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ON FUZZY PARATOPOLOGICAL GROUP AND DECISION MAKING DURING ROBOT CRASH

In this article, we introduce fuzzy paratopological group, fuzzy semitopological group and fuzzy quasitopological group with illustrative examples and properties. These new notions belong to fuzzy topological group. Here, we prove that each fuzzy regular paratopological group is completely regular by using fuzzy uniformities. Moreover, we prove some results related to fuzzy semitopological group and fuzzy quasitopological group. In addition, we provide an application in the area of decision making during robot crash by using our above stated notions and nano topology.

Keywords: fuzzy quasitopological group, fuzzy paratopological group, fuzzy quasi uniformity, robotics, decision making.

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Introduction

Nature is full of uncertainties. Both living and non-living things in this universe are somehow connected with uncertainties. Prior to 1965, probability theory was the only mathematical tool to deal with uncertainties. But it was Zadeh [1], who had the courage to introduce the concept of fuzzy sets. Fuzzy set theory distinguishes grey zones from black and white reality of nature. Fuzzy set is a mapping from a universe X to $[0, 1]$. It is an extended form of classical set theory. The primary kind of mathematical structures are algebraic structure, ordered structure and topological structure. Theory of fuzzy topological space was introduced by Chang [2] in 1968. Later in 1976, Lowen [3] redefined it in a new way.

A prevalent carrier of topological structure and ordered structure is fuzzy topology. Augmented an algebraic structure with this, fuzzy groups in 1971 and fuzzy topological groups in 1979 were formulated by Rosenfeld [4] and Foster [5] respectively. In 1997, Liu and Luo [6] elucidated quasi-coincidence relation which ascertains quasi-coincident neighbourhood systems. With this neighbourhood system, fuzzy topology is a generalization of topology but it has its own notable peculiarity. Liang and Hai [7] redefined fuzzy topological group in terms of quasi neighbourhoods, which yields topological group as a particular case of fuzzy topological group.

An imperative field of analysis and topology is uniform structures because it indulges an apportion ambience to ligament metrics with topological structures. Hutton [8] expounded fuzzy uniform spaces, quasi fuzzy uniformity and interpreted that every fuzzy topological space is quasi-uniformizable. Lowen [9] dispensed the conception of fuzzy uniform space and Katsaras [10] proved that every uniformity on a set resembles a fuzzy uniformity. Moreover, various approximations are mandatory to mathematically model any real life situation. One of the approximation tools of general topology is nano topology [11]. A nano topological space is defined in terms of lower and upper approximations with respect to a relation on a subset X of a universe μ.

In this work, we define fuzzy paratopological group, fuzzy semitopological group, fuzzy quasitopological group and discuss their properties. We prove that each regular fuzzy paratopological group is completely regular by using fuzzy uniformity structure. At the end, we model a decision making during robot crash using the concept of nano topology and fuzzy topological group.

§ 1. Preliminaries

Throughout this paper, the symbol a^{-1} (respectively, $\overline{\mathsf{A}}$ and A°) denotes inverse of a (respectively, closure of A and interior of A). Let K be a set, $\mathsf{I} = [0, 1]$ be the unit interval and I^{K} be the set of all functions from K to I. A fuzzy set on K is an element of I^{K} with membership function ρ and a fuzzy topology [12] $\mathfrak{T}$ on K is a subset of I^{K} satisfying (i) $\emptyset$, $\mathsf{K} \in \mathfrak{T}$, (ii) $\mathsf{A} \cap \mathsf{B} \in \mathfrak{T}$, whenever $\mathsf{A}, \mathsf{B} \in \mathfrak{T}$ and (iii) $\cup \mathsf{A}_{\alpha} \in \mathfrak{T}$, whenever $\mathsf{A}_{\alpha} \in \mathfrak{T}$. Fuzzy topological space is a set K together with a fuzzy topology $\mathfrak{T}$. Every member of $\mathfrak{T}$ is called $\mathfrak{T}$-open fuzzy set and complement of each member of $\mathfrak{T}$ is $\mathfrak{T}$-closed fuzzy set. A fuzzy set in K is a fuzzy point [12] if and only if it takes the value 0 for all $m \in \mathsf{K}$ except one, say, $k \in \mathsf{K}$. If its value at k is t, then it is denoted by k_t, where the point k is called its support. The fuzzy point k_t is contained in S [12] if and only if $t \leq \mathsf{S}(k)$. A fuzzy point k_t (fuzzy set M respectively) is quasi-coincident with S [12] if and only if $t + \mathsf{S}(k) > 1$ (there exists $k \in \mathsf{K}$ such that $\mathsf{S}(k) + \mathsf{M}(k) > 1$ respectively) and it is denoted by $k_t q \mathsf{S}$ ($\mathsf{M} q \mathsf{S}$ respectively). A fuzzy set S is Q-open neighbourhood of k_t [12] if and only if there exists $\mathsf{M} \in \mathfrak{T}$ such that $k_t q \mathsf{M}$ and $\mathsf{M} \subseteq \mathsf{S}$. A function $g \colon (\mathsf{K}, \mathfrak{T}) \mapsto (\mathsf{M}, \mathfrak{S})$ is fuzzy continuous [2] if and only if for every $\mathsf{N} \in \mathfrak{S}$, $g^{-1}(\mathsf{N}) \in \mathfrak{T}$. A fuzzy group [4] in K is a fuzzy set S satisfying $\mathsf{S}(ab) \geq \min\{\mathsf{S}(a), \mathsf{S}(b)\}$ for all $a, b \in \mathsf{K}$ and a fuzzy group S is normal if $\mathsf{S}(ab) = \mathsf{S}(ba)$ for every $a, b \in \mathsf{K}$. Let K be a group with S, T two fuzzy sets and m_t, a fuzzy point in K. We define $\mathsf{ST}(k) = \sup\limits_{k_1 k_2 = k} \min(\mathsf{S}(k_1), \mathsf{T}(k_2))$, $m_t \mathsf{T}(k) = \sup\limits_{m k_2 = k} \min(t, \mathsf{T}(k_2))$, $\mathsf{S} m_t(k) = \sup\limits_{k_1 m = k} \min(\mathsf{S}(k_1), t)$ and $\mathsf{T}^{-1}(k) = \mathsf{T}(k^{-1})$. A fuzzy topological space K is regular [13] if every Q-open set E can be expressed as union of Q-open sets F_{α}'s such that $\overline{\mathsf{F}_{\alpha}} \subseteq \mathsf{E}$ for all α. K is completely regular [13], if for all $k \in \mathsf{K}$ and a Q-open neighbourhood E of k, there exists a continuous function $\phi \colon \mathsf{K} \mapsto [0, 1]$ with $\phi(k) = 1$ and $\phi(t) = 0 \ \forall t \notin \mathsf{E}$. A pair $(\mathsf{K}, \mathfrak{T})$ denotes a group with a fuzzy topology $\mathfrak{T}$. The collection of Q-open neighbourhoods in K is denoted by $\mathsf{Q}(\mathsf{K})$. The following theorem is a preliminary result which is mandatory for sequel.

Theorem 1 (see [14]). *Let $\zeta \colon (\mathsf{K}, \mathfrak{T}) \mapsto (\mathsf{P}, \mathfrak{S})$ be a function. Then, the following are equivalent:*

(i) *ζ is fuzzy continuous,*

(ii) *for every closed set E in P, $\zeta^{-1}(\mathsf{E})$ is closed in K,*

(iii) *for each member C of a subbase for $\mathfrak{S}$, $\zeta^{-1}(\mathsf{C})$ is open,*

(iv) *for any fuzzy set F in K, $\zeta(\overline{\mathsf{F}}) \subseteq \overline{\zeta(\mathsf{F})}$,*

(v) *for any fuzzy set D in P, $\overline{\zeta^{-1}(\mathsf{D})} \subseteq \zeta^{-1}(\overline{\mathsf{D}})$.*

Definition 1. Let μ be any non-empty finite set of objects called universe, $\mathbf{R}$ be an equivalence relation on μ. Then, the pair $(\mu, \mathbf{R})$ is said to be the approximation space. Let $X \subseteq \mu$.

(i) The lower approximation of X [15] with respect to $\mathbf{R}$ is the set of all objects, whose equivalence class lies in X and it is denoted by $\mathbf{L_R}(X) = \bigcup\limits_{a \in \mu} \{\mathbf{R}(a) \mid \mathbf{R}(a) \subseteq X\}$, where $\mathbf{R}(a)$ denotes the equivalence class determined by a,

(ii) The upper approximation of X [15] with respect to $\mathbf{R}$ is the set of all objects, whose equivalence class intersects X with respect to $\mathbf{R}$ and it is denoted by

$$\mathbf{U_R}(X) = \bigcup_{a \in \mu} \{\mathbf{R}(a) \mid \mathbf{R}(a) \cap X \neq \emptyset\},$$

(iii) The boundary region of X [15] with respect to $\mathbf{R}$ is the set difference of upper approximation with lower approximation and it is denoted by $\mathbf{B_R}(X) = \mathbf{U_R}(X) - \mathbf{L_R}(X)$,

(iv) The nano topology of X [11] is given by $\tau_{\mathbf{R}}(X) = \{\emptyset, \mathbf{L}_{\mathbf{R}}(X), \mathbf{U}_{\mathbf{R}}(X), \mathbf{B}_{\mathbf{R}}(X), \mu\}$ with basis, $\beta_{\mathbf{R}}(X) = \{\mathbf{L}_{\mathbf{R}}(X), \mathbf{B}_{\mathbf{R}}(X), \mu\}$.

Definition 2 (see [16]). Let μ be a universe, A be a non-empty finite set of attributes. V_a is the attribute value set of an attribute $a \in A$ and $f_a \colon \mu \mapsto V_a$ is called attribute function. For any subset B of A, there is a binary relation on μ corresponding to B given by

$$\mathbf{R}(B) = \{(x, y) \in \mu \times \mu \mid f_a(x) = f_a(y) \text{ or } f_a(x) = * \text{ or } f_a(y) = * \text{ for any } a \in B\}.$$

Then, $\mathbf{R}(B)$ is a tolerance relation on μ (reflexive and symmetric). $S_B(x)$ is the maximal set of objects which are possibly indiscernible with x by the tolerance relation on μ denoted by

$$S_B(x) = \{y \in \mu \mid (x, y) \in \mathbf{R}(B)\},$$

where $x \in \mu$.

§ 2. Fuzzy paratopological group and related structures

In this section, we define generalized structures of fuzzy topological group and discuss their properties with illustrative examples.

Definition 3. The pair $(\mathsf{K}, \mathfrak{T})$ is a fuzzy paratopological group (fuzzy semitopological group respectively) if (1) $\forall\, m, n \in \mathsf{K}$ and $\mathsf{D} \in \mathsf{Q}(\mathsf{K})$ of fuzzy point $(mn)_k$, $\exists\,\mathsf{Q}$-open neighbourhoods $\mathsf{E}, \mathsf{F} \in \mathsf{Q}(\mathsf{K})$ of m_k and n_k such that $\mathsf{EF} \subseteq \mathsf{D}$ ($m_k\mathsf{F} \subseteq \mathsf{D}$, $\mathsf{E}n_k \subseteq \mathsf{D}$ respectively).

A fuzzy topological group (fuzzy quasitopological group respectively) is a fuzzy paratopological group (fuzzy semitopological group respectively) with the proviso (2) $\forall\, t \in \mathsf{K}$ and $\mathsf{P} \in \mathsf{Q}(\mathsf{K})$ of fuzzy point t_k^{-1}, $\exists\, \mathsf{S} \in \mathsf{Q}(\mathsf{K})$ of t_k such that $\mathsf{S}^{-1} \subseteq \mathsf{P}$.

Example 1. Consider the symmetric group S_3. Define fuzzy sets on S_3 by

$$\begin{aligned}
\mathsf{A} &= \{(e,0), ((123),0), ((132),0), ((12),0), ((13),0), ((23),0)\},\\
\mathsf{B} &= \{(e,1), ((123),1), ((132),1), ((12),1), ((13),1), ((23),1)\},\\
\mathsf{C} &= \{(e,1), ((123),1), ((132),1), ((12),0), ((13),0), ((23),0)\},\\
\mathsf{D} &= \{(e,0), ((123),1), ((132),1), ((12),0), ((13),0), ((23),0)\},
\end{aligned}$$

and we get fuzzy topologies $\mathfrak{T}_1 = \{\mathsf{A}, \mathsf{B}\}$, $\mathfrak{T}_2 = \{\mathsf{A}, \mathsf{B}, \mathsf{C}\}$, $\mathfrak{T}_3 = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\}$. Now, the tuples $(S_3, \mathfrak{T}_1)$, $(S_3, \mathfrak{T}_2)$ are all fuzzy topological groups since the possible choice of Q-open neighbourhoods are B, C and the neighbourhood itself satisfies conditions (1), (2) in Definition 3. $(S_3, \mathfrak{T}_3)$ is not a fuzzy semitopological group (fuzzy quasitopological group respectively), if we consider D as Q-open neighbourhood of $((123), 1)$, $((132), 1)$, it is not possible to find a Q-open neighbourhood satisfying condition (1) in Definition 3.

Proposition 1. *Let $(\mathsf{K}, \mathfrak{T})$ be a fuzzy topological space. Then, condition (1) of Definition 3 holds if and only if the map $p \colon (\mathsf{K}, \mathfrak{T}) \times (\mathsf{K}, \mathfrak{T}) \mapsto (\mathsf{K}, \mathfrak{T})$ defined by $(m, n) \mapsto mn$ is fuzzy continuous.*

P r o o f. Suppose condition (1) holds, then $\forall\, \mathsf{D} \in \mathsf{Q}(\mathsf{K})$ of $(mn)_k$ $\exists\, \mathsf{E}, \mathsf{F} \in \mathsf{Q}(\mathsf{K})$ of m_k and n_k such that $\mathsf{EF} \subseteq \mathsf{D}$. For a fuzzy point $(m, n)_k \in (\mathsf{K}, \mathfrak{T}) \times (\mathsf{K}, \mathfrak{T})$,

$$\rho_{(\mathsf{E},\mathsf{F})}(m, n) + k = \min(\rho_{\mathsf{E}}(m), \rho_{\mathsf{F}}(n)) + k = \min(\rho_{\mathsf{E}}(m) + k, \rho_{\mathsf{F}}(n) + k) > 1,$$

we have $(m, n)_k q(\mathsf{E}, \mathsf{F})$ in $(\mathsf{K}, \mathfrak{T}) \times (\mathsf{K}, \mathfrak{T})$. Hence, (E, F) is a Q-open neighbourhood of $(m, n)_k$. But, $p(\mathsf{E}, \mathsf{F}) = \mathsf{EF} \subseteq \mathsf{D}$ and so, p is fuzzy continuous at the fuzzy point $(m, n)_k$. Since $(m, n)_k$ is arbitrary, thus p is fuzzy continuous.

Conversely, suppose p is fuzzy continuous. Then, p is fuzzy continuous at any fuzzy point $(m,n)_k$. So, $\forall\,\mathsf{D}\in\mathsf{Q}(\mathsf{K})$ of $(mn)_k$ $\exists$ a Q-open neighbourhood $\mathsf{G}_{(m,n)_k}$ such that $p[\mathsf{G}_{(m,n)_k}]\subseteq\mathsf{D}$. Then, there must be $\mathsf{E},\mathsf{F}\in\mathsf{Q}(\mathsf{K})$ such that $(\mathsf{E},\mathsf{F})\subseteq\mathsf{G}_{(m,n)_k}$ and $(m,n)_k q(\mathsf{E},\mathsf{F})$. Moreover, $m_k q\mathsf{E}$ and $n_k q\mathsf{F}$. Thus, $\exists\,\mathsf{E},\mathsf{F}\in\mathsf{Q}(\mathsf{K})$ of m_k and n_k such that $\mathsf{EF}=p(\mathsf{E},\mathsf{F})\subseteq p(\mathsf{G}_{(mn)_k})\subseteq\mathsf{D}$. Hence, condition (1) holds. □

Proposition 2. *In a fuzzy topological space* $(\mathsf{K},\mathfrak{T})$*, condition* (2) *of Definition* 3 *holds if and only if the map* $i\colon(\mathsf{K},\mathfrak{T})\mapsto(\mathsf{K},\mathfrak{T})$ *by* $m\mapsto m^{-1}$ *is fuzzy continuous.*

P r o o f. Suppose condition (2) holds, then $\forall\,\mathsf{P}\in\mathsf{Q}(\mathsf{K})$ of t_k^{-1}, $\exists\,\mathsf{S}\in\mathsf{Q}(\mathsf{K})$ of t_k such that $\mathsf{S}^{-1}\subseteq\mathsf{P}$. For a fuzzy point $t_k\in(\mathsf{K},\mathfrak{T})$, $\rho_{(\mathsf{S})}(t_k)+k>1$. So, we have $t_k q\mathsf{S}$ in $(\mathsf{K},\mathfrak{T})$. Hence, $\mathsf{S}\in\mathsf{Q}(\mathsf{K})$ of t_k and $i(\mathsf{S})=\mathsf{S}^{-1}\subseteq\mathsf{P}$. Thus, i is fuzzy continuous at t_k. Since t_k is arbitrary, thus i is fuzzy continuous. Conversely, suppose i is fuzzy continuous. Then, i is continuous at any fuzzy point t_k. So, $\forall\,\mathsf{P}\in\mathsf{Q}(\mathsf{K})$ of t_k^{-1} $\exists\,\mathsf{S}\in\mathsf{Q}(\mathsf{K})$ of t_k such that $i(\mathsf{S})\subseteq\mathsf{P}$. Hence, condition (2) holds. □

Proposition 3. *Let* $(\mathsf{K},\mathfrak{T})$ *be a fuzzy topological space. Then, conditions* (1) *and* (2) *of Definition* 3 *hold if and only if* $\forall\,\mathsf{D}\in\mathsf{Q}(\mathsf{K})$ *of* $(mn^{-1})_k$*,* $\exists\,\mathsf{E},\mathsf{F}\in\mathsf{Q}(\mathsf{K})$ *of* m_k *and* n_k *such that* $\mathsf{EF}^{-1}\subseteq\mathsf{D}$.

P r o o f. Suppose conditions (1) and (2) hold, then by condition (1) we have, $\forall\,m,n\in\mathsf{K}$, $\mathsf{D}\in\mathsf{Q}(\mathsf{K})$ of $(mn^{-1})_k$, $\exists\,\mathsf{E},\mathsf{F}\in\mathsf{Q}(\mathsf{K})$ of m_k and n_k^{-1} such that $\mathsf{EF}\subseteq\mathsf{D}$. By condition (2), $\exists\,\mathsf{C}\in\mathsf{Q}(\mathsf{K})$ of n_k such that $\mathsf{C}^{-1}\subseteq\mathsf{F}$, whence $\mathsf{EC}^{-1}\subseteq\mathsf{EF}\subseteq\mathsf{D}$.

Conversely, suppose $\forall\,\mathsf{O}\in\mathsf{Q}(\mathsf{K})$ of $(mn^{-1})_k$ $\exists\,\mathsf{M},\mathsf{N}\in\mathsf{Q}(\mathsf{K})$ of m_k and n_k such that $\mathsf{MN}^{-1}\subseteq\mathsf{O}$. Since M is quasi-open neighbourhood of m_k, we have $m_k q\mathsf{M}$ and so, $\rho_{\mathsf{M}}(m)+k>1$. Thus, $\rho_{\mathsf{M}}(m)>1-k=k'$. Now, assume $\overline{k}=\rho_{\mathsf{M}}(m),\mathsf{A}=\mathsf{N}\cap\overline{k}^*$, where $\overline{k}^*$ is the constant fuzzy set with membership $\rho(a)=\overline{k}$ for every $a\in\mathsf{K}$. Then, $\mathsf{A}\in\mathsf{Q}(\mathsf{K})$ with $n_k q\mathsf{A}$ and $\mathsf{MA}^{-1}\subseteq\mathsf{MN}^{-1}\subseteq\mathsf{O}$. Since $\rho_{m_{\overline{k}}\mathsf{A}^{-1}}(t)=\sup\limits_{t_1t_2=t}\min(\rho_{m_{\overline{k}}}(t_1),\rho_{\mathsf{A}^{-1}}(t_2))=\min(\overline{k},\rho_{\mathsf{A}^{-1}}(t))=$ $=\rho_{\mathsf{A}^{-1}}(t)$, it follows that $m_{\overline{k}}\mathsf{A}^{-1}\subseteq\mathsf{MA}^{-1}\subseteq\mathsf{O}$. Therefore, condition (2) of Definition 3 holds. Next, $\forall\,\mathsf{D}\in\mathsf{Q}(\mathsf{K})$ of $(mn)_k=(m(n^{-1})^{-1})_k$, by assumption, $\exists\,\mathsf{E},\mathsf{F}\in\mathsf{Q}(\mathsf{K})$ of m_k and n_k^{-1} such that $\mathsf{EF}^{-1}\subseteq\mathsf{D}$. Using condition (2), $\exists\,\mathsf{C}\in\mathsf{Q}(\mathsf{K})$ of n_k such that $\mathsf{C}^{-1}\subseteq\mathsf{F}$. Thus, $\mathsf{EC}=\mathsf{E}(\mathsf{C}^{-1})^{-1}\subseteq\mathsf{EF}^{-1}\subseteq\mathsf{D}$. Hence, condition (1) holds. □

Proposition 4. *Let* $(\mathsf{K},\mathfrak{T})$ *be a fuzzy semitopological group (fuzzy quasitopological group), then the mappings* $l\colon m\mapsto nm$*,* $r\colon m\mapsto mn$ $(i\colon m\mapsto m^{-1})$ *are homeomorphisms of* $(\mathsf{K},\mathfrak{T})$ *onto itself, where* $m,n\in\mathsf{K}$.

P r o o f. Since $(\mathsf{K},\mathfrak{T})$ is a fuzzy semitopological group, thus $\forall\,\mathsf{D}\in\mathsf{Q}(\mathsf{K})$ of $(l(m))_k=(nm)_k$ $\exists\,\mathsf{F}\in\mathsf{Q}(\mathsf{K})$ of m_k such that $n_k\mathsf{F}\subseteq\mathsf{D}$. Assume $\mathsf{F}_{\overline{k}}=\mathsf{F}\cap k^*$. Then, $m_k q\mathsf{F}_{\overline{k}}$ and $n_k\mathsf{F}_{\overline{k}}\subseteq\mathsf{D}$. Since $\rho_{l(\mathsf{F}_{\overline{k}})}(t)=\sup\limits_{s\in l^{-1}(t)}\rho_{\mathsf{F}_{\overline{k}}}(s)=\rho_{\mathsf{F}_{\overline{k}}}(n^{-1}t)$, it follows that

$$\rho_{n_k\mathsf{F}_{\overline{k}}}(t)=\sup_{t_1t_2=t}\min(\rho_{\mathsf{F}_{\overline{k}}}(t_2),k)\geq\min(\rho_{\mathsf{F}_{\overline{k}}}(n^{-1}t),k)=\rho_{\mathsf{F}_{\overline{k}}}(n^{-1}t).$$

Hence, $l(\mathsf{F}_{\overline{k}})\subseteq n_k\mathsf{F}_{\overline{k}}\subseteq\mathsf{D}$. So, l is continuous at m_k. Since m_k is arbitrary, thus l is continuous. We can define $l^{-1}\colon nm\mapsto m$ as left translation of nm by n^{-1}. Since each left translation is continuous, we have l^{-1} is continuous. Thus, l is a fuzzy homeomorphism. Similarly, we can prove that r is a fuzzy homeomorphism. By proposition 2, i is fuzzy continuous. Since $i^{-1}(m)=i(m^{-1})$, thus i^{-1} is fuzzy continuous. Hence, i is a fuzzy homeomorphism. □

Corollary 1. *Let* $(\mathsf{K},\mathfrak{T})$ *be a fuzzy quasitopological group. If* D *is open* (*closed respectively*) *in* $\mathfrak{T}$*, then* $\mathsf{CD},\mathsf{DC},\mathsf{D}^{-1}$ ($k\,\mathsf{D},k\mathsf{D},\mathsf{D}^{-1}$ *respectively*) *are all open* (*closed respectively*) *sets in* $\mathfrak{T}$*, where* $k\in\mathsf{K}$ *and* C *is a nonfuzzy subset of* K.

Lemma 1. *Let* $(\mathsf{K}, \mathfrak{T})$ *be a fuzzy paratopological group and* E, F *be fuzzy subsets of* K. *Then,*

(i) $\overline{t\mathsf{E}t^{-1}} = t\overline{\mathsf{E}}t^{-1}$, *where* $t \in \mathsf{K}$,

(ii) *if* $\overline{\mathsf{E}} \times \overline{\mathsf{F}} \subseteq \overline{\mathsf{E} \times \mathsf{F}}$, *then* $\overline{\mathsf{E}}\,\overline{\mathsf{F}} \subseteq \overline{\mathsf{EF}}$.

P r o o f.

(i) By Corollary 1, $t\overline{\mathsf{E}}t^{-1}$ is a closed set. Since $\overline{t\mathsf{E}t^{-1}}$ is the smallest closed set containing $t\mathsf{E}t^{-1}$, so $\overline{t\mathsf{E}t^{-1}} \subseteq t\overline{\mathsf{E}}t^{-1}$. Define $\phi\colon (\mathsf{K}, \mathfrak{T}) \mapsto (\mathsf{K}, \mathfrak{T})$ by $\phi(k) = tkt^{-1}$. By Proposition 4, ϕ is a fuzzy homeomorphism. Again, by Theorem 1, $\phi(\overline{\mathsf{E}}) \subseteq \overline{\phi\mathsf{E}}$. Thus, $t\overline{\mathsf{E}}t^{-1} \subseteq \overline{t\mathsf{E}t^{-1}}$.

(ii) By Proposition 1, the map $g\colon (\mathsf{K}, \mathfrak{T}) \times (\mathsf{K}, \mathfrak{T}) \mapsto (\mathsf{K}, \mathfrak{T})$ by $g(m, n) = mn$ is fuzzy continuous. Given that $\overline{\mathsf{E}} \times \overline{\mathsf{F}} \subseteq \overline{\mathsf{E} \times \mathsf{F}}$, and so $g(\overline{\mathsf{E}} \times \overline{\mathsf{F}}) \subseteq g(\overline{\mathsf{E} \times \mathsf{F}})$. Since g is fuzzy continuous, by Theorem 1 we have, $g(\overline{\mathsf{E} \times \mathsf{F}}) \subseteq \overline{g(\mathsf{E} \times \mathsf{F})}$. Thus, $\overline{\mathsf{E}}\,\overline{\mathsf{F}} \subseteq \overline{\mathsf{EF}}$. This completes the proof. □

Theorem 2. *Suppose* $(\mathsf{K}, \mathfrak{T})$ *is a fuzzy paratopological group and* M *a fuzzy (normal respectively) subgroup of* K *with* $\overline{\mathsf{M}} \times \overline{\mathsf{M}} \subseteq \overline{\mathsf{M} \times \mathsf{M}}$. *Then,* $\overline{\mathsf{M}}$ *is a fuzzy (normal respectively) subgroup.*

P r o o f. Let M be a fuzzy subgroup and $s \in \mathsf{K}$. Then,

$$\mathsf{M}(t) = \mathsf{M}(ss^{-1}t) \geq \min(\mathsf{M}(s), \mathsf{M}(s^{-1}t)).$$

Thus, $\mathsf{M}(t) \geq \sup_{t=mn} \min(\mathsf{M}(m), \mathsf{M}(n))$. Hence, $\mathsf{MM}(t) \leq \mathsf{M}(t)\ \forall\, t \in \mathsf{K}$, i. e., $\mathsf{MM} \subseteq \mathsf{M}$. Thus, $\overline{\mathsf{MM}} \subseteq \overline{\mathsf{M}}$. By Lemma 1, $\overline{\mathsf{M}}\,\overline{\mathsf{M}} \subseteq \overline{\mathsf{MM}}$. Hence, $\overline{\mathsf{M}}\,\overline{\mathsf{M}} \subseteq \overline{\mathsf{MM}} \subseteq \overline{\mathsf{M}}$, i. e., $\overline{\mathsf{M}}(mn) \geq (\overline{\mathsf{M}}\overline{\mathsf{M}})(mn) = \sup_{mn=ts} \min(\overline{\mathsf{M}}(t), \overline{\mathsf{M}}(s)) \geq \min(\overline{\mathsf{M}}(m), \overline{\mathsf{M}}(n))$. Since M is a fuzzy subgroup, thus $\mathsf{M}(a) = \mathsf{M}(a^{-1}) = \mathsf{M}^{-1}(a)$, for every $a \in \mathsf{K}$. Hence, $\overline{\mathsf{M}} = \overline{\mathsf{M}^{-1}}$. Now, $\overline{\mathsf{M}^{-1}}(a) = \left(\bigcap_{\mathsf{M}^{-1} \subseteq \mathsf{E}_i} \mathsf{E}_i\right)(a) = \liminf_{\mathsf{M}^{-1} \subseteq \mathsf{E}_i}(a) = \liminf_{\mathsf{M} \subseteq \mathsf{E}_i^{-1}}(a^{-1}) = \overline{\mathsf{M}}(a^{-1}) = (\overline{\mathsf{M}})^{-1}(a)$. Thus, $\overline{\mathsf{M}}(a) = (\overline{\mathsf{M}})^{-1}(a) = \overline{\mathsf{M}}(a^{-1})$. Hence, $\overline{\mathsf{M}}$ is a fuzzy subgroup of K.

Now, let R be a fuzzy normal subgroup. Then, $\mathsf{R}(mn) = \mathsf{R}(nm)\ \forall\, m, n \in \mathsf{K}$. Hence, $t\mathsf{R}t^{-1}(s) = \min(t\mathsf{R}(st), t^{-1}(t^{-1})) = t\mathsf{R}(st) = \min(t(t), \mathsf{R}(t^{-1}st)) = \mathsf{R}(t^{-1}st) = \mathsf{R}(stt^{-1}) = \mathsf{R}(s)\ \forall\, s, t \in \mathsf{K}$. So, $t\mathsf{R}t^{-1} = \mathsf{R}$. Thus, $\overline{t\mathsf{R}t^{-1}} = \overline{\mathsf{R}}\ \forall\, t \in \mathsf{K}$. By Lemma 1, $t\overline{\mathsf{R}}t^{-1} = \overline{t\mathsf{R}t^{-1}}$. Hence, $\overline{\mathsf{R}} = t\overline{\mathsf{R}}t^{-1}\ \forall\, t \in \mathsf{K}$. Thus, $\overline{\mathsf{R}}(ts) = t\overline{\mathsf{R}}t^{-1}(ts) = \min(t(t), \overline{\mathsf{R}}t^{-1}(t^{-1}ts)) = \overline{\mathsf{R}}t^{-1}(s) = \min(\overline{\mathsf{R}}(st), t^{-1}(t^{-1})) = \overline{\mathsf{R}}(st)\ \forall\, s, t \in \mathsf{K}$. Hence, $\overline{\mathsf{R}}$ is a fuzzy normal subgroup in K. □

Theorem 3. *Let* $(\mathsf{K}, \mathfrak{T})$ *be a fuzzy paratopological group,* $\mathfrak{S} = \{\mathsf{E}\}$ *be a* Q*-open neighbourhood base of* e_k *and* $\mathfrak{S}_{\overline{k}} = \{\mathsf{E}_{\overline{k}} = \mathsf{E} \cap \overline{k^*} \mid \mathsf{E} \in \mathfrak{S},\ \overline{k} = \sup_{\mathsf{E} \in \mathfrak{S}} \rho_{\mathsf{E}}(e)\}$, *where* $\overline{k}^*$ *is the constant fuzzy set with fuzzy value* $\overline{k}$. *Then,*

(i) $\{t_{\overline{k}}\mathsf{E}_t\}$ *is a* Q*-open neighbourhood base of* t_k,

(ii) *if* $\mathsf{E}_{\overline{k}}, \mathsf{F}_{\overline{k}} \in \mathfrak{S}_{\overline{k}}$, *then* $\exists\, \mathsf{D}_{\overline{k}} \in \mathfrak{S}_{\overline{k}}$ *such that* $\mathsf{D}_{\overline{k}} \subseteq \mathsf{E}_{\overline{k}} \cap \mathsf{F}_{\overline{k}}$,

(iii) *if* $\mathsf{E}_{\overline{k}} \in \mathfrak{S}_{\overline{k}}$, *then* $\exists\, \mathsf{F}_{\overline{k}} \in \mathfrak{S}_{\overline{k}}$ *such that* $\mathsf{F}_{\overline{k}}\mathsf{F}_{\overline{k}} \subseteq \mathsf{E}_{\overline{k}}$,

(iv) *for any* $\mathsf{E}_{\overline{k}} \in \mathfrak{S}_{\overline{k}}$ *and* $t \in \mathsf{K}$, $\exists\, \mathsf{F}_{\overline{k}} \in \mathfrak{S}_{\overline{k}}$ *such that* $t_{\overline{k}}^{-1}\mathsf{F}_{\overline{k}}t_{\overline{k}} \subseteq \mathsf{E}_{\overline{k}}$,

(v) *for any* $\mathsf{E}_{\overline{k}} \in \mathfrak{S}_{\overline{k}}$, *if* $t_f q \mathsf{E}_{\overline{k}}$, *then* $\exists\, \mathsf{F}_{\overline{f}} \in \mathfrak{S}_{\overline{f}}$ *such that* $\mathsf{E}_{\overline{k}}\mathsf{F}_{\overline{f}} \subseteq \mathsf{E}_{\overline{k}}$,

(vi) *moreover, if* $(\mathsf{K}, \mathfrak{T})$ *is fuzzy quasitopological group with* $\mathsf{E}_{\overline{k}} \in \mathfrak{S}_{\overline{k}}$, *then* $\exists\, \mathsf{F}_{\overline{k}} \in \mathfrak{S}_{\overline{k}}$ *such that* $\mathsf{F}_{\overline{k}}^{-1} \subseteq \mathsf{E}_{\overline{k}}$.

P r o o f.

(i) From $\rho_{t_{\overline{k}}\mathsf{E}_{\overline{k}}}(t)+k=\min(\overline{k}\mid \mathsf{E}_{\overline{k}}(e))+k>1$, we have $t_{\overline{k}}qt_{\overline{k}}\mathsf{E}_{\overline{k}}$. Since $(\mathsf{K},\mathfrak{T})$ is a fuzzy paratopological group, $\forall\ \mathsf{E}_{\overline{k}}\in \mathsf{Q}(\mathsf{K})$ of $e_{\overline{k}}=(t_{\overline{k}}^{-1}t_{\overline{k}})\ \exists\ \mathsf{C},\mathsf{D}\in\mathsf{Q}(\mathsf{K})$ of t_k^{-1} and t_k such that $\mathsf{CD}\subseteq \mathsf{E}_{\overline{k}}$. Suppose $\mathsf{C}_{\overline{k}}=\mathsf{C}\cap\overline{k^*}, \mathsf{D}_{\overline{k}}=\mathsf{D}\cap\overline{k^*}$, then $\mathsf{C}_{\overline{k}},\mathsf{D}_{\overline{k}}\in\mathsf{Q}(\mathsf{K})$ of t_k^{-1} and t_k respectively. Let $\rho_{\mathsf{C}_{\overline{k}}}(t^{-1})=r$, then $1-k<r\leq\overline{k}$. From $t_r^{-1}\mathsf{D}_{\overline{k}}\subseteq \mathsf{C}_{\overline{k}}\mathsf{D}_{\overline{k}}\subseteq\mathsf{CD}\subseteq\mathsf{E}_{\overline{k}}$, we have $t_{\overline{k}}t_r^{-1}\mathsf{D}_{\overline{k}}\subseteq t_{\overline{k}}\mathsf{E}_{\overline{k}}$ and so $e_r\mathsf{D}_{\overline{k}}\subseteq t_{\overline{k}}\mathsf{E}_{\overline{k}}$. Let $\mathsf{D}_{\overline{r}}=\mathsf{D}\cap r^*$, where r^* is the constant fuzzy set with fuzzy value r. Then, $\mathsf{D}_{\overline{r}}=e_r\mathsf{D}_{\overline{r}}\subseteq e_{\overline{k}}\mathsf{D}_{\overline{k}}\subseteq t_{\overline{k}}\mathsf{E}_{\overline{k}}$. Thus, $\mathsf{D}_{\overline{k}}\in\mathsf{Q}(\mathsf{K})$ of t_k. Then, we have $t_{\overline{k}}\mathsf{E}_{\overline{k}}\in\mathsf{Q}(\mathsf{K})$ of t_k and $\{t_{\overline{k}}\mathsf{E}_{\overline{k}}\}$ is a family of $\mathsf{Q}(\mathsf{K})$ of t_k. Since $(te)_k=t_k$, $\forall\,\mathsf{E}\in\mathsf{Q}(\mathsf{K})$ of t_k $\exists\,\mathsf{C}\in\mathsf{Q}(\mathsf{K})$ of t_k and $\mathsf{D}_{\overline{k}}\in\mathfrak{S}_{\overline{k}}$ of e_k such that $\mathsf{CD}_{\overline{k}}\subseteq\mathsf{E}$. Let $\mathsf{C}_{\overline{k}}=\mathsf{C}\cap\overline{k^*}$, $\rho_{\mathsf{C}_{\overline{k}}}(t)=\hat{k}$ and $\mathsf{D}_{\hat{k}}=\mathsf{D}\cap\hat{k}^*$. It is obvious that $\mathsf{D}_{\hat{k}}\in\mathsf{Q}(\mathsf{K})$ of e_k and $\mathsf{C}_{\overline{k}}\mathsf{D}_{\hat{k}}\subseteq\mathsf{C}_{\overline{k}}\mathsf{D}_{\overline{k}}\subseteq\mathsf{CD}_{\overline{k}}\subseteq\mathsf{E}$. From

$$\begin{aligned}\rho_{t_{\overline{k}}\mathsf{D}_{\hat{k}}}(s)&=\min(\overline{k},\rho_{\mathsf{D}_{\hat{k}}}(t^{-1}s))=\rho_{\mathsf{D}_{\hat{k}}}(t^{-1}s)=\sup_{s_1s_2=s}\min(\rho_{\mathsf{C}_{\overline{k}}}(s_1),\rho_{\mathsf{D}_{\hat{k}}}(s_2))\\&\geq\min(\hat{k},\rho_{\mathsf{D}_{\hat{k}}}(t^{-1}s))=\rho_{\mathsf{D}_{\hat{k}}}(t^{-1}s),\end{aligned}$$

it follows that $t_k\mathsf{D}_{\hat{k}}\subseteq\mathsf{C}_{\overline{k}}\mathsf{D}_{\hat{k}}\subseteq\mathsf{E}$. Moreover, $\mathsf{D}_{\hat{k}}\in\mathsf{Q}(\mathsf{K})$ of e_k and there must be a $\mathsf{G}_{\overline{k}}\in\mathfrak{S}_{\overline{k}}$ such that $\mathsf{G}_{\overline{k}}\subseteq\mathsf{D}_{\hat{k}}$ thus $t_{\overline{k}}\mathsf{G}_{\overline{k}}\subseteq t_{\overline{k}}\mathsf{D}_{\hat{k}}\subseteq\mathsf{E}$ which shows that $\{t_{\overline{k}}\mathsf{E}_{\overline{k}}\}\in\mathsf{Q}(\mathsf{K})$ base of t_k. Now, assume $m_cqt_{\overline{k}}\mathsf{E}_{\overline{k}}$. From $\rho_{t_{\overline{k}}\mathsf{E}_{\overline{k}}}(m)=\min(\overline{k},\rho_{\mathsf{E}_{\overline{k}}}(t^{-1}m))=\rho_{\mathsf{E}_{\overline{k}}}(t^{-1}m)>1-c$, we have that $\mathsf{E}_{\overline{k}}\in\mathsf{Q}(\mathsf{K})$ of $(t^{-1}m)_c$. Then, $\exists\,$Q-neighbourhood $\mathsf{E}_{\overline{c}}\in\mathfrak{S}_{\overline{c}}$ of e_c such that $(t^{-1}m)_c\mathsf{E}_{\overline{c}}\subseteq\mathsf{E}_{\overline{k}}$, so that $t_{\overline{k}}t_{\overline{c}}^{-1}m_{\overline{c}}\mathsf{E}_{\overline{c}}\subseteq t_{\overline{k}}\mathsf{E}_{\overline{k}}$. If $\overline{k}\geq\overline{c}$, then $m_{\overline{c}}\mathsf{E}_{\overline{c}}\subseteq t_{\overline{k}}\mathsf{E}_{\overline{k}}$. If $\overline{k}<\overline{c}$, then from $t_{\overline{c}}(t^{-1}m)_{\overline{c}}\mathsf{E}_{\overline{c}}\subseteq t_{\overline{c}}\mathsf{E}_{\overline{k}}$ we obtain $m_c\mathsf{E}_{\overline{c}}\subseteq t_{\overline{c}}\mathsf{E}_{\overline{k}}=t_{\overline{k}}\mathsf{E}_{\overline{k}}$. Since $m_{\overline{c}}\mathsf{E}_{\overline{c}}\in\mathsf{Q}(\mathsf{K})$ of m_c, it follows that $t_{\overline{k}}\mathsf{E}_{\overline{k}}\in\mathsf{Q}(\mathsf{K})$ of m_c. Hence, $t_{\overline{k}}\mathsf{E}_{\overline{k}}\in\mathsf{Q}(\mathsf{K})$.

(ii) Let $\mathsf{E}_{\overline{k}},\mathsf{F}_{\overline{k}}\in\mathfrak{S}_{\overline{k}}$. Since $\mathfrak{S}_{\overline{k}}$ is a $\mathsf{Q}(\mathsf{K})$ base of e_k, so $\rho_{\mathsf{E}_{\overline{k}}\cap\mathsf{F}_{\overline{k}}}(e)+\overline{k}=\min(\rho_{\mathsf{E}_{\overline{k}}}(e),\rho_{\mathsf{F}_{\overline{k}}}(e))+$ $+\,k>1$. Hence, $\mathsf{E}_{\overline{k}}\cap\mathsf{F}_{\overline{k}}\in\mathsf{Q}(\mathsf{K})$ of e_k and so, $\exists\ \mathsf{D}_{\overline{k}}\in\mathfrak{S}_{\overline{k}}$ such that $\mathsf{D}_{\overline{k}}\subseteq\mathsf{E}_{\overline{k}}\cap\mathsf{F}_{\overline{k}}$.

(iii) Let $\mathsf{E}_{\overline{k}}\in\mathfrak{S}_{\overline{k}}$. Using the fact $(ee)_k=e_k$ and by fuzzy continuity of multiplication, $\exists\,\mathsf{D}_{\overline{k}},\mathsf{F}_{\overline{k}}\in\mathsf{Q}(\mathsf{K})$ of e_k such that $\mathsf{D}_{\overline{k}}\mathsf{F}_{\overline{k}}\subseteq\mathsf{E}_{\overline{k}}$. Assume $\mathsf{C}_{\overline{k}}=\mathsf{D}_{\overline{k}}\cap\mathsf{F}_{\overline{k}}$, then $\mathsf{C}_{\overline{k}}\mathsf{C}_{\overline{k}}\subseteq\mathsf{D}_{\overline{k}}\mathsf{F}_{\overline{k}}\subseteq\mathsf{E}_{\overline{k}}$.

(iv) From $(t^{-1}t)_k=e_k$, $\forall\ \mathsf{E}_{\overline{k}}\in\mathfrak{S}_{\overline{k}}\ \exists\ \mathsf{C},\mathsf{D}\in\mathsf{Q}(\mathsf{K})$ of t_k^{-1} and t_k respectively such that $\mathsf{CD}\subseteq\mathsf{E}_{\overline{k}}$. We may suppose that $\mathsf{C}=t_{\overline{k}}^{-1}\mathsf{F}_{\overline{k}}$, where $\mathsf{F}_{\overline{k}}\in\mathfrak{S}_{\overline{k}}$. By (i), we have $\{\mathsf{E}_{\overline{k}}t_{\overline{k}}\mid\mathsf{E}_{\overline{k}}\in\mathfrak{S}_{\overline{k}}\}$ is a $\mathsf{Q}(\mathsf{K})$ base of t_k. We may suppose that $\mathsf{D}=\mathsf{G}_{\overline{k}}t_{\overline{k}}$, where $\mathsf{G}_{\overline{k}}\in\mathfrak{S}_{\overline{k}}$. Then, we have $e_kq\mathsf{F}_{\overline{k}}\mathsf{G}_{\overline{k}}$. Let $\min(\rho_{\mathsf{F}_{\overline{k}}}(e),\rho_{\mathsf{G}_{\overline{k}}}(e))=\hat{k}$, $\mathsf{C}_{\hat{k}}=\mathsf{G}_{\overline{k}}\cap\hat{k}$. Then, $e_{\hat{k}}\subseteq\mathsf{F}_{\overline{k}}$ and $e_{\hat{k}}\mathsf{C}_{\hat{k}}\subseteq\mathsf{F}_{\overline{k}}\mathsf{G}_{\overline{k}}$. But, $e_{\hat{k}}\mathsf{C}_{\hat{k}}=\mathsf{C}_{\hat{k}}\in\mathsf{Q}(\mathsf{K})$ of e_k so that $\mathsf{F}_{\overline{k}}\mathsf{G}_{\overline{k}}\in\mathsf{Q}(\mathsf{K})$ of e_k, therefore $\exists\ \mathsf{C}_{\overline{k}}\in\mathfrak{S}_{\overline{k}}$ such that $\mathsf{C}_{\overline{k}}\subseteq\mathsf{F}_{\overline{k}}\mathsf{G}_{\overline{k}}$. Consequently, $t_{\overline{k}}^{-1}\mathsf{C}_{\overline{k}}t_{\overline{k}}\subseteq t_{\overline{k}}^{-1}\mathsf{F}_{\overline{k}}\mathsf{G}_{\overline{k}}t_{\overline{k}}=\mathsf{CD}\subseteq\mathsf{E}_{\overline{k}}$.

(v) Let $\mathsf{E}_{\overline{k}}\in\mathfrak{S}_{\overline{k}}$. Since $\{t_f\mathsf{E}_{\overline{f}}\}\in\mathsf{Q}(\mathsf{K})$ base of t_f, then we have $t_fq\mathsf{E}_{\overline{k}}$. So $t_{\overline{f}}\mathsf{F}_{\overline{f}}\subseteq\mathsf{E}_{\overline{k}}$, where $\mathsf{F}_{\overline{f}}\in\mathfrak{S}_{\overline{f}}$.

(vi) The result holds due to fuzzy continuity of inversion. □

§ 3. Fuzzy uniformities

In this section, we characterize regular fuzzy paratopological groups by using fuzzy uniform structures.

Definition 4. [9] A fuzzy uniformity on K is a non-empty subset $\mathfrak{M}\subseteq\mathsf{I}^{\mathsf{K}\times\mathsf{K}}$ which satisfies:

(i) $m\cap n\in\mathfrak{M}$; where $m,n\in\mathfrak{M}$,

(ii) $\forall$ family $(m_\epsilon)_{\epsilon\in[0,1)}$ of elements of $\mathfrak{M}$, $\sup\limits_{\epsilon\in[0,1)}(m_\epsilon-\epsilon)\in\mathfrak{M}$,

(iii) for each $m \in \mathfrak{M}, \epsilon > 0$, $\exists n \in \mathfrak{M}$ with $n \circ n \leq m + \epsilon$ (where $m \circ n$ is defined by $(m \circ n)(a,b) = \sup_{c \in \mathsf{K}}\{m(a,c) \cap n(c,b)\}$),

(iv) if $m \in \mathfrak{M}$, then $m^{-1} \in \mathfrak{M}$.

The pair $(\mathsf{K}, \mathfrak{M})$ is called a fuzzy uniform space and members of $\mathfrak{M}$ are called fuzzy entourages. $\mathfrak{M}$ is said to be fuzzy quasi uniformity if conditions (i)–(iii) of the above definition hold.

Given two fuzzy entourages $\mathsf{M}, \mathsf{N} \in \mathsf{K} \times \mathsf{K}$. Let $\mathsf{MN} = \{(a,c) \in \mathsf{K} \times \mathsf{K} \mid \exists b \in \mathsf{K} \ni (a,b) \in \mathsf{M}$ and $(b, c \in \mathsf{N})$ be the composition and $\mathsf{M}^{-1} = \{(b,a) \mid (a,b) \in \mathsf{M}\}$ be the inverse fuzzy entourage to M. For a point $k \in \mathsf{K}$, the set $\mathsf{B}(k, \mathsf{M}) = \{n \in \mathsf{K} \mid (k,n) \in \mathsf{M}\}$ is called the M-ball centered at k and for a subset $\mathsf{N} \subseteq \mathsf{K}$, the set $\mathsf{B}(\mathsf{N}, \mathsf{M}) = \bigcup_{n \in \mathsf{N}} \mathsf{B}(n, \mathsf{M})$ is the M-neighbourhood of N. A quasi uniformity $\mathfrak{M}$ on K is normal if $\overline{\mathsf{N}} \subseteq \overline{\mathsf{B}}^{\circ}(\mathsf{N}, \mathsf{M})$ for any subset $\mathsf{N} \subseteq \mathsf{K}$ and any fuzzy entourage $\mathsf{M} \in \mathfrak{M}$. A subfamily $\mathfrak{B} \subseteq \mathfrak{M}$ is called a base of fuzzy quasi uniformity $\mathfrak{M}$ if each fuzzy entourage $\mathsf{M} \in \mathfrak{M}$ contains some fuzzy entourage $\mathsf{N} \in \mathfrak{B}$.

In any fuzzy paratopological group K, we can define two trivial fuzzy quasi uniformities:

1. Left fuzzy quasi uniformity $\mathfrak{L}$ generated by the base

$$\{(m,n) \in \mathsf{K} \times \mathsf{K} \mid n \in m\mathsf{M} \text{ and } \mathsf{M} \in \mathfrak{S}_e\},$$

2. Right fuzzy quasi uniformity $\mathfrak{R}$ generated by the base

$$\{(m,n) \in \mathsf{K} \times \mathsf{K} \mid n \in \mathsf{M}m \text{ and } \mathsf{M} \in \mathfrak{S}_e\}.$$

Proposition 5. *In a fuzzy paratopological group* K*, the fuzzy quasi uniformities* $\mathfrak{L}$ *and* $\mathfrak{R}$ *are normal.*

P r o o f. Let $\mathsf{N} \subseteq \mathsf{K}$ and $\mathsf{M} \in \mathfrak{S}_e$. We claim that $\overline{\mathsf{N}} \subseteq \overline{\mathsf{B}}^{\circ}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}})$, where $\mathfrak{L}_{\mathsf{M}} = \{(m,n) \in \mathsf{K} \times \mathsf{K} \mid n \in m\mathsf{M}\}$. By the fuzzy continuity of right translation in K, we have $\overline{\mathsf{N}}\mathsf{M} \subseteq \overline{\mathsf{N}\mathsf{M}} = \overline{\mathsf{B}}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}})$. Since the left translation is fuzzy continuous, the set $\overline{\mathsf{N}}\mathsf{M}$ is open in K and is contained in interior of $\overline{\mathsf{B}}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}})$. Thus, $\overline{\mathsf{N}} \subseteq \overline{\mathsf{N}}\mathsf{M} \subseteq \overline{\mathsf{B}}^{\circ}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}})$ and hence, left fuzzy quasi uniformity is normal. Similarly, we can prove that right fuzzy quasi uniformity is normal. □

Theorem 4. *Each regular fuzzy paratopological group is completely regular.*

P r o o f. Let $\mathfrak{L}_{\mathsf{M}} = \{(m,n) \in \mathsf{K} \times \mathsf{K} \mid n \in m\mathsf{M}\} \in \mathfrak{L}$ and $\mathfrak{R}_{\mathsf{M}} = \{(m,n) \in \mathsf{K} \times \mathsf{K} \mid n \in \mathsf{M}m\} \in$ $\in \mathfrak{R}$ be the fuzzy entourages determined by M. Define a sequence of fuzzy entourage $(\mathfrak{L}_{\mathsf{M}_n}) \in \mathfrak{L}^{\mathbb{N}}$ such that $\mathfrak{L}_{\mathsf{M}_0} \subset \mathfrak{L}_{\mathsf{M}}$ and $\mathfrak{L}_{\mathsf{M}_n}\mathfrak{L}_{\mathsf{M}_n} \subseteq \mathfrak{L}_{\mathsf{M}_{n-1}}$ for each $n \in \mathbb{N}$. Let us denote the set of binary fractions in the interval $(0,1)$ by $\mathbb{K} = \left\{\frac{a}{2^n} \mid a, n \in \mathbb{N}, 0 < a < 2^n\right\}$. Each element $k \in \mathbb{K}$ can be expressed uniquely as $k = \sum_{n=1}^{\infty} \frac{k_n}{p^n}$, where $k_n \in \{0,1\}$. Since $k > 0$, we can define $m_k = \max\{n \in \mathbb{N} \mid k_n \neq 0\}$ and so, $k = \sum_{i=1}^{m_k} \frac{k_i}{2^i}$. For each fuzzy entourage $\mathfrak{L}_{\mathsf{M}} \in \mathfrak{L}$, we put $\mathfrak{L}_{\mathsf{M}^1} = \mathfrak{L}_{\mathsf{M}}$ and $\mathfrak{L}_{\mathsf{M}^0} = \Delta_{\mathsf{K}}$, where Δ_{K} is the diagonal of K. For every $k \in \mathbb{K}$ consider the fuzzy entourage $\mathfrak{L}_{\mathsf{M}^k} = \mathfrak{L}_{\mathsf{M}_1^{k_1}} \ldots \mathfrak{L}_{\mathsf{M}_{m_k}^{k_{m_k}}} \in \mathfrak{B}$ which determines the closed neighbourhood $\overline{\mathsf{B}}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}^k})$ of N. Let $k, l \in \mathbb{K}$ with $l < k$ and $(k_n), (\mathbf{L}_n)$ be the binary sequences of k, l respectively. By the fact, $l < k$, there exists $m \in \mathbb{N}$ such that $0 = \mathbf{L}_m < k_m = 1$ and $\mathbf{L}_i = k_i$, for all $i < m$. It follows that $m_l \neq m \leq m_k$. If $m_l < m$, then by the normality of $\mathfrak{L}$, we have

$$\begin{aligned}\overline{\mathsf{B}}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}^l}) &= \overline{\mathsf{B}}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}_1^{\mathbf{L}_1}} \dots \mathfrak{L}_{\mathsf{M}_{m_l}^{\mathbf{L}_{m_l}}}) \\ &= \overline{\mathsf{B}}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}_1^{k_1}} \dots \mathfrak{L}_{\mathsf{M}_{k_l}^{k_{m_l}}}) \\ &\subseteq \overline{\mathsf{B}}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}_1^{k_1}} \dots \mathfrak{L}_{\mathsf{M}_{m-1}^{k_{m-1}}}) \\ &\subseteq \overline{\mathsf{B}}^{\circ}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}_1^{k_1}} \dots \mathfrak{L}_{\mathsf{M}_{m-1}^{k_{m-1}}} \mathfrak{L}_{\mathsf{M}_m^{k_m}}) \\ &\subseteq \overline{\mathsf{B}}^{\circ}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}_1^{k_1}} \dots \mathfrak{L}_{\mathsf{M}_{m_k}^{k_{m_k}}}) \\ &= \overline{\mathsf{B}}^{\circ}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}^k}).\end{aligned}$$

If $m < m_l$, then the inclusion $\mathfrak{L}_{\mathsf{M}_n}\mathfrak{L}_{\mathsf{M}_n} \subseteq \mathfrak{L}_{\mathsf{M}_{n-1}}$ for $m < n \leq m_l$, guarantees that $\mathfrak{L}_{\mathsf{M}_{m+1}} \dots \mathfrak{L}_{\mathsf{M}_{m_l}} \mathfrak{L}_{\mathsf{M}_{m_l+1}} \subseteq \mathfrak{L}_{\mathsf{M}_m}$ and then

$$\begin{aligned}\overline{\mathsf{B}}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}^l}) &= \overline{\mathsf{B}}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}_1^{\mathbf{L}_1}} \dots \mathfrak{L}_{\mathsf{M}_{m_l}^{\mathbf{L}_{m_l}}}) \\ &\subseteq \overline{\mathsf{B}}^{\circ}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}_1^{\mathbf{L}_1}} \dots \mathfrak{L}_{\mathsf{M}_{m_l}^{\mathbf{L}_{m_l}}} \mathfrak{L}_{\mathsf{M}_{m_l+1}}) \\ &= \overline{\mathsf{B}}^{\circ}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}_1^{\mathbf{L}_1}} \dots \mathfrak{L}_{\mathsf{M}_{m-1}^{\mathbf{L}_{m-1}}} \mathfrak{L}_{\mathsf{M}_m^0} \mathfrak{L}_{\mathsf{M}_{m+1}^{\mathbf{L}_{m+1}}} \dots \mathfrak{L}_{\mathsf{M}_{m_l}^{\mathbf{L}_{m_l}}} \mathfrak{L}_{\mathsf{M}_{m_l}+1}) \\ &\subseteq \overline{\mathsf{B}}^{\circ}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}_1^{\mathbf{L}_1}} \dots \mathfrak{L}_{\mathsf{M}_{m-1}^{\mathbf{L}_{m-1}}} \mathfrak{L}_{\mathsf{M}_m}) \\ &= \overline{\mathsf{B}}^{\circ}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}_1^{k_1}} \dots \mathfrak{L}_{\mathsf{M}_{m-1}^{k_{m-1}}} \mathfrak{L}_{\mathsf{M}_m^{k_m}}) \\ &\subseteq \overline{\mathsf{B}}^{\circ}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}_1^{k_1}} \dots \mathfrak{L}_{\mathsf{M}_{m_k}^{k_{m_k}}}) \\ &= \overline{\mathsf{B}}^{\circ}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}^k}).\end{aligned}$$

So, $\overline{\mathsf{B}}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}^l}) \subseteq \overline{\mathsf{B}}^{\circ}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}^k})$. Now, define the function $\psi_{\mathfrak{L}} \colon \mathsf{K} \mapsto [0,1]$ by $\psi_{\mathfrak{L}}(a) = \inf(\{1\} \cup$ $\cup \{l \in \mathbb{K} \mid a \in \overline{\mathsf{B}}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}^l})\})$ for $a \in \mathsf{K}$. Now, $\mathsf{N} \subset \psi_{\mathfrak{L}}^{-1}(0)$ and $\psi_{\mathfrak{L}}^{-1}([0,1)) \subset \bigcup_{l \in \mathbb{K}} \overline{\mathsf{B}}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}^l}) =$ $= \bigcup_{k \in \mathbb{K}} \overline{\mathsf{B}}^{\circ}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}^k}) \subset \overline{\mathsf{B}}^{\circ}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}_0}) \subset \overline{\mathsf{B}}^{\circ}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}})$. Let $\alpha \in (0,1)$, then by the equalities $\psi_{\mathfrak{L}}^{-1}([0,\alpha)) = \bigcup_{l \in \mathbb{K} \colon l < \alpha} \overline{\mathsf{B}^{\circ}}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}^l})$ and $\psi_{\mathfrak{L}}^{-1}((\alpha,1]) = \bigcup_{\alpha < k \in \mathbb{K}} \mathsf{K} \setminus \overline{\mathsf{B}}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}^k})$, the sets $\psi_{\mathfrak{L}}^{-1}([0,\alpha))$ and $\psi_{\mathfrak{L}}^{-1}((\alpha,1])$ are open and so, the map $\psi_{\mathfrak{L}}$ is continuous. Similarly, by using right fuzzy quasi uniformity $\mathfrak{R}$ we obtain a continuous function $\psi_{\mathfrak{R}}$ such that $\mathsf{N} \subseteq \psi_{\mathfrak{L}}^{-1}(0) \subseteq \psi_{\mathfrak{L}}^{-1}([0,1)) \subseteq$ $\subseteq \overline{\mathsf{B}}^{\circ}(\mathsf{N}, \mathfrak{L}_{\mathsf{M}}) = \overline{\mathsf{NM}}^{\circ}$ and $\mathsf{N} \subseteq \psi_{\mathfrak{R}}^{-1}(0) \subseteq \psi_{\mathfrak{R}}^{-1}([0,1)) \subseteq \overline{\mathsf{B}}^{\circ}(\mathsf{N}, \mathfrak{R}_{\mathsf{M}}) = \overline{\mathsf{MN}}^{\circ}$. Now, define $\psi = \psi_{\mathfrak{L}}.\psi_{\mathfrak{R}}$ which is continuous with $\mathsf{N} \subseteq \psi^{-1}(0) \subseteq \psi^{-1}([0,1)) \subseteq \overline{\mathsf{MN}}^{\circ} \cap \overline{\mathsf{NM}}^{\circ}$. Hence, K is completely regular. □

§ 4. Robot crash and decision making

In this section, we model a robotic crash using nano topology and fuzzy topological group. Let K be the group of actions performed by a robot via its various parts $r_1, r_2, \dots, r_n$ and $\rho(x)$ be the membership function of getting struck due to performing the action x. Here, the membership function relays on two factors; the action x and the robotic part r_i performing it. The membership value increases as the count of r_i which causes robot stuck due to performing x increases. The membership values play a vital role in the proposed model in which equivalence classes of considering robots will be based on it. Now, let $\mathfrak{T}$ be the fuzzy topology generated by the fuzzy sets A_i, where each A_i assumes non-zero values only for actions carried out by r_i and assumes 0 for actions of $r_j, i \neq j$. Then, $(\mathsf{K}, \mathfrak{T})$ is fuzzy paratopological group, fuzzy quasitopological group and fuzzy semitopological group. Thus, $(\mathsf{K}, \mathfrak{T})$ holds for all the results discussed in the above sections and we model robotic crash using this $(\mathsf{K}, \mathfrak{T})$. Here, we model a situation where a robot crashed while performing a task and the reason of this crash is yet not found. In this situation, we are to find the particular part to check, if we know the possibility of robot crash due to its each part. The algorithm of this model is given below.

Algorithm:

Step 1: Given a finite universe μ, a finite set A of attributes that is divided into two classes, S of condition attributes and D of decision attributes, an equivalence relation $\mathbf{R}$ on μ corresponding to S and a subset X of μ, represents the data as table, columns of which are labeled by attributes and rows by elements of μ. The entries of the table are attribute values. We denote the set of equivalence classes under the equivalence relation $\mathbf{R}$ as $\mathbf{R}_{\mathsf{S}}$.

Step 2: Find the lower approximation $\mathbf{L}_{\mathsf{S}}(X)$, the upper approximation$\mathbf{U}_{\mathsf{S}}(X)$, and the boundary region $B_{\mathsf{S}}(X)$ of X with respect to $\mathbf{R}_{\mathsf{S}}$.

Step 3: Generate the nano topology $\tau_{\mathsf{S}}(X)$ on μ and its basis $\beta_{\mathsf{S}}(X)$ corresponding to the conditional attribute set S.

Step 4: Remove an attribute a from S and find the lower approximation, the upper approximation and the boundary region of X with respect to the equivalence relation on $\mathsf{S} \setminus \{a\}$.

Step 5: Generate the nano topology $\tau_{\mathsf{S}\setminus\{a\}}(X)$ on μ and its basis $\beta_{\mathsf{S}\setminus\{a\}}(X)$.

Step 6: Repeat steps 3 and 4 for all attributes in S.

Step 7: Now, the core is the collection of those attributes in S for which $\beta_{\mathsf{S}\setminus\{a\}}(X) \neq \beta_{\mathsf{S}}(X)$.

Thus, by using the above algorithm, we may remove some conditional attributes and obtain the core attributes for further decision making related to robot crash. The pseudocode of the above algorithm is given below.

Algorithm 1 Pseudocode of the algorithm

Require: A finite universe μ, a set of condition attributes S, a set of decision attributes D, an equivalence relation R on μ corresponding to S, a subset X of S, the set of equivalence classes under the equivalence relation R as R_S

Calculate: Lower approximation $L_S(X)$, upper approximation $U_S(X)$ and the boundary region $B_S(X)$ of X with respect to R_S

Generate: The nano topology of X given by $\tau_S(X) = \{\emptyset, L_S(X), U_S(X), B_S(X), \mu\}$ with basis $\beta_S(X) = \{L_S(X), B_S(X), \mu\}$

```
for all a ∈ S do
    generate the nano topology τ_{S\{a}}(X) with basis β_{S\{a}}(X)
    if β_S(X) = β_{S\{a}}(X) then
        reject a
    else
        if β_S(X) ≠ β_{S\{a}}(X) then
            accept 'a' as an element of "core"
        end if
    end if
end for
```

Example 2. Let $\mu = \{R_1, R_2, \ldots, R_7\}$ be the collection of robots and S be the set of condition attributes that robot crashed due to its five parts viz. manipulator, endeffector, locomotive device, controller and the sensors, together with decision attribute 's' that the robot crashed or not.

Robots	Manipulator	Endeffector	Locomotive device	Controller	Sensors	s
R_1	0.2	0.1	0.2	0.1	0.1	Yes
R_2	0.1	0.1	0.2	0.2	0.2	No
R_3	0.2	0.2	0.2	0.1	0.1	No
R_4	0.1	0.3	0.2	0.2	0.1	Yes
R_5	0.2	0.1	0.2	0.1	0.1	Yes
R_6	0.2	0.4	0.1	0.3	0.1	Yes
R_7	0.1	0.3	0.1	0.2	0.2	No

<u>**Case 1:**</u> Let $X = \{R_1, R_4, R_5, R_6\}$ be the set of robots which crashed. Then, the set of equivalence classes under the relation coincidence is given by $\mathbf{R}_{\mathsf{S}} = \{\{R_1, R_5\}, \{R_2\}, \{R_3\}, \{R_4\}, \{R_6\}, \{R_7\}\}$. Now, the upper approximation, the lower approximation, the boundary region and the basis of nano topology are $\mathbf{U}_{\mathsf{S}}(X) = \{R_1, R_4, R_5, R_6\} = \mathbf{L}_{\mathsf{S}}(X), B_{\mathsf{S}}(X) = \emptyset$ and $\beta_{\mathsf{S}}(X) = \{\emptyset, \{R_1, R_4, R_5, R_6\}, \mu\}$ respectively.

Step 1:

<table>
<tr><th>Removing attribute a</th><th>$\mathbf{R}_{\mathsf{S}\setminus\{a\}}$</th><th>$\mathbf{U}_{\mathsf{S}\setminus\{a\}}(X)$</th><th>$\mathbf{L}_{\mathsf{S}\setminus\{a\}}(X)$</th><th>$B_{\mathsf{S}\setminus\{a\}}(X)$</th><th>$\beta_{\mathsf{S}\setminus\{a\}}(X)$</th></tr>
<tr><td>endeffector</td><td>$\{\{R_1, R_3, R_5\}, \{R_2\}, \{R_4\}, \{R_6\}, \{R_7\}\}$</td><td>$\{R_1, R_3, R_4, R_5, R_6\}$</td><td>$\{R_4, R_6\}$</td><td>$\{R_1, R_3, R_5\}$</td><td>$\{\{R_4, R_6\}, \{R_1, R_3, R_5\}, \mu\}$</td></tr>
<tr><td>manipulator</td><td rowspan="4">$\{\{R_1, R_5\}, \{R_2\}, \{R_3\}, \{R_4\}, \{R_6\}, \{R_7\}\}$</td><td rowspan="4">$\{R_1, R_4, R_5, R_6\}$</td><td rowspan="4">$\{R_1, R_4, R_5, R_6\}$</td><td rowspan="4">$\emptyset$</td><td rowspan="4">$\{\{R_1, R_4, R_5, R_6\}, \emptyset, \mu\}$</td></tr>
<tr><td>locomotive device</td></tr>
<tr><td>controller</td></tr>
<tr><td>sensors</td></tr>
</table>

Therefore, $\beta_{\mathsf{S}}(X) \neq \beta_{\mathsf{S}\setminus\{\text{endeffector}\}}(X)$ and $\beta_{\mathsf{S}}(X) = \beta_{\mathsf{S}\setminus\{a\}}(X)$, where a is other than endeffector. Thus, the attributes manipulator, locomotive device, controller, and sensors are omitted. Thus, the only optimal core attribute is endeffector. Hence, endeffector is the core attribute; which causes the robots crashed.

<u>**Case 2:**</u> Let $X = \{R_2, R_3, R_7\}$ be the set of robots which did not crash. Then, the upper approximation, the lower approximation, the boundary region and the basis of nano topology are $\mathbf{U}_{\mathsf{S}}(X) = \{R_2, R_3, R_7\} = \mathbf{L}_{\mathsf{S}}(X), B_{\mathsf{S}}(X) = \emptyset$ and $\beta_{\mathsf{S}}(X) = \{\mu, \{R_2, R_3, R_7\}, \emptyset\}$ respectively.

Step 1:

<table>
<tr><th>Removing attribute a</th><th>$\mathbf{R}_{\mathsf{S}\setminus\{a\}}$</th><th>$\mathbf{U}_{\mathsf{S}\setminus\{a\}}(X)$</th><th>$\mathbf{L}_{\mathsf{S}\setminus\{a\}}(X)$</th><th>$B_{\mathsf{S}\setminus\{a\}}(X)$</th><th>$\beta_{\mathsf{S}\setminus\{a\}}(X)$</th></tr>
<tr><td>endeffector</td><td>$\{\{R_1, R_3, R_5\}, \{R_2\}, \{R_4\}, \{R_6\}, \{R_7\}\}$</td><td>$\{R_1, R_2, R_3, R_5, R_7\}$</td><td>$\{R_2, R_7\}$</td><td>$\{R_1, R_3, R_5\}$</td><td>$\{\{R_2, R_7\}, \{R_1, R_3, R_5\}, \mu\}$</td></tr>
<tr><td>manipulator</td><td rowspan="4">$\{\{R_1, R_5\}, \{R_2\}, \{R_3\}, \{R_4\}, \{R_6\}, \{R_7\}\}$</td><td rowspan="4">$\{R_2, R_3, R_7\}$</td><td rowspan="4">$\{R_2, R_3, R_7\}$</td><td rowspan="4">$\emptyset$</td><td rowspan="4">$\{\{R_2, R_3, R_7\}, \emptyset, \mu\}$</td></tr>
<tr><td>locomotive device</td></tr>
<tr><td>controller</td></tr>
<tr><td>sensors</td></tr>
</table>

Therefore, $\beta_{\mathsf{S}}(X) \neq \beta_{\mathsf{S}\setminus\{\text{endeffector}\}}(X)$ and $\beta_{\mathsf{S}}(X) = \beta_{\mathsf{S}\setminus\{a\}}(X)$, where a is other than endeffector. Thus, the attributes manipulator, locomotive device, controller, and sensors are omitted and the only optimal core attribute is endeffector. Hence, endeffector is the core attribute which keeps the robots being not crashed. In both cases, either a crash is happened or not we have to monitor endeffectors optimally. Suppose, if the core attributes for both cases varies, say a_i for Case 1 and b_i for Case 2, then we have to focus on attributes b_i until a crash is happened and we have to verify the attributes a_i if the robots crashed.

§ 5. Robot crash with missing data

In the model of the above section, the core attribute is endeffector in both the cases. Is it always same for any sort of data? What we have to do if some possibilities are missing? Let the possibility data as given below.

Robots	*Manipulator*	*Endeffector*	*Locomotive device*	*Controller*	*Sensors*	s
R_1	0.2	0.1	$*$	0.1	$*$	Yes
R_2	0.1	0.1	$*$	$*$	0.2	No
R_3	$*$	0.2	0.2	0.1	$*$	No
R_4	0.1	$*$	0.2	$*$	0.1	Yes
R_5	$*$	0.1	$*$	0.1	$*$	Yes
R_6	0.2	$*$	0.1	$*$	0.1	Yes
R_7	$*$	$*$	0.1	0.2	0.2	No

In this type of issue, equivalence classes are reduced into tolerance classes, and by finding the upper approximation, the lower approximation and the boundary region as in the above model we have two cases to deal with.

<u>**Case 1:**</u> Let $X = \{R_1, R_4, R_5, R_6\}$ be the set of robots which crashed. Then, the set of tolerance classes with respect to the tolerance relation coincidence are given by

$$\mathbf{R}_{\mathsf{S}} = \{\{R_1, R_5, R_6\}, \{R_2, R_5\}, \{R_2, R_7\}, \{R_3, R_4\}, \{R_4, R_5\}\}.$$

Now, the upper approximation, the lower approximation, the boundary region, and the basis of nano topology are

$$\mathbf{U}_{\mathsf{S}}(X) = \{R_1, R_2, R_3, R_4, R_5, R_6, R_7\}, \qquad B_{\mathsf{S}}(X) = \{R_2, R_3, R_7\},$$
$$\mathbf{L}_{\mathsf{S}}(X) = \{R_1, R_4, R_5, R_6\}, \qquad \beta_{\mathsf{S}}(X) = \{\{R_1, R_4, R_5, R_6\}, \{R_2, R_3, R_7\}, \mu\}$$

respectively.

Step 1:

<table>
<tr><th>Removing attribute a</th><th>$\mathbf{R}_{\mathsf{S}\setminus\{a\}}$</th><th>$\mathbf{U}_{\mathsf{S}\setminus\{a\}}(X)$</th><th>$\mathbf{L}_{\mathsf{S}\setminus\{a\}}(X)$</th><th>$B_{\mathsf{S}\setminus\{a\}}(X)$</th><th>$\beta_{\mathsf{S}\setminus\{a\}}(X)$</th></tr>
<tr><td>manipulator</td><td>$\{\{R_1, R_2, R_5\}, \{R_3, R_4\}, \{R_1, R_5, R_6\}, \{R_2, R_7\}, \{R_1, R_4, R_5\}\}$</td><td rowspan="3">$\{R_1, R_2, R_3, R_4, R_5, R_6\}$</td><td rowspan="2">$\{R_1, R_4, R_5, R_6\}$</td><td rowspan="2">$\{R_2, R_3\}$</td><td rowspan="2">$\{\{R_1, R_4, R_5, R_6\}, \{R_2, R_3\}, \mu\}$</td></tr>
<tr><td>locomotive device</td><td>$\{\{R_1, R_5, R_6\}, \{R_3, R_6\}, \{R_3, R_4\}, \{R_4, R_5\} \{R_2, R_5\}, \{R_2, R_7\}\}$</td></tr>
<tr><td>endeffector</td><td>$\{\{R_1, R_5, R_6\}, \{R_1, R_3, R_5\} \{R_3, R_4, R_5\}, \{R_2, R_3, R_5\} \{R_2, R_7\}\}$</td><td>$\{R_1, R_5, R_6\}$</td><td>$\{R_2, R_3, R_4\}$</td><td>$\{\{R_2, R_3, R_4\} \{R_1, R_5, R_6\}, \mu\}$</td></tr>
<tr><td>controller</td><td>$\{\{R_1, R_5, R_6\}, \{R_1, R_5, R_7\} \{R_2, R_5, R_7\}, \{R_3, R_4\} \{R_4, R_5\}\}$</td><td rowspan="2">$\{R_1, R_2, R_3, R_4, R_5, R_6, R_7\}$</td><td>$\{R_1, R_4, R_5, R_6\}$</td><td>$\{R_2, R_3, R_7\}$</td><td>$\{\{R_1, R_4, R_5, R_6\} \{R_2, R_3, R_7\}, \mu\}$</td></tr>
<tr><td>sensors</td><td>$\{\{R_1, R_5, R_6\}, \{R_2, R_4, R_5\} \{R_2, R_7\}, \{R_6, R_7\} \{R_3, R_4\}\}$</td><td>$\{R_1, R_5, R_6\}$</td><td>$\{R_2, R_3, R_4, R_7\}$</td><td>$\{\{R_1, R_5, R_6\} \{R_2, R_3, R_4, R_7\}, \mu\}$</td></tr>
</table>

Since $\beta_{\mathsf{S}\setminus\{a\}}(X) = \beta_{\mathsf{S}}(X)$ for $a \in$ {manipulator, locomotive device}, thus the attributes manipulator and locomotive device are omitted.

Step 2: Let $M =$ {endeffector, controller, sensors}, then

$$\beta_M(X) = \{\{R_1, R_4, R_5, R_6\}, \{R_2, R_3\}, \mu\}.$$

Removing attribute a	$\mathbf{R}_{M\setminus\{a\}}$	$\mathbf{U}_{M\setminus\{a\}}(X)$	$\mathbf{L}_{M\setminus\{a\}}(X)$	$B_{M\setminus\{a\}}(X)$	$\beta_{M\setminus\{x\}}(X)$
endeffector	$\{\{R_1, R_3, R_4, R_5, R_6\}$ $\{R_1, R_2, R_3, R_5\}, \{R_2, R_7\}\}$	$\{R_1, R_2, R_3, R_4, R_5, R_6\}$	$\emptyset$	$\{R_1, R_2, R_3, R_4, R_5, R_6\}$	$\{\emptyset, \{R_1, R_2, R_3, R_4, R_5, R_6\}, \mu\}$
sensors	$\{\{R_1, R_2, R_4, R_5, R_6\}$ $\{R_2, R_4, R_6, R_7\}$ $\{R_3, R_4, R_6\}\}$	$\{R_1, R_2, R_3, R_4, R_5, R_6, R_7\}$		$\{R_1, R_2, R_3, R_4, R_5, R_6, R_7\}$	$\{\emptyset, \{R_1, R_2, R_3, R_4, R_5, R_6, R_7\}, \mu\}$
controller	$\{\{R_1, R_4, R_5, R_6\}, \{R_3, R_7\},$ $\{R_1, R_2, R_5, R_7\}, \{R_3, R_4, R_6\}\}$		$\{R_1, R_4, R_5, R_6\}$	$\{R_2, R_3, R_7\}$	$\{\{R_1, R_4, R_5, R_6\}, \{R_2, R_3, R_7\}, \mu\}$

Since, $\beta_M(X) \neq \beta_{M\setminus\{a\}}(X)$, $\forall\, a \in M$, where $M =$ {endeffector, controller, sensors}. Thus, endeffectors, controllers, sensors are core attributes.

<u>**Case 2:**</u> Let $X = \{R_2, R_3, R_7\}$ be the set of robots which did not crash. Then, the upper approximation, the lower approximation, the boundary region, and the basis of nano topology are

$$\mathbf{U}_{\mathsf{S}}(X) = \{R_2, R_3, R_4, R_5, R_7\}, \qquad B_{\mathsf{S}}(X) = \{R_2, R_3, R_4, R_5, R_7\},$$
$$\mathbf{L}_{\mathsf{S}}(X) = \emptyset, \qquad \beta_{\mathsf{S}}(X) = \{\emptyset, \{R_2, R_3, R_4, R_5, R_7\}, \mu\}$$

respectively.

Step 1:

Removing attribute a	$\mathbf{R}_{\mathsf{S}\setminus\{a\}}$	$\mathbf{U}_{\mathsf{S}\setminus\{a\}}(X)$	$\mathbf{L}_{\mathsf{S}\setminus\{a\}}(X)$	$B_{\mathsf{S}\setminus\{a\}}(X)$	$\beta_{\mathsf{S}\setminus\{a\}}(X)$
manipulator	$\{\{R_1, R_2, R_5\}, \{R_3, R_4\},$ $\{R_1, R_5, R_6\}, \{R_2, R_7\},$ $\{R_1, R_4, R_5\}\}$	$\{R_1, R_2, R_3, R_4, R_5, R_7\}$	$\{R_2, R_7\}$	$\{R_1, R_3, R_4, R_5\}$	$\{\{R_1, R_3, R_4, R_5\}, \{R_2, R_7\}, \mu\}$
endeffector	$\{\{R_1, R_5, R_6\}, \{R_1, R_3, R_5\}$ $\{R_3, R_4, R_5\}, \{R_2, R_3, R_5\}$ $\{R_2, R_7\}\}$				
locomotive device	$\{\{R_1, R_5, R_6\}, \{R_3, R_6\},$ $\{R_3, R_4\}, \{R_4, R_5\}$ $\{R_2, R_5\}, \{R_2, R_7\}\}$	$\{R_2, R_3, R_4, R_5, R_6, R_7\}$		$\{R_3, R_4, R_5, R_6\}$	$\{\{R_3, R_4, R_5, R_6\}, \{R_2, R_7\}, \mu\}$
sensors	$\{\{R_1, R_5, R_6\}, \{R_2, R_4, R_5\}$ $\{R_2, R_7\}, \{R_6, R_7\}$ $\{R_3, R_4\}\}$			$\{R_4, R_5, R_6, R_7\}$	$\{\{R_4, R_5, R_6, R_7\}$ $\{R_2, R_7\}, \mu\}$
controller	$\{\{R_1, R_5, R_6\}, \{R_1, R_5, R_7\}$ $\{R_2, R_5, R_7\}, \{R_3, R_4\}$ $\{R_4, R_5\}\}$	$\{R_1, R_2, R_3, R_4, R_5, R_7\}$	$\emptyset$	$\{R_1, R_2, R_3, R_4, R_5, R_7\}$	$\{\emptyset, \{R_1, R_2, R_3, R_4, R_5, R_7\}, \mu\}$

Thus, $\beta_{\mathsf{S}}(X) \neq \beta_{\mathsf{S}\setminus\{a\}}(X)$, $\forall\, a \in \mathsf{S}$ and hence, all conditional attributes are the core attributes. Hence, we obtain core attributes for further decision making. Thus, it can be seen that ideas of fuzzy paratopological groups, fuzzy quasitopological groups and fuzzy semitopological groups along with nano topology can be used as one of the tools in decision making and modelling regarding robot crash.

Conclusion

We generalized the concept of fuzzy topological group to fuzzy semitopological group, fuzzy paratopological group and fuzzy quasitopological group in terms of Q-open neighbourhoods and characterize their properties. We discussed some results related to them and proved that each regular fuzzy paratopological group is completely regular. As an application of our notions and results, we provided a situation of robot crash and related decision making. We hope that our article may find its due importance in future.

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О нечеткой паратопологической группе и принятии решений при аварии робота

Ключевые слова: нечеткая квазитопологическая группа, нечеткая паратопологическая группа, нечеткая квазиравномерность, робототехника, принятие решений.

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В этой статье мы вводим нечеткую паратопологическую группу, нечеткую полутопологическую группу и нечеткую квазитопологическую группу, приводим примеры и свойства. Эти новые понятия относятся к нечеткой топологической группе. С помощью нечетких однородностей доказано, что каждая нечеткая регулярная паратопологическая группа полностью регулярна. Помимо этого, мы доказываем некоторые результаты, связанные с нечеткой полутопологической группой и нечеткой квазитопологической группой. Кроме того, используя приведенные нами понятия и нанотопологию, мы представляем приложение в области принятия решений во время аварии робота.

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