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A PROBLEM OF SIMPLE GROUP PURSUIT WITH POSSIBLE DYNAMICAL DISTURBANCE IN DYNAMICS AND PHASE CONSTRAINTS

In finite-dimensional Euclidian space, we treat the problem of pursuit of one evader by a group of pursuers, which is described by a system of the form

$$\dot{z}_i = a_i(t)u_i - v, \quad u_i \in U_i, \quad v \in V,$$

where the functions $\alpha_i(t)$ are equal to 1 for all t, except for a certain interval of a given length, on which they are equal to zero (to each pursuer there corresponds its own interval). This fact can be interpreted in such a way that each of the pursuers has a possible failure of the control device at any previously unknown moment in time, and the length of the time interval needed to fix the failure is given, while in the process of fixing the failure the pursuers have no possibility to carry out a capture. The target sets are convex compact sets, and the evader does not leave the bounds of the convex polyhedral set. We obtain sufficient conditions for solvability of the pursuit problem.

Keywords: differential games, pursuer, evader, capture, phase constraints, breakdown.

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Introduction

In modeling various controlled processes in real dynamical systems, situations are possible in which the control system is acted upon by uncontrolled disturbances and the control system itself may fail to operate. It is quite natural to consider the problems of control in such situations within the framework of differential game theory.

Among the first studies devoted to the problems of control with possible disturbances in the dynamics is the work of M. S. Nikolskii [1]. The situation of conflict interaction of two persons on a finite time interval, which is described by a linear system of differential equations under the condition that the control device of the pursuer may fail to operate at any previously unknown moment in time, and the length of the time interval necessary to eliminate the breakdown is given, was discussed in [2, 3]. In those papers, sufficient conditions for steering the phase trajectory to a given terminal set were obtained using the direct methods of L. S. Pontryagin.

The linear problem of impulse control on a finite time interval in the presence of an uncontrolled disturbance and a possible breakdown was considered in [4]. The goal of control in this work was to ensure that the value of a given linear function belongs to a given interval at the final time. Sufficient conditions for solvability of this problem were obtained.

In [5], a problem of control with a disturbance and a possible breakdown on a finite time interval was considered. The terminal component of the payoff depended on the absolute value of the linear function of phase coordinates at a finite time, the integral component of the payoff was specified by the integral of the degree of control, and the control was constructed by the principle of a guaranteed result. The construction of a u-stable bridge in a linear differential holding game with a breakdown on a finite time interval is presented in [6]. The problem of controlling a parabolic system with breakdowns and possible disturbances in the dynamics was dealt with in [7, 8]. The authors of [9, 10] considered linear differential games of two persons under the condition that a breakdown of the control devices of the pursuer occurs at some a priori unknown time instant. The method of resolving functions was used to obtain sufficient conditions for solvability of the pursuit problem.

It should be noted that differential games with possible disturbances in the dynamics supplement the class of hybrid differential games, in which the dynamics of one or both players changes only once [11–13].

In this paper we address the problem of a simple pursuit of one evader by a group of pursuers under the condition that breakdowns of the control devices of each of the pursuers are possible at a priori unknown moments of time, and the time required for repair is specified. The goal sets are convex compacts in phase space. Sufficient conditions for solvability of the pursuit problem are obtained.

§ 1. Formulation of the problem

In the space $\mathbb{R}^k$ $(k \geqslant 2)$, we consider the differential game $\Gamma(n+1)$ of $n+1$ persons: n pursuers $P_1, \ldots, P_n$ and evader E. The law of motion of evader E has the form

$$\dot{y} = v, \quad y(0) = y^0, \quad v \in V. \tag{1.1}$$

The law of motion of each of pursuers P_i has the form

$$\dot{x}_i = a_{\theta_i}(t) u_i, \quad x_i(0) = x_i^0, \quad u_i \in U_i, \tag{1.2}$$

where the functions a_{θ_i} have the form

$$a_{\theta_i}(t) = \begin{cases} 0, & t \in [\theta_i, \theta_i + h_i], \\ 1, & t \notin [\theta_i, \theta_i + h_i]. \end{cases}$$

Here $i \in I = \{1, \ldots, n\}$, $y, x_i, u_i, v, y^0, x_i^0 \in \mathbb{R}^k$, and U_i, V are convex compacts of $\mathbb{R}^k$. Suppose that the time instants θ_i are the instants when the control devices of pursuers P_i fail to operate or that θ_i are the instants when the maintenance of the control devices of pursuers P_i begins. The quantities $h_i \geqslant 0$ are the repair time or the maintenance time. The terminal sets M_i are convex compacts of $\mathbb{R}^k$, and $z_i^0 \notin M_i$ for all $i \in I$.

It follows from the definition of the functions a_{θ_i} that, in the process of eliminating the breakdown or in the process of maintenance, pursuers P_i have no opportunity to perform a capture. It is assumed that the time instants θ_i, $i \in I$, are unknown to pursuers P_i, $i \in I$, and the breakdown or the necessity of maintenance can begin at any instant. The quantities h_i, $i \in I$, are known to all participants.

In addition, it is assumed that, in the process of the game, evader E does not move out of the convex polyhedral set

$$\Omega = \big\{ y \;\big|\; y \in \mathbb{R}^k,\ (p_j, y) \leqslant \mu_j,\ j \in J = \{1, \ldots, r\} \big\}$$

with a nonempty interior, where $p_1, \ldots, p_r$ are the unit vectors of $\mathbb{R}^k$, $\mu_1, \ldots, \mu_r$ are real numbers, and (a, b) is the scalar product of the vectors a and b. We assume that $\Omega = \mathbb{R}^k$ with $r = 0$.

Let us introduce new variables $z_i = x_i - y$ and pass from the systems (1.1) and (1.2) to the new system

$$\dot{z}_i = a_{\theta_i}(t) u_i - v, \quad z_i(0) = z_i^0 = x_i^0 - y^0, \quad u_i \in U_i, \quad v \in V, \tag{1.3}$$

A measurable function $v\colon [0, +\infty) \to \mathbb{R}^k$ will be called admissible if $v(t) \in V$ and $y(t) \in \Omega$ for all $t \geqslant 0$. By the prehistory $v_t(\cdot)$ of the function v at time t we will mean the restriction of the function v to the interval $[0, t]$.

Let Ξ_i denote a one-parameter family of maps $\mathcal{F}_i(t, v_t(\cdot))$ which are defined for each $t > 0$ on the set of measurable functions $v(\cdot)$ such that $v(s) \in V$, $y(s) \in \Omega$, $0 \leqslant s \leqslant t$, and which take values in U_i and possess the property of superpositional measurability: the function $u_i(t) = \mathcal{F}_i(t, v_t(\cdot))$ is measurable at $t \geqslant 0$ for any measurable function $v(\cdot)$, $v(s) \in V$, $y(s) \in \Omega$, $s \geqslant 0$.

Definition 1.1 (see [2]). The strategy $\mathcal{U}_i$ of pursuer P_i is the pair

$$\mathcal{U}_i = \big(\mathcal{F}_i^1(t, v_t(\cdot)), \mathcal{F}_i^{2\theta_i}(t, v_t(\cdot))\big),$$

where $\mathcal{F}_i^1(t, v_t(\cdot)) \in \Xi_i$, $\mathcal{F}_i^{2\theta_i}(t, v_t(\cdot)) \in \Xi_i$ for $\theta_i \geqslant 0$.

The solution to the Cauchy problem (1.3) with $\mathcal{U}_i$ fixed, $\theta_i > 0$ and with the measurable function $v(t) \in V$, $t \geqslant 0$ is understood as follows.

At $0 \leqslant t \leqslant \theta_i$ this solution is identical to the solution $w_i^1(t)$ to the following Cauchy problem:

$$\dot{w}_i^1(t) = \mathcal{F}_i^1(t, v_t(\cdot)) - v(t), \quad w_i(0) = z_i^0.$$

At $t \geqslant \theta_i$ it is identical to the solution $w_i^2(t)$ to the following Cauchy problem:

$$\dot{w}_i^2(t) = a_{\theta_i}(t)\mathcal{F}_i^{2\theta_i}(t, v_t(\cdot)) - v(t), \quad w_i^2(\theta_i) = w_i^1(\theta_i).$$

§ 2. Auxiliary facts

Definition 2.1. The Minkowski difference of sets A and B is the set

$$A \overset{*}{-} B = \big\{c \;\big|\; c + B \subset A\big\}.$$

Definition 2.2. Let A be a bounded subset of $\mathbb{R}^k$. The support function of set A is a function $c\colon \mathbb{R}^k \to \mathbb{R}$ of the form

$$c(A, \varphi) = \sup_{a \in A}(a, \varphi).$$

Lemma 2.1. *Let A and B be the convex compact sets of $\mathbb{R}^k$, $A \overset{*}{-} B \neq \varnothing$, $0 \in B$. Then,*

$$A \overset{*}{-} B \subset A.$$

The validity of this statement follows from the equality

$$A \overset{*}{-} B = \bigcap_{b \in B} \big(A - b\big).$$

Lemma 2.2. *Let A and B be the convex compacts of $\mathbb{R}^k$, $A \overset{*}{-} B \neq \varnothing$. Then,*

$$\big(A \overset{*}{-} B\big) + B \subset A.$$

P r o o f. From formula (6) [14, p. 431], it follows that, for all $\varphi \in \mathbb{R}^k$, the following inequality holds:

$$c\big(A \overset{*}{-} B, \varphi\big) \leqslant c(A, \varphi) - c(B, \varphi). \tag{2.1}$$

From Property 3 [15, p. 35] of the support function, it follows that, for all $\varphi \in \mathbb{R}^k$, the following equation holds:

$$c\big((A \overset{*}{-} B) + B, \varphi\big) = c\big(A \overset{*}{-} B, \varphi\big) + c(B, \varphi). \tag{2.2}$$

From (2.1) and (2.2), we find that, for all $\varphi \in \mathbb{R}^k$, the following inequality holds:

$$c\big((A \overset{*}{-} B) + B, \varphi\big) \leqslant c(A, \varphi). \tag{2.3}$$

Since $A \overset{*}{-} B + B$ is a convex compact set and inequality (2.3) holds, it follows from the corollary from [15, p. 43] that

$$\big(A \overset{*}{-} B\big) + B \subset A.$$

This proves the lemma. □

§ 3. Capture without phase restrictions

In this section we assume that there are no phase restrictions, i. e., $\Omega = \mathbb{R}^k$.

3.1. Capture with loss compensation

Assumption 3.1. For all $i \in I$, the inclusion $0 \in U_i \overset{*}{-} V$ holds.

Assumption 3.2. For each $i \in I$, there exists a $t_i > 0$, for which the inclusion $Vh_i \subset (U_i \overset{*}{-} V)t_i$ holds.

We introduce the following notation:

$$\tau_i = \min\big\{t \geqslant 0 \,\big|\, Vh_i \subset \big(U_i \overset{*}{-} V\big)t\big\}, \quad \lambda\big(z_i^0, v\big) = \sup\big\{\lambda \geqslant 0 \,\big|\, \lambda(M_i - z_i^0) \cap (U_i - v) \neq \varnothing\big\},$$
$$\delta = \min_{v \in V} \max_{i \in I} \lambda(z_i^0, v).$$

Lemma 3.1. *Suppose that Assumption 3.1 holds, $\delta > 0$, and $c_i, i \in I$, are positive numbers. Then, there exists $T > 0$ such that, for any admissible function $v(\cdot)$ and any set of intervals $[a_i, b_i]$, $b_i - a_i \leqslant c_i$, $i \in I$, there is a number $l \in I$, for which*

$$\int_{\mathcal{F}_l(T)} \lambda\big(z_l^0, v(s)\big)\, ds \geqslant 1, \quad \textit{where} \quad \mathcal{F}_i(t) = [0, t] \cap [a_i, b_i].$$

P r o o f. Let $t > c = \sum\limits_{i=1}^{n} c_i$, $v(\cdot)$ be an arbitrary admissible function. Consider the set $\mathcal{F}(t) = [0, t] \setminus \bigcup\limits_{i=1}^{n} [a_i, b_i]$. Then, $\mathcal{F}(t) \neq \varnothing$ and

$$\sum_{i \in I} \int_{\mathcal{F}_i(t)} \lambda\big(z_i^0, v(s)\big)\, ds \geqslant$$
$$\geqslant \sum_{i \in I} \int_{\mathcal{F}(t)} \lambda\big(z_i^0, v(s)\big)\, ds \geqslant \int_{\mathcal{F}(t)} \max_{i \in I} \lambda\big(z_i^0, v(s)\big)\, ds \geqslant \int_{\mathcal{F}(t)} \delta\, ds \geqslant \delta(t - c).$$

Taking $T > \frac{n}{\delta} + c$, we obtain

$$\sum_{i \in I} \int_{\mathcal{F}_i(t)} \lambda\big(z_i^0, v(s)\big)\, ds \geqslant n.$$

Hence, there exists an $l \in I$, for which

$$\int_{\mathcal{F}_l(t)} \lambda\big(z_l^0, v(s)\big)\, ds \geqslant 1.$$

This proves the lemma. □

Definition 3.1. A capture occurs in the game $\Gamma(n+1)$ if there exist $T_0 > 0$ and strategies $\mathcal{U}_i$, $i \in I$, of pursuers P_i, $i \in I$, such that for any vector $\theta = (\theta_1, \dots, \theta_n)$, $\theta_i + h_i + \tau_i \leqslant T_0$ and any admissible function $v(\cdot)$ there is a number $l \in I$, for which $z_l(T_0) \in M_l$.

Theorem 3.1. *Suppose that Assumptions 3.1 and 3.2 are satisfied and $\delta > 0$. Then, a capture occurs in the game $\Gamma(n+1)$.*

P r o o f. Let T_0 be the time instant satisfying Lemma 3 with $c_i = h_i + \tau_i$. Prove that a capture occurs at time T_0. Let $v(\cdot)$ be an arbitrary admissible control of the evader. Consider the functions

$$h_i(t) = 1 - \int_0^t \lambda_0\big(s, z_i^0, v(s)\big)\, ds,$$

where

$$\lambda_0\big(s, z_i^0, v(s)\big) = \begin{cases} \lambda\big(z_i^0, v(s)\big), & s \notin [\theta_i, \theta_i + h_i + \tau_i], \\ 0, & s \in [\theta_i, \theta_i + h_i + \tau_i]. \end{cases}$$

From the definition of the time instant T_0, it follows that there exist a number $l \in I$ and time $T_l \in [0, T_0]$, for which $h_l(T_l) = 0$. We will construct the control of pursuers P_i, $i \in I$, as follows.

Define the multivalued maps

$$U_i^1(v) = \big\{u_i \in U_i \;\big|\; u_i - v \in \lambda\big(z_i^0, v\big)\big(M_i - z_i^0\big)\big\}.$$

By the theorem of a measurable choice [16], the multivalued maps $U_i^1(v)$, $i \in I$, have measurable selectors $u_i^1(v)$, $i \in I$, such that the function $u_i^1(v(t))$ is measurable. For all $t \in [0, T_0] \setminus [\theta_i, \theta_i + h_i + \tau_i]$, for which $h_i(t) > 0$, we assume the control of pursuers P_i, $i \in I$, to be equal to

$$u_i(t) = u_i^1(v(t)).$$

Let T_i be the first time instant for which $h_i(T_i) = 0$. For all $t \in [T_i, T_0] \setminus [\theta_i, \theta_i + h_i + \tau_i]$, we assume the control of pursuer P_i to be equal to

$$u_i(t) = v(t).$$

Thus, it remains to define the control of pursuers P_i, $i \in I$, on the interval $[\theta_i + h_i, \theta_i + h_i + \tau_i]$. At time $\theta_i + h_i$, pursuer P_i knows the control of evader E on the interval $[\theta_i, \theta_i + h_i]$. Therefore, pursuer P_i knows the vector

$$\omega_i = \int_{\theta_i}^{\theta_i + h_i} v(s)\, ds.$$

By virtue of Assumption 3.2, we have

$$\omega_i \in \int_{\theta_i + h_i}^{\theta_i + h_i + \tau_i} \big(U_i \overset{*}{-} V\big)\, ds.$$

By the definition of the integral of the multivalued map, there exists a measurable selector $\omega_i^0(s) \in (U_i \overset{*}{-} V)$ such that

$$\omega_i = \int_{\theta_i}^{\theta_i+h_i} \omega_i^0(s)\,ds.$$

Let us define the control of pursuer P_i, $i \in I$, on the interval $[\theta_i + h_i, \theta_i + h_i + \tau_i]$, assuming that

$$u_i^3(t) = v(t) + \omega_i^0(t).$$

If $T_l \leqslant \theta_l$, then, from the system (1.3), it follows that

$$z_l(T_0) = z_l^0 + \int_0^{T_l} \big(u_l(s) - v(s)\big)\,ds + \int_{T_l}^{\theta_l} \big(u_l(s) - v(s)\big)\,ds - \int_{\theta_l}^{\theta_l+h_l} v(s)\,ds + \\ + \int_{\theta_l+h_l}^{\theta_l+h_l+\tau_l} \big(u_l(s) - v(s)\big)\,ds + \int_{\theta_l+h_l+\tau_l}^{T_0} \big(u_l(s) - v(s)\big)\,ds.$$

From the definition of the controls of pursuers, we find that the following relations hold:

$$\int_0^{T_l} \big(u_l(s) - v(s)\big)\,ds = \int_0^{T_l} \big(u_l^1(v(s)) - v(s)\big)\,ds \in \int_0^{T_l} \lambda\big(z_l^0, v(s)\big)\big(M_l - z_l^0\big)\,ds,$$

$$\int_{T_l}^{\theta_l} \big(u_l(s) - v(s)\big)\,ds + \int_{\theta_l+h_l+\tau_l}^{T_0} \big(u_l(s) - v(s)\big)\,ds = 0,$$

$$-\int_{\theta_l}^{\theta_l+h_l} v(s)\,ds + \int_{\theta_l+h_l}^{\theta_l+h_l+\tau_l} \big(u_l^3(s) - v(s)\big)\,ds = -\omega_l + \int_{\theta_l+h_l}^{\theta_l+h_l+\tau_l} \omega_l^0(s)\,ds = 0.$$

Therefore,

$$z_l(T_0) \in z_l^0 + \int_0^{T_l} \lambda\big(z_l^0, v(s)\big)\big(M_l - z_l^0\big)\,ds = \\ = z_l^0\left(1 - \int_0^{T_l} \lambda\big(z_l^0, v(s)\big)\,ds\right) + \int_0^{T_l} \lambda\big(z_l^0, v(s)\big) M_l\,ds = M_l.$$

Consequently, in this case a capture occurs in the game $\Gamma(n+1)$.

If $T_l \geqslant \theta_l + h_l + \tau_l$, then, from the system (1.3), it follows that

$$z_l(T_0) = z_l^0 + \int_0^{\theta_l} \big(u_l(s) - v(s)\big)ds - \int_{\theta_l}^{\theta_l+h_l} v(s)ds + \int_{\theta_l+h_l}^{\theta_l+h_l+\tau_l} \big(u_l(s) - v(s)\big)ds + \\ + \int_{\theta_l+h_l+\tau_l}^{T_l} \big(u_l(s) - v(s)\big)ds + \int_{T_l}^{T_0} \big(u_l(s) - v(s)\big)ds.$$

From the definition of the controls of the pursuers, we find that the following relations hold:

$$\int_0^{\theta_l} \big(u_l(s) - v(s)\big)\,ds + \int_{\theta_l+h_l+\tau_l}^{T_l} \big(u_l(s) - v(s)\big)\,ds = \\ = \int_0^{\theta_l} \big(u_l^1(v(s)) - v(s)\big)\,ds + \int_{\theta_l+h_l+\tau_l}^{T_l} \big(u_l^1(v(s)) - v(s)\big)\,ds \in \\ \in \left(\int_0^{\theta_l} \lambda\big(z_l^0, v(s)\big)\,ds + \int_{\theta_l+h_l+\tau_l}^{T_l} \lambda\big(z_l^0, v(s)\big)\,ds\right)\big(M_l - z_l^0\big)\,ds,$$

$$-\int_{\theta_l}^{\theta_l+h_l} v(s)\,ds + \int_{\theta_l+h_l}^{\theta_l+h_l+\tau_l} \big(u_l^3(s) - v(s)\big)\,ds = -\omega_l + \int_{\theta_l+h_l}^{\theta_l+h_l+\tau_l} \omega_l^0(s)\,ds = 0,$$

$$\int_{T_l}^{T_0} \big(u_l(s) - v(s)\big)\,ds = 0.$$

Therefore,

$$z_l(T_0) \in z_l^0 + \left(\int_0^{\theta_l} \lambda\big(z_l^0, v(s)\big)\,ds + \int_{\theta_l+h_l+\tau_l}^{T_l} \lambda\big(z_l^0, v(s)\big)\,ds\right)\big(M_l - z_l^0\big)\,ds =$$
$$= z_l^0\left(1 - \int_0^{\theta_l} \lambda\big(z_l^0, v(s)\big)\,ds - \int_{\theta_l+h_l+\tau_l}^{T_l} \lambda\big(z_l^0, v(s)\big)\,ds\right) +$$
$$+ \left(\int_0^{\theta_l} \lambda\big(z_l^0, v(s)\big)\,ds + \int_{\theta_l+h_l+\tau_l}^{T_l} \lambda\big(z_l^0, v(s)\big)\,ds\right)M_l = M_l.$$

Consequently, a capture occurs in the game $\Gamma(n+1)$ in this case as well. This proves the theorem. □

Remark 3.1. Let $U_i = V$ and $h_i = 0$ for all $i \in I$. Then, Assumptions 3.1 and 3.2 are satisfied automatically. Therefore, if $\delta > 0$, then a capture occurs in the game $\Gamma(n+1)$. Thus, the theorem of B. N. Pshenichnyi [17] and its generalization [18, Theorem 1.1, p. 55] are consequences of Theorem 3.1.

3.2. Capture with a new aim

Assumption 3.3. $0 \in V$ and for all $i \in I$ the sets $M_i \overset{*}{-} (-h_i V) \neq \varnothing$.

Introduce the following notation: $M_i^1 = M_i \overset{*}{-} (-h_i V)$,

$$\lambda_1\big(z_i^0, v\big) = \sup\big\{\lambda \geqslant 0 \mid \lambda\big(M_i^1 - z_i^0\big) \cap (U_i - v) \neq \varnothing\big\}, \qquad \delta_0 = \min_v \max_{i\in I} \lambda_1\big(z_i^0, v\big).$$

Lemma 3.2. *Suppose that Assumption 3.1 holds and $\delta_0 > 0$. Then, there exists a $T > 0$ such that, for any admissible function $v(\cdot)$ and any $\theta_i > 0$, $\theta_i + h_i \leqslant T$, there is a number $l \in I$, for which*

$$\int_0^{\theta_l} \lambda_1\big(z_l^0, v(s)\big)\,ds + \int_{\theta_l+h_l}^{T} \lambda_1\big(z_l^0, v(s)\big)\,ds \geqslant 1.$$

The lemma is proved along the same lines as Lemma 3.1.

Definition 3.2. A capture occurs in the game $\Gamma(n+1)$ if there exist a $T_0 > 0$ and strategies $\mathcal{U}_i$, $i \in I$, of pursuers $P_i, i \in I$, such that for any vector $\theta = (\theta_1, \dots \theta_n)$ and any admissible function $v(\cdot)$ there is a number $l \in I$, for which $z_l(T_0) \in M_l$.

Theorem 3.2. *Suppose that Assumptions 3.1 and 3.3 hold and $\delta_0 > 0$. Then a capture occurs in the game $\Gamma(n+1)$.*

P r o o f. Let T be the time instant, for which the statement of Lemma 3.2 holds, and let $v(\cdot)$ be the admissible control of the evader. Consider the functions

$$h_i(t) = 1 - \int_0^t \lambda_1^0\big(s, z_i^0, v(s)\big)\,ds,$$

where

$$\lambda_1^0\big(s, z_i^0, v\big) = \begin{cases} \lambda_1\big(z_i^0, v\big), & s \notin [\theta_i, \theta_i + h_i], \\ 0, & s \in [\theta_i, \theta_i + h_i]. \end{cases}$$

From Lemma 3.2, it follows that there exist a number $l \in I$ and time $T_l \in [0, T]$, for which $h_l(T_l) = 0$.

Define the multivalued maps

$$U_i^1(v) = \{u_i \in U_i \mid u_i - v \in \lambda_1(z_i^0, v)(M_i^1 - z_i^0)\}.$$

By the theorem of a measurable choice [16], the multivalued maps $U_i^1(v)$, $i \in I$, have measurable selectors $u_i^1(v)$, $i \in I$, such that the function $u_i^1(v(t))$ is measurable. For all $t \in [0, T_0] \setminus [\theta_i, \theta_i + h_i + \tau_i]$, for which $h_i(t) > 0$, we assume the control of pursuers P_i, $i \in I$, to be equal to

$$u_i(t) = u_i^1(v(t)).$$

If $T_l \leqslant \theta_l$, then

$$z_l(T_l) = z_l^0 + \int_0^{T_l} (u_l(s) - v(s))\, ds = z_l^0 + \int_0^{T_l} u_l^1(s)\, ds \in z_l^0 + \int_0^{T_l} \lambda_1(z_l^0, v(s))(M_l^1 - z_l^0)\, ds =$$

$$= z_l^0 \left(1 - \int_0^{T_l} \lambda_1(z_l^0, v(s))\, ds\right) + \int_0^{T_l} \lambda_1(z_l^0, v(s)) M_l^1\, ds = M_l^1 = M_l \overset{*}{-} (-h_l V).$$

Since $0 \in V$, by virtue of Lemma 2.1, $M_l \overset{*}{-} (-h_l V) \subset M_l$. Therefore, $z_l(T_l) \in M_l$, which implies that a capture occurs in the game $\Gamma(n+1)$.

Let $T_l \geqslant \theta_l + h_l$. Then,

$$z_l(T_l) = z_l^0 + \int_0^{\theta_l} (u_l(s) - v(s))\, ds - \int_{\theta_l}^{\theta_l + h_l} v(s)\, ds + \int_{\theta_l + h_l}^{T_l} (u_l(s) - v(s))\, ds =$$

$$= z_l^0 + \int_0^{\theta_l} u_l^1(s)\, ds - \int_{\theta_l}^{\theta_l + h_l} v(s)\, ds + \int_{\theta_l + h_l}^{T_l} u_l^1(s)\, ds \in$$

$$\in z_l^0 + \int_0^{\theta_l} \lambda_1(z_l^0, v(s))(M_l^1 - z_l^0)\, ds - \int_{\theta_l}^{\theta_l + h_l} v(s)\, ds + \int_{\theta_l + h_l}^{T_l} \lambda_1(z_l^0, v(s))(M_l^1 - z_l^0)\, ds =$$

$$= z_l^0 \left(1 - \int_0^{\theta_l} \lambda_1(z_l^0, v(s))\, ds - \int_{\theta_l + h_l}^{T_l} \lambda_1(z_l^0, v(s))\, ds\right) +$$

$$+ \left(\int_0^{\theta_l} \lambda_1(z_l^0, v(s))\, ds + \int_{\theta_l + h_l}^{T_l} \lambda_1(z_l^0, v(s))\, ds\right) M_l^1 - \int_{\theta_l}^{\theta_l + h_l} v(s)\, ds \in M_l^1 - \int_{\theta_l}^{\theta_l + h_l} v(s)\, ds.$$

Since $-\int_{\theta_l}^{\theta_l + h_l} v(s)\, ds \in (-h_l V)$, applying Lemma 2.2, we obtain

$$z_l(T_l) \in M_l^1 + (-h_l V) = (M_i \overset{*}{-} (-h_i V)) + (-h_l V) \subset M_l.$$

This implies that a capture occurs in the game $\Gamma(n+1)$. This proves the theorem. □

Remark 3.2. Let $U_i = V = \{v \mid \|v\| \leqslant 1\}$ and $h_i = 0$ for all $i \in I$. Then Assumptions 3.1 and 3.2 hold automatically. Therefore, if $\delta_0 > 0$, then a capture occurs in the game $\Gamma(n+1)$. Thus, the theorem of B. N. Pshenichnyi [17] and its generalization [18, Theorem 1.1, p. 55] are consequences of Theorem 3.1.

§ 4. Capture with phase restrictions

4.1. Capture with loss compensations and phase restrictions

Denote

$$\delta_1 = \min_{v \in V} \max\left\{\max_{i \in I} \lambda(z_i^0, v), \max_{j \in J} (p_j, v)\right\}.$$

Lemma 4.1. *Suppose that Assumption 3.1 holds and that* $r = 1$, $\delta_1 > 0$. *Then there exists* $T > 0$ *such that for any admissible function* $v(\cdot)$ *and any set of intervals* $[a_i, b_i]$, $b_i - a_i \leqslant c_i$, $i \in I$, *there is a number* $l \in I$, *for which*

$$\int_{\mathcal{F}_l(T)} \lambda\big(z_l^0, v(s)\big)\, ds \geqslant 1, \quad \textit{where} \ \ \mathcal{F}_i(t) = [0,t] \cap [a_i, b_i].$$

P r o o f. Let $t > c = \sum\limits_{i=1}^{n} c_i$, $v(\cdot)$ be an arbitrary admissible function, and let $\mathcal{F}(t) = [0,t] \setminus \bigcup\limits_{i=1}^{n} [a_i, b_i]$. Then $\big(p_1, y(t)\big) \leqslant \mu_1$ for all $t \geqslant 0$. Therefore, for all $t \geqslant 0$ the following inequality holds:

$$\int_0^t \big(p_1, v(s)\big)\, ds \leqslant \mu_0 = \mu_1 - \big(p_1, y^0\big). \tag{4.1}$$

Define the sets

$$T_1(t) = \big\{\tau \mid \tau \in [0,t],\ \big(p_1, v(\tau)\big) \geqslant \delta_1\big\}, \qquad T_2(t) = \big\{\tau \mid \tau \in [0,t],\ \big(p_1, v(\tau)\big) \leqslant \delta_1\big\},$$
$$\mathcal{F}^2(t) = T_2(t) \setminus \bigcup_{i=1}^{n} [a_i, b_i].$$

From (4.1), it follows that, for all $t \geqslant 0$, the following inequality holds:

$$\int_{T_1(t)} \big(p_1, v(s)\big)\, ds + \int_{T_2(t)} \big(p_1, v(s)\big)\, ds \leqslant \mu_0.$$

Since $\big(p_1, v\big) \geqslant -1$ for all $v \in V$, it holds that

$$\mu_0 \geqslant \delta_1 \int_{T_1(t)} ds - \int_{T_2(t)} ds, \qquad t = \int_{T_1(t)} ds + \int_{T_2(t)} ds.$$

This implies that

$$\int_{T_2(t)} ds \geqslant \frac{\delta_1 t - \mu_0}{\delta_1 + 1}.$$

Next, we have

$$\sum_{i \in I} \int_{\mathcal{F}_i(t)} \lambda\big(z_i^0, v(s)\big)\, ds \geqslant \sum_{i \in I} \int_{\mathcal{F}(t)} \lambda\big(z_i^0, v(s)\big)\, ds \geqslant \int_{\mathcal{F}(t)} \max_{i \in I} \lambda\big(z_i^0, v(s)\big)\, ds \geqslant$$
$$\geqslant \int_{\mathcal{F}(t)} \delta_1\, ds \geqslant \int_{\mathcal{F}^2(t)} \delta_1\, ds \geqslant \delta_1 \int_{T_2(t)} ds - \delta_1 c \geqslant \delta_1 \bigg(\frac{\delta_1 t - \mu_0}{\delta_1 + 1} - c\bigg).$$

Taking $T > 0$ so that the inequality

$$\delta_1 \bigg(\frac{\delta_1 T - \mu_0}{\delta_1 + 1} - c\bigg) \geqslant n$$

is satisfied, we find that

$$\sum_{i=1}^{n} \int_{\mathcal{F}_i(T)} \lambda\big(z_i^0, v(s)\big)\, ds \geqslant n.$$

Consequently, there exists an $l \in I$ satisfying the condition of the lemma. This proves the lemma. $\square$

Theorem 4.1. *Suppose that Assumptions 3.1 and 3.2 hold and that* $r = 1$ *and* $\delta_1 > 0$. *Then, a capture occurs in the game* $\Gamma(n+1)$. (*Capture is understood in the sense of Definition 3.1*).

This theorem is proved along the same lines as Theorem 3.1 using Lemma 3.2.
Let $\operatorname{co} A$ denote the convex hull of set A,

$$V_1 = \big\{v \in V \mid \max_{i\in I} \lambda\big(z_i^0, v\big) = 0\big\}.$$

Theorem 4.2. *Suppose that Assumptions 3.1 and 3.2 hold and that* $r > 1$, $\delta_1 > 0$, $\max_{j\in J}\big(p_j, v\big) > 0$ *for all* $v \in \overline{\operatorname{co} V_1}$. *Then, a capture occurs in the game* $\Gamma(n+1)$.

P r o o f. By Theorem 1.10.5 from [19, p. 33], there exist nonnegative γ_j, $j \in J$, the sum of which is equal to 1 and

$$\inf_{v\in\overline{\operatorname{co} V_1}} \sum_{j=1}^{r} \gamma_j\big(p_j, v\big) > 0.$$

Take $p = \gamma_1 p_1 + \ldots + \gamma_r p_r$, $\mu = \gamma_1\mu_1 + \ldots + \gamma_r\mu_r$ and consider the set

$$\Omega_0 = \big\{y \mid \big(p, y\big) \leqslant \mu\big\}.$$

Then $\Omega \subset \Omega_0$ and

$$\delta_1^0 = \min_{v\in V} \max\Big\{\max_{i\in I} \lambda\big(z_i^0, v\big), \big(p, v\big)\Big\} > 0.$$

Hence, a capture occurs in the game $\Gamma(n+1)$ with phase restrictions Ω_0. Hence, a capture occurs in the initial game as well. This proves the theorem. $\square$

4.2. Capture with a new aim and phase restrictions

We introduce the following notation:

$$\delta_2 = \min_{v} \max\Big\{\max_{i\in I} \lambda_1\big(z_i^0, v\big), \max_{j\in J}\big(p_j, v\big)\Big\}, \quad V_2 = \big\{v \in V \mid \max_{i\in I} \lambda_1\big(z_i^0, v\big) = 0\big\}.$$

Lemma 4.2. *Suppose that Assumption 3.1 holds and that* $r = 1$ *and* $\delta_2 > 0$. *Then, there exists a* $T > 0$ *such that, for any admissible function* $v(\cdot)$ *and any* $\theta_i > 0, \theta_i + h_i \leqslant T$, *there is a number* $l \in I$, *for which*

$$\int_0^{\theta_l} \lambda_1\big(z_l^0, v(s)\big)\, ds + \int_{\theta_l + h_l}^{T} \lambda_1\big(z_l^0, v(s)\big)\, ds \geqslant 1.$$

The lemma is proved along the same lines as Lemma 4.1.

Theorem 4.3. *Suppose that Assumptions 3.1 and 3.3 hold and that* $r = 1$ *and* $\delta_2 > 0$. *Then a capture occurs in the game* $\Gamma(n+1)$. (*Capture is understood in the sense of Definition 3.2*).

This theorem is proved along the same lines as Theorem 3.2 using Lemma 4.2.

Theorem 4.4. *Suppose that Assumptions 3.1 and 3.3 hold and that* $r > 1$, $\delta_2 > 0$, $\max_{j\in J}\big(p_j, v\big) > 0$ *for all* $v \in \overline{\operatorname{co} V_2}$. *Then, a capture occurs in the game* $\Gamma(n+1)$.

This theorem is proved along the same lines as Theorem 4.2.

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Задача простого группового преследования с возможным нарушением в динамике и фазовыми ограничениями

Ключевые слова: дифференциальная игра, преследователь, убегающий, поимка, фазовые ограничения, поломка.

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В конечномерном евклидовом пространстве рассматривается задача преследования группой преследователей одного убегающего, описываемая системой вида

$$\dot{z}_i = a_i(t)u_i - v, \quad u_i \in U_i, \quad v \in V,$$

где функции $a_i(t)$ равны 1 при всех t, за исключением некоторого отрезка заданной длины, на котором они равны нулю (для каждого преследователя свой отрезок). Этот факт можно трактовать так, что у каждого из преследователей возможен отказ в работе управляющего устройства в любой заранее неизвестный момент времени, а длина промежутка времени, необходимого на устранение поломки, задана, при этом в процессе устранения поломки преследователи не имеют возможности осуществлять поимку. Целевые множества — выпуклые компакты, убегающий не покидает пределы выпуклого многогранного множества. Получены достаточные условия разрешимости задачи преследования.

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