284824
30	0.09801714	10.06841523	10.06841510	3.91115654	3.91115602
35	−0.14673047	9.76866931	9.76866920	3.83479058	3.83479058
40	−0.38268343	9.38881646	9.38881637	3.68080629	3.68080592
45	−0.59569930	8.97465464	8.97465457	3.48641255	3.48641231
50	−0.77301045	8.57871328	8.57871323	3.29049234	3.29049221
55	−0.90398929	8.25510529	8.25510524	3.12801253	3.12801246
60	−0.98078528	8.05231840	8.05231834	3.02620388	3.02620385
63	−0.99879546	8.00329876	8.00329870	3.00165117	3.00165116

Furthermore, Figure 2 shows the graphics of equation (4.2) in the four cases above with $n = 64$ and the boundary conditions $a = 8$, $b = 10$, where the dots are the results of the PSM and the lines are the graphics computed by the Mathematica 10.4.

Besides, from the numerical results in Tables 3–4, we evaluate the highest differences between the two columns $u_n(x_i)$ and NDSolve; they are very small (10^{-6}) and they have shown in the following Table 5.

§ 5. Conclution

In this work, we have developed two new iterative procedures combining the PSM and the CDM to find the approximate solutions of the autonomous nonlinear systems of two types (0.1) and (0.2). Additionally, we have demonstrated two examples including the Van der Pol oscillator equations. The PSM's numerical results are compared to the numerical results computed by Mathematica 10.4; they show convergence and reliability.

The accuracy of numerical results in the problems depends on the order of the Chebyshev polynomial; this means that, if n increases, then the accuracy of results will be better.

This method and the iterative procedures might be applied to other nonlinear systems such as the Rayleigh equations, the Lienard equations, and the Emden–Fowler equations.

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Table 4. Numerical results of the equations (4.2) in the last two cases

		The third case		The fourth case	
i	x_i	$u_n(x_i)$	NDSolve	$u_n(x_i)$	NDSolve
1	0.99969882	5.00084439	5.00084439	2.99914787	2.99914855
8	0.98078528	5.05317779	5.05315987	2.94677891	2.94677943
16	0.92387953	5.20258661	5.20258709	2.80231341	2.80231348
24	0.83146961	5.42064933	5.42065034	2.60648078	2.60648020
32	0.70710678	5.66966371	5.66966532	2.40983652	2.40983519
40	0.55557023	5.91100374	5.91100590	2.25844220	2.25844019
48	0.38268343	6.11320104	6.11320358	2.18384099	2.18383851
56	0.19509032	6.25682072	6.25682340	2.20037879	2.20037613
64	0	6.33555542	6.33555796	2.30774821	2.30774564
72	−0.19509032	6.35428066	6.35428287	2.49489845	2.49489623
80	−0.38268343	6.32547006	6.32547181	2.74301190	2.74301016
88	−0.55557023	6.26533456	6.26533582	3.02752510	3.02752386
96	−0.70710678	6.19060057	6.19060138	3.31993597	3.31993520
104	−0.83146961	6.11630385	6.11630430	3.58997046	3.58997005
112	−0.92387953	6.05456196	6.05456216	3.80842676	3.80842661
120	−0.98078528	6.01406872	6.01406873	3.95063905	3.95063899
127	−0.99969882	6.00022201	6.00022201	3.99922103	3.99922104

Table 5. The highest differences between two columns $u_n(x_i)$ and NDSolve of the equation (4.2)

Cases	The highest differences
The first case	1.37032×10^{-7}
The second case	5.29304×10^{-7}
The third case	2.67477×10^{-6}
The fourth case	2.66773×10^{-6}

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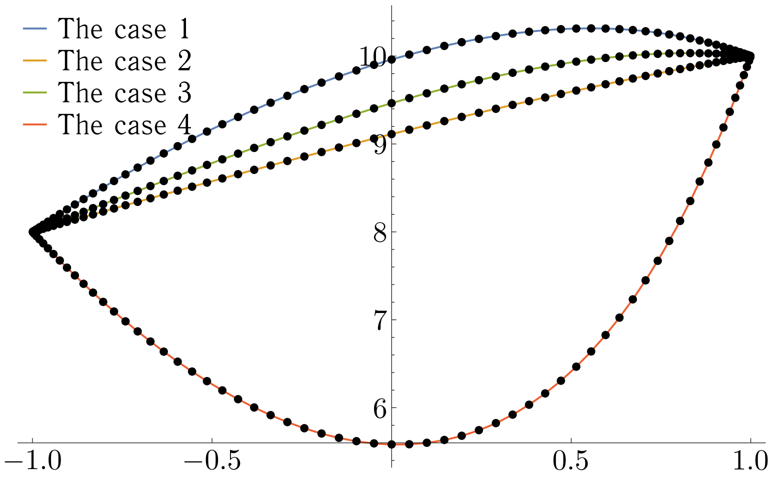

Fig 2. Graphics of the equation (24) in the 4 cases above with $n = 64$, $a = 8$ and $b = 10$. The dots are the numerical results of the PSM, also the lines are graphics computed of the Mathematica 10.4

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Псевдоспектральный метод для автономных нелинейных дифференциальных уравнений второго порядка

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Ключевые слова: псевдоспектральный метод, матрица дифференцирования Чебышева, полином Чебышева, автономные уравнения, нелинейные дифференциальные уравнения, осциллятор Ван-дер-Поля.

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Автономные нелинейные дифференциальные уравнения представляют собой систему обыкновенных дифференциальных уравнений, которые часто применяются в различных областях механики, квантовой физики, химического машиностроения, физики и прикладной математики. Здесь рассматриваются автономные нелинейные дифференциальные уравнения второго порядка $u''(x) - u'(x) = f[u(x)]$ и $u''(x) + f[u(x)]u'(x) + u(x) = 0$ на промежутке $[-1, 1]$ с заданными граничными значениями $u[-1]$ и $u[1]$. Для решения этих задач используется псевдоспектральный метод, основанный на матрице дифференцирования Чебышева с точками Чебышева–Гаусса–Лобатто. Для нахождения приближенных решений построены две новые итерационные процедуры. В этой статье был использован язык программирования Mathematica версии 10.4 для представления алгоритмов, численных результатов и рисунков. В качестве примера численного моделирования исследовано известное уравнение Ван дер Поля и получены

хорошие результаты. Впоследствии возможно применение полученных результатов к другим нелинейным системам, таким как уравнения Рэлея, уравнения Льенара и уравнения Эмдена–Фаулера.

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