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SPECTRAL PROPERTIES AND NON-HERMITIAN SKIN EFFECT IN THE HATANO–NELSON MODEL

At present, non-Hermitian topological systems continue to be actively studed. In a rigorous approach, we study one of the key non-Hermitian systems — the Hatano–Nelson model H. We find the Green function for this Hamiltonian. Using the Green function, we analytically obtain the eigenvalues and eigenfunctions of H for finite and semi-infinite chains, as well as for an infinite chain with a local potential. We discuss the non-Hermitian skin effect for the models mentioned above. We also describe the boundary between localized and resonant eigenfunctions (for the zero spectral parameter, this is the boundary between non-Hermitian topological phases).

Keywords: Hatano–Nelson model, eigenvalues, eigenfunctions, non-Hermitian skin effect.

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Introduction

At present, non-Hermitian topological models continue to be actively investigated in physical literature [1–5]. Unlike isolated quantum systems, non-Hermitian Hamiltonians provide an effective physical description of open quantum systems, including atoms and photons systems [6, 7]. In this case, one-dimensional lattice models are most often considered, in particular, the Hatano–Nelson (HN) model [1, 4, 5, 8].

The characteristic feature of topological systems, such as topological insulators and superconductors, described by Hermitian Hamiltonians, is the bulk-boundary correspondence [9, 10]. This means that the number of zero-energy states, localized at the edges of a finite Hermitian system, is determined by the topological invariant related to the bulk of the system. In the non-Hermitian case, the bulk-boundary correspondence may be violated, which is associated with the non-Hermitian skin effect (NHSE) [1,2,4,5]. For the one-dimensional open system this means that instead of two edge states, as in the Hermitian case, there will be one, but after the transposition of the Hamiltonian matrix, another state appears at the opposite edge [2–4, 11].

There are interesting mathematical questions in this area of research that have not yet attracted the attention of mathematicians. In the paper, in the rigorous approach, we investigate the HN Hamiltonian H. For the first time, we found the analytically expressions for the Green function of this Hamiltonian for the infinite chain. Using the Green function and the potential that breaks connections between sites, we analytically obtain the eigenvalues and eigenfunctions of H for finite and semi-infinite chains (cf. the similar approach [12]). At the singular point, the Green function does not exist, so we find the eigenfunctions at this point using the matrix approach. We also discuss the features of the NHSE for the models mentioned above. In addition, we describe the boundary between localized and resonant eigenfunctions in the parameter space (for the zero energy this is the boundary between non-Hermitian topological phases).

§ 1. Dyson equation for various HN chains

1.1. Dyson equation

The infinite lattice model HN is determined by the Hamiltonian H which acts on functions according the formula

$$(H\psi)(n) = (t-\gamma)\psi(n-1) + (t+\gamma)\psi(n+1), \tag{1.1}$$

$n = 0, \pm 1, \ldots$; where $t \pm \gamma$ are the real asymmetric hopping amplitudes (see Fig. 1). In this section, we suppose that $t \neq \pm\gamma$, i. e., a particle can move in two directions. Further, we assume that $t > 0$.

We will look for eigenvalues and eigenfunctions of the Hamiltonian H using the homogeneous Dyson equation

$$\psi = -(H-E)^{-1}V\psi, \tag{1.2}$$

where E is the energy, V is some local potential (see below). The equation (1.2) comes from the equation $(H+V)\psi = E\psi$. Let's write the equation (1.2) using the Green function (A.6) (see Appendix A) of the Hamiltonian H in the form

$$\begin{aligned}\psi(n) = -\frac{1}{2(t+\gamma)e^{ip_+} - E} \times \\ \times \sum_{n'} \Big(e^{ip_+|n-n'|)}\theta(n-n') + e^{ip_-|n-n'|}\big(1-\theta(n-n')\big)\Big)(V\psi)(n'),\end{aligned} \tag{1.3}$$

where

$$\theta(n) = \begin{cases} 1, & n \geqslant 0, \\ 0, & n < 0, \end{cases}$$

and the functions e^{ip_+}, e^{ip_-} are defined by equalities (A.3), (A.4). In this case, the condition (A.7) must be satisfied.

1.2. Finite chain

To form the finite HN chain, we use the potential V acting on functions according to the equality

$$(V\psi)(n) = -\big((t-\gamma)\psi(0)\delta_{n,1} + (t+\gamma)\psi(1)\delta_{n,0} + (t-\gamma)\psi(N)\delta_{n,N+1} + (t+\gamma)\psi(N+1)\delta_{n,N}\big). \tag{1.4}$$

This potential destroys links between sites with numbers 0 and 1, and also with numbers N and $N+1$ in the infinite HN chain.

The equation (1.3) with the potential (1.4) has the form

$$\begin{aligned}\psi(n) = \frac{1}{2(t+\gamma)e^{ip_+} - E}\Big((t-\gamma)\psi(0)\theta(n-1)e^{ip_+|n-1|} + (t+\gamma)\psi(1)\theta(n)e^{ip_+|n|} + \\ + (t-\gamma)\psi(N)\theta(n-N-1)e^{ip_+|n-N-1|} + (t+\gamma)\psi(N+1)\theta(n-N)e^{ip_+|n-N|} + \\ + (t-\gamma)\psi(0)\big(1-\theta(n-1)\big)e^{ip_-|n-1|} + (t+\gamma)\psi(1)\big(1-\theta(n)\big)e^{ip_-|n|} + \\ + (t-\gamma)\psi(N)\big(1-\theta(n-N-1)\big)e^{ip_-|n-N-1|} + (t+\gamma)\psi(N+1)\big(1-\theta(n-N)\big)e^{ip_-|n-N|}\Big).\end{aligned} \tag{1.5}$$

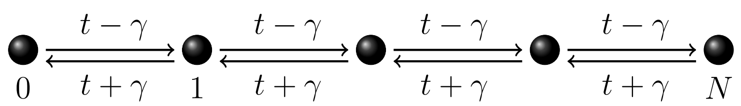

Fig. 1. Model HN. Sites of the chain and hopping amplitudes are shown

By taking $n = 0, 1, N, N+1$ in (1.5), with the help of (A.2) and (A.5), we get the matrix equation, whose solutions will allow us to find from (1.5) eigenfunctions of the Hamiltonian $H + V$:

$$\begin{pmatrix} (t+\gamma)e^{-ip_-} & t+\gamma & (t-\gamma)e^{ip_-(N+1)} & (t+\gamma)e^{ip_-N} \\ t-\gamma & (t+\gamma)e^{-ip_-} & (t-\gamma)e^{ip_-N} & (t+\gamma)e^{ip_-(N-1)} \\ (t-\gamma)e^{ip_+(N-1)} & (t+\gamma)e^{ip_+N} & (t-\gamma)e^{-ip_+} & t+\gamma \\ (t-\gamma)e^{ip_+N} & (t+\gamma)e^{ip_+(N+1)} & t-\gamma & (t-\gamma)e^{-ip_+} \end{pmatrix} \times \tag{1.6}$$
$$\times \big(\psi(0), \psi(1), \psi(N), \psi(N+1)\big)^T = 0.$$

Using (A.2), (A.3), and (A.4), we rewrite (1.6) as

$$\begin{pmatrix} \sqrt{E^2-4(t^2-\gamma^2)} & 0 & 0 & 0 \\ t-\gamma & (t+\gamma)e^{-ip_-} & (t-\gamma)e^{ip_-N} & (t+\gamma)e^{ip_-(N-1)} \\ (t-\gamma)e^{ip_+(N-1)} & (t+\gamma)e^{ip_+N} & (t-\gamma)e^{-ip_+} & t+\gamma \\ 0 & 0 & 0 & \sqrt{E^2-4(t^2-\gamma^2)} \end{pmatrix} \times \tag{1.7}$$
$$\times \big(\psi(0), \psi(1), \psi(N), \psi(N+1)\big)^T = 0.$$

Hence $\psi(0) = 0$, $\psi(N+1) = 0$. The matrix determinant in (1.7) is equal to zero if

$$(t^2-\gamma^2)(e^{-i(p_++p_-)} - e^{i(p_++p_-)N}) = 0,$$

or equivalently (see (A.3) and (A.4)),

$$e^{i(p_++p_-)(N+1)} = \left(\frac{(E \pm \sqrt{E^2-4(t^2-\gamma^2)})^2}{4(t^2-\gamma^2)}\right)^{N+1} = \left(w \pm \sqrt{w^2-1}\right)^{2(N+1)} = 1, \tag{1.8}$$

where $w = E/(2\sqrt{t^2-\gamma^2})$. The Zhukovsky function $w = \frac{1}{2}(z+1/z)$ is inverse to the two-valued function $z = w \pm \sqrt{w^2-1}$. Rewrite (1.8) as

$$z = w \pm \sqrt{w^2-1} = 1^{\frac{1}{2(N+1)}}.$$

Therefore, E takes the values

$$E_m = \sqrt{t^2-\gamma^2}\left(1^{\frac{1}{2(N+1)}} + 1^{-\frac{1}{2(N+1)}}\right) = 2\sqrt{t^2-\gamma^2}\cos\frac{\pi m}{N+1}, \quad m = 0, 1, \dots, N+1. \tag{1.9}$$

Thus, there are $N+2$ values of E, for which (1.6) has a nonzero solution. If N is odd, then for $m = (N+1)/2$ we have zero value $E = 0$. If $t^2 - \gamma^2 > 0$, then all values of E are real, and if $t^2 - \gamma^2 < 0$, then all values of E except $E = 0$ (in the case of odd N) are complex.

Note that $E = \pm 2\sqrt{t^2-\gamma^2}$ for $m = 0, N+1$, but then condition (A.7) is not satisfied. Therefore, further we consider $m = 1, \dots, N$.

Substituting into (A.3) and (A.4) the values of E from (1.9), we find

$$e^{ip_+} = \sqrt{\frac{t-\gamma}{t+\gamma}}e^{\pm\frac{i\pi m}{N+1}}, \quad e^{ip_-} = \sqrt{\frac{t+\gamma}{t-\gamma}}e^{\pm\frac{i\pi m}{N+1}}, \tag{1.10}$$

where $m = 1, \dots, N$. We choose the normalization, assuming that $\psi(N) = 1$. Then, since $\psi(0) = \psi(N+1) = 0$, from (1.7) and (1.10), we get

$$\psi(1) = (-1)^{m+1} \left(\frac{t+\gamma}{t-\gamma} \right)^{(N-1)/2}, \tag{1.11}$$

$m = 1, \dots, N$. Using (1.5), taking into account (1.10), (1.11) and equality

$$2(t+\gamma)e^{ip_+} - E = \pm 2i\sqrt{t^2-\gamma^2} \sin\left(\pi m/(N+1)\right),$$

we find eigenfunctions for the considered finite chain, which have the form

$$\psi_m(n) = \frac{(-1)^{m+1} \sin\frac{\pi m n}{N+1}}{2\sin\frac{\pi m}{N+1}} \left(\frac{t+\gamma}{t-\gamma} \right)^{\frac{N-n}{2}}, \quad m = 1, \dots, N. \tag{1.12}$$

For odd N and $m = (N+1)/2$, we obtain the eigenfunction for $E = 0$

$$\psi_{\frac{N+1}{2}}(n) = \frac{1}{2}(-1)^{\frac{N-1}{2}} \sin\frac{\pi n}{2} \left(\frac{t+\gamma}{t-\gamma} \right)^{\frac{N-n}{2}}. \tag{1.13}$$

Let us study the thermodynamic limit [5] of the normalized eigenfunction (1.13) corresponding to $E = 0$, as $N \to \infty$. Since N is odd, we have

$$\left\| \psi_{\frac{N+1}{2}} \right\|^2 = C(1 - a^{N+1}), \tag{1.14}$$

where $a = |(t+\gamma)/(t-\gamma)|$, $C = 1/(4(1-a^2))$.

Let $a < 1$, then, by (1.14), $\|\psi_{\frac{N+1}{2}}(n)\|^2 \to C$ as $N \to \infty$. Therefore, according to (1.13),

$$\frac{\left|\psi_{\frac{N+1}{2}}(n)\right|^2}{\|\psi_{\frac{N+1}{2}}\|^2} \to 0, \quad N \to \infty, \quad n = 1, 2, \dots.$$

This means that the normalized function $\psi_{\frac{N+1}{2}}(n)$ is equal to zero in the limit $N \to \infty$, i. e., the state disappears.

For $a > 1$, from (1.13) and (1.14), we get, for arbitrary $n \geqslant 1$,

$$\frac{\left|\psi_{\frac{N+1}{2}}(n)\right|^2}{\|\psi_{\frac{N+1}{2}}\|^2} = \frac{C_1 a^{N-n}}{a^{N+1} - 1} \to C_1 a^{-(n+1)}, \quad N \to \infty,$$

where $C_1 = (a^2-1)(1-(-1)^n)/2$. Thus, in the limit, we obtain an exponentially decreasing function. Note that it is not an eigenfunction of the Hamiltonian H in the case of a semi-infinite chain (see below).

1.3. Semi-infinite chain

To obtain a semi-infinite chain with site numbers $n = 1, 2, \dots$, we leave only the terms containing $\delta_{n,0}$ and $\delta_{n,1}$ in the expression for the potential (1.4). Thus, now the potential breaks the links between sites with numbers $n = 0$ and $n = 1$. Instead of (1.5), we get the following form of the eigenfunction:

$$\begin{aligned}\psi(n) = \frac{1}{2(t+\gamma)e^{ip_+} - E} \Big(&(t-\gamma)\psi(0)\theta(n-1)e^{ip_+|n-1|} + (t+\gamma)\psi(1)\theta(n)e^{ip_+|n|} + \\ &+ (t-\gamma)\psi(0)\left(1-\theta(n-1)\right)e^{ip_-|n-1|} + (t+\gamma)\psi(1)\left(1-\theta(n)\right)e^{ip_-|n|} \Big).\end{aligned} \tag{1.15}$$

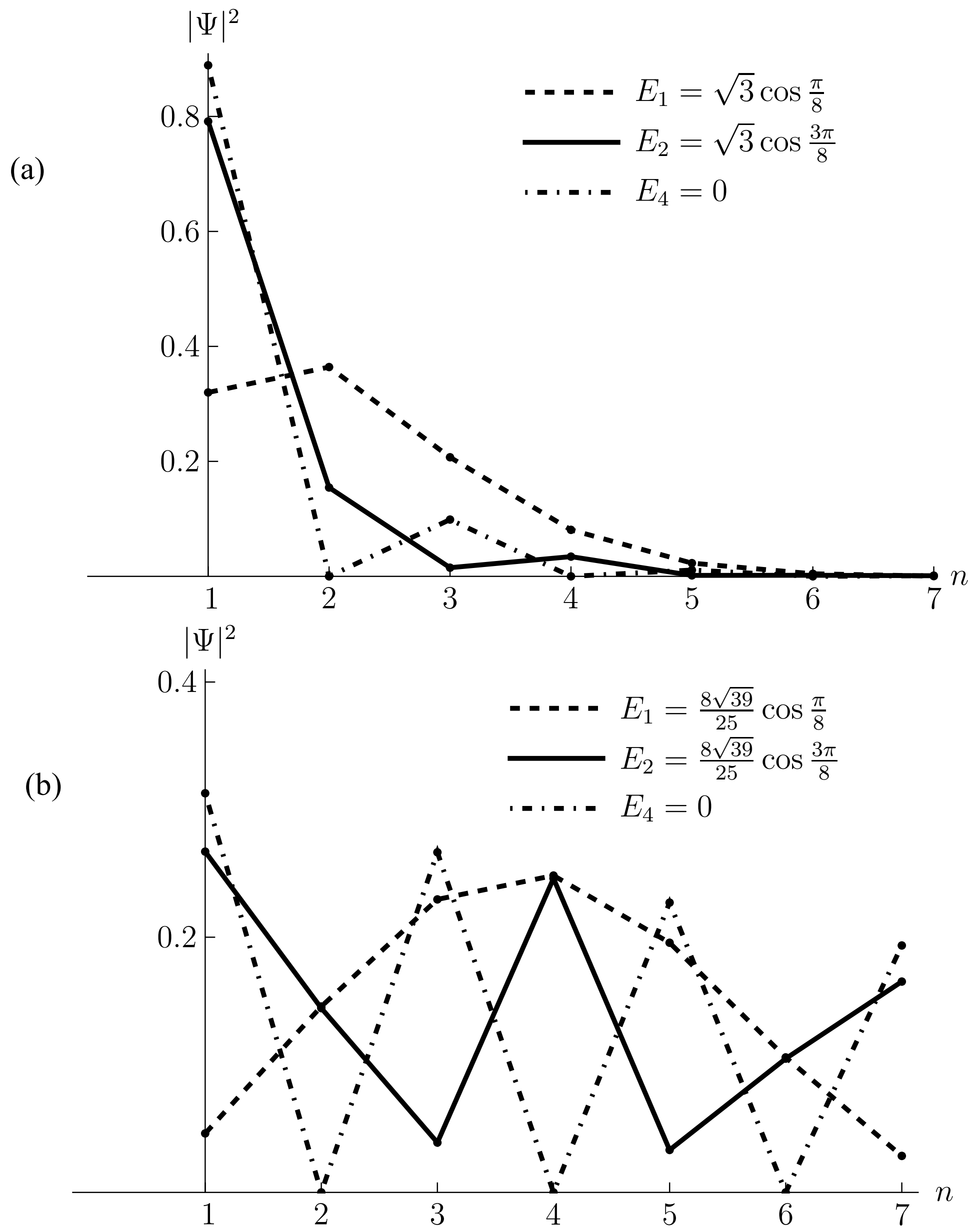

Fig. 2. Normalized eigenfunctions for $N = 7, m = 1, 3, 4$ (see (1.12)). (a) Here $t = 1, \gamma = 0.5$. The states are localized near the left boundary (see also the next figure). (b) Here $t = 1, \gamma = 0.04$. The states are delocalized

Substituting $n = 0$ and $n = 1$ into (1.15), we obtain, as above (cf. (1.6)), the matrix equation

$$\begin{pmatrix} (t+\gamma)e^{-ip_-} & t+\gamma \\ t-\gamma & (t+\gamma)e^{-ip_-} \end{pmatrix} \begin{pmatrix} \psi(0) \\ \psi(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1.16}$$

The matrix determinant in (1.16) is zero if

$$e^{ip_-} = \pm\sqrt{\frac{t+\gamma}{t-\gamma}}. \tag{1.17}$$

Then, from (1.17), (A.2), and (A.5), we obtain

$$E = \pm 2\sqrt{t^2 - \gamma^2}.$$

This contradicts condition (A.7). Thus, within the framework of the Dyson equation, there are no states in the semi-infinite chain.

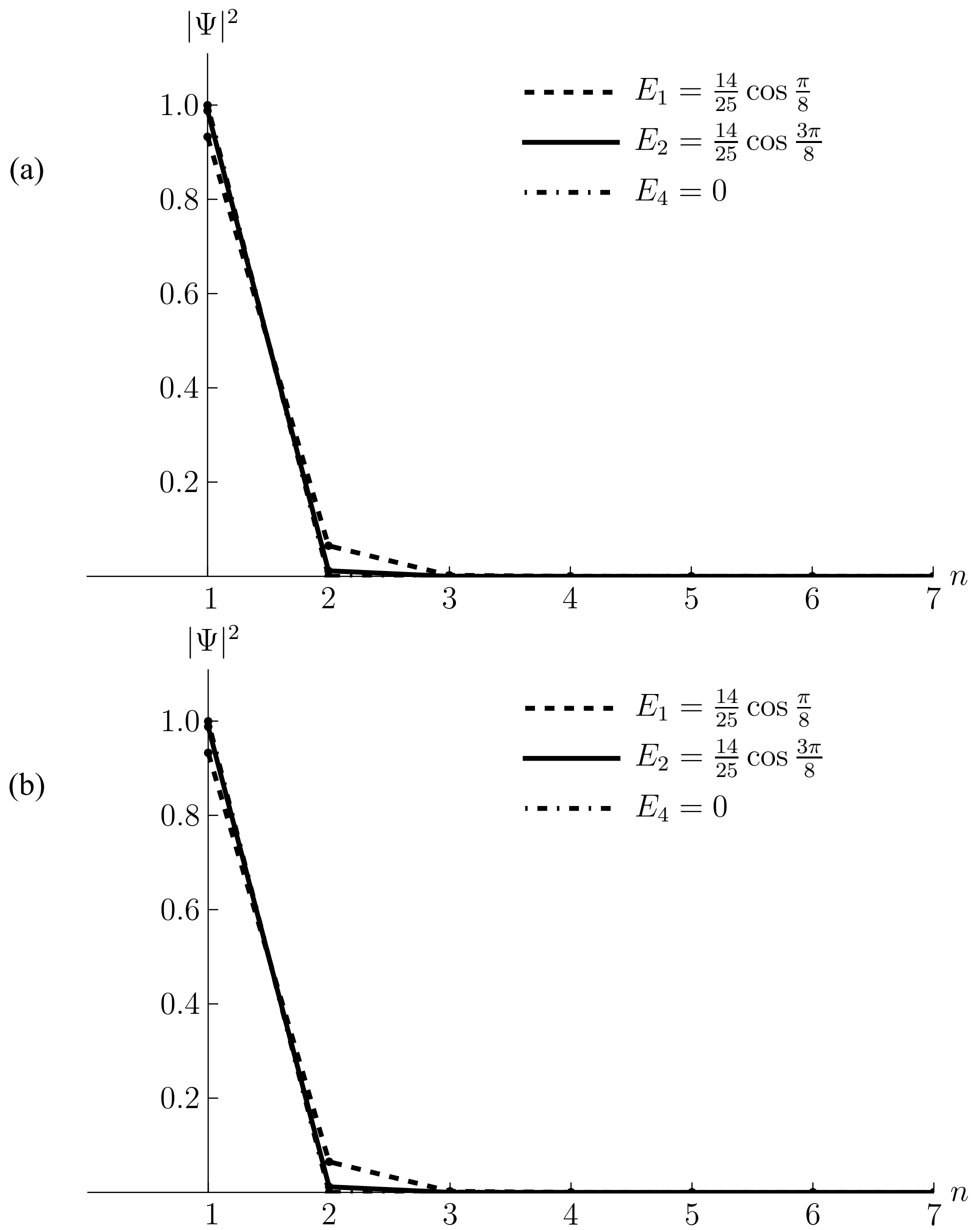

Fig. 3. Normalized eigenfunctions for $N = 7, m = 1, 3, 4$ (see (1.12)). (a) Here $t = 1, \gamma = 0.96$. (b) Here $t = 1$, $\gamma = -0.96$. For $\pm\gamma \approx t$ the states are highly localized near the left or right ends of the chain, respectively. This indicates the presence of the NHSE

1.4. An infinite chain with an impurity

Consider the Dyson equation (1.2) for the HN Hamiltonian and an infinite chain with the potential of the form $V(n) = v\delta_{n,0}$ simulating an impurity, where $v \neq 0$ is a real number. According to (1.3), this equation has the form

$$\psi(n) = -\frac{v}{2(t+\gamma)e^{ip_+} - E} \times \\ \times \sum_{n'} \left(e^{ip_+|n-n'|}\theta(n-n') + e^{ip_-|n-n'|}(1-\theta(n-n'))\right)\delta_{n',0}\psi(n'). \tag{1.18}$$

From (1.18), we have

$$\psi(0) = \mp\frac{v}{\sqrt{E^2 - 4(t^2-\gamma^2)}}\psi(0)$$

and, assuming that $\psi(0) \neq 0$ (otherwise $\psi(n) = 0$),

$$\mp\sqrt{E^2 - 4(t^2 - \gamma^2)} = v. \tag{1.19}$$

By virtue of (1.19),

$$E = \pm\sqrt{v^2 + 4(t^2 - \gamma^2)}. \tag{1.20}$$

From (1.20), we find the boundary $v = \pm 2\sqrt{\gamma^2 - t^2}$ between real and purely imaginary E. We have $E = 0$ on the boundary. Due to the reality of v, the condition $\gamma^2 > t^2$ is satisfied. In the most important case $E = 0$, from (A.3) and (A.5), we have

$$e^{ip_+} = \pm\sqrt{\frac{\gamma - t}{\gamma + t}}, \quad e^{ip_-} = \mp\sqrt{\frac{\gamma + t}{\gamma - t}}, \tag{1.21}$$

and from (1.18), (1.20), and (1.21), we obtain

$$\psi(n) = \begin{cases} e^{ip_+ n}, & n \geqslant 0, \\ e^{-ip_- n}, & n < 0, \end{cases} = \begin{cases} \left(\pm\sqrt{\dfrac{\gamma - t}{\gamma + t}}\right)^n, & n \geqslant 0, \\ \left(\mp\sqrt{\dfrac{\gamma - t}{\gamma + t}}\right)^n, & n < 0. \end{cases} \tag{1.22}$$

It follows from (1.22) that, for $\gamma > 0$, the function $\psi(n)$ decreases exponentially for $n \to +\infty$ and increases exponentially for $n \to -\infty$, and, for $\gamma < 0$, on the contrary, it increases exponentially for $n \to +\infty$ and decreases exponentially for $n \to -\infty$. Therefore, the function describes a localized state on one semi-axis and a virtual state on other [13].

§ 2. Finding the states in the matrix approach

The Green function (A.6) has a singular point $E = \pm 2\sqrt{t^2 - \gamma^2}$. In the previous section, we did not investigate states at this point, because the Green function used in that section does not exist in it. The case $t \pm \gamma = 0$ was also not studied. Let us partially fill this gap by considering the equation $(H - E)\psi(n) = 0$ for finite and semi-infinite chains in the matrix form, which follows from (1.1):

$$\begin{pmatrix} -E & t+\gamma & 0 & 0 & \dots \\ t-\gamma & -E & t+\gamma & 0 & \dots \\ 0 & t-\gamma & -E & t+\gamma & \dots \\ 0 & 0 & t-\gamma & -E & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} \psi(1) \\ \psi(2) \\ \psi(3) \\ \psi(4) \\ \dots \end{pmatrix} = 0, \tag{2.1}$$

where the matrix can be finite or infinite.

For a chain with N sites, when $t = \gamma$, there is a unique eigenfunction $\psi(n) = \delta_{n,1}$, corresponding to $E = 0$, localized at the left boundary site. With the same parameters after transposing the matrix in (2.1), there is also one eigenfunction $\psi(n) = \delta_{n,N}$, localized at the right boundary site. This means the presence of the NHSE, see [1, 2, 4, 11]. Note that the function (1.13) after normalization tends to $\delta_{n,1}$ as $t - \gamma \to 0$ and tends to $\delta_{n,N}$ as $t + \gamma \to 0$.

In the case of the semi-infinite chain, when $t = \gamma$, there is the following family of solutions of the equation (2.1): $\psi(n) = (E/2t)^{n-1}$, describing only localized states if $|E/2t| < 1$. In this case, $E = 0$ corresponds to the state $\psi(n) = \delta_{n,1}$. After transposing the matrix to (2.1), all states disappear, so there is no NHSE in the semi-infinite chain.

§ 3. Discussion of results

Consider the case of a finite chain. Assume that the non-Hermitian parameter γ is sufficiently close to t or $-t$. Then, thanks to the factor

$$\left(\frac{t+\gamma}{t-\gamma}\right)^{\frac{N-n}{2}}$$

in (1.12), the eigenfunction will either decrease or increase with some error, depending on the sign of γ (see Fig. 2, *a*). For a finite chain, if γ is replaced by $-\gamma$ (i. e., when transposing the matrix in (2.1)), then the localization of the state near the boundary changes to the localization at the opposite boundary. This demonstrates the NHSE [1].

If γ is small enough, then this effect is lost due to the sine in the numerator (1.12) (see the second graph in Fig. 2).The maximum localization of the eigenfunction corresponding to $E = 0$ at the single boundary site is achieved in the extreme case $\gamma = t$.

In the case of the infinite chain with the impurity, there is the eigenfunction corresponding to the zero energy, which decreases exponentially as $n \to +\infty$ and increases exponentially as $n \to -\infty$, for $\gamma > 0$, and another eigenfunction that increases exponentially as $n \to +\infty$ and decreases exponentially as $n \to -\infty$, for $\gamma < 0$. It is natural to consider regions with the exponential increase of the wave functions, by analogy with resonant states [13, 14], as regions in which the states decay. When the Hamiltonian matrix is transposed, the regions of decay and localization of two wave functions are interchanged, which can be considered as an analogue of the NHSE.

The functions $e^{ip_\pm|n-n'|}$, which are part of the Green function (A.6), play a basic role in the describing the behavior of wave functions as $n \to \pm\infty$. For the main case $E = 0$, the decrease or increase of the wave functions is determined by virtue of (A.3) and (A.4) by the quantities

$$r = \left|\frac{t \pm \gamma}{t \mp \gamma}\right|.$$

According to (1.3), only one of the $e^{ip_\pm}$ functions remains in the Dyson equation for arbitrary n. If the set of n corresponding to one of these functions forms an infinite interval, then the inequality $r < 1$ means the exponential decrease in this interval of wave functions, and the inequality $r > 1$ defines an exponential increase, which corresponds to virtual decaying states [13,14]. In this case, the transition from $r < 1$ to $r > 1$, i. e., the sign change of the non-Hermitian parameter γ, means the disappearance of the state (cf. the case of the Hermitian Hamiltonian [15]). In particular, for $E = 0$, the equality $r = 1$ defines the boundary between non-Hermitian topological phases.

Note that, for real E, the boundary between localized and resonant states has the form $\gamma = 0$, and for purely imaginary E, this boundary is described by the equality $t = 0$. For other values of E, the boundary is the ellipse of the form

$$\left(\frac{\operatorname{Re} E}{2t}\right)^2 + \left(\frac{\operatorname{Im} E}{2\gamma}\right)^2 = 1. \tag{3.1}$$

(See Appendix A).

§ 4. Conclusions

In the rigorous approach, we studied the HN model for finite and semi-infinite chains, as well as for an infinite chain with an impurity. With the help of the Green function, we have analytically found the eigenvalues and eigenfunctions of the HN Hamiltonian for this chains. We studied the NHSE for various chains. We also have described the boundary between localized and resonant eigenfunction (for $E = 0$ this is the boundary between non-Hermitian topological phases).

Appendix A

We assume that $t \neq \pm\gamma$. We find the Green function $G(n-n', E)$ of the Hamiltonian H for the infinite chain from the equation

$$((H-E)G)(n,E) = \delta_{n,0}, \tag{A.1}$$

where $\delta_{n,m}$ is the Kronecker delta. We put

$$G(n,E) = C \begin{cases} e^{ip_+|n|}, & n \geqslant 0, \\ e^{ip_-|n|}, & n < 0, \end{cases}$$

where $C, p_\pm = \text{const}$. For $n > 0$, by (1.1) and (A.1), we have the equality

$$(H-E)Ce^{ip_+n} = C\left((t-\gamma)e^{ip_+(n-1)} + (t+\gamma)e^{ip_+(n+1)} - Ee^{ip_+n}\right) =$$
$$= Ce^{ip_+n}\left((t-\gamma)e^{-ip_+} + (t+\gamma)e^{ip_+} - E\right) = 0,$$

from which we get

$$(t-\gamma)e^{-ip_+} + (t+\gamma)e^{ip_+} - E = 0. \tag{A.2}$$

and

$$e^{ip_+} = \frac{E \pm \sqrt{E^2 - 4(t^2-\gamma^2)}}{2(t+\gamma)}. \tag{A.3}$$

For $n < 0$, in (A.2), γ will be replaced by $-\gamma$ and vice versa, therefore,

$$e^{ip_-} = \frac{E \pm \sqrt{E^2 - 4(t^2-\gamma^2)}}{2(t-\gamma)}. \tag{A.4}$$

For opposite signs in front of the roots in (A.3) and (A.4), according to (1.1), we have

$$((H-E)G(n,E))|_{n=0} = (t-\gamma)G(-1,E) + (t+\gamma)G(1,E) - EC = 0,$$

which contradicts (A.1). For the same signs in (A.3) and (A.4), we obtain

$$e^{i(p_+-p_-)} = \frac{t-\gamma}{t+\gamma}. \tag{A.5}$$

From (1.1), (A.1), and (A.5), we have

$$((H-E)G)(n,E)|_{n=0} = C\left((t-\gamma)e^{ip_-|-1|} + (t+\gamma)e^{ip_+|1|} - E\right) =$$
$$= C\left(2(t+\gamma)e^{ip_+} - E\right) = 1.$$

Hence, due to (A.1) and (A.5),

$$C = \left(2(t+\gamma)e^{ip_+} - E\right)^{-1} = \left(2(t-\gamma)e^{ip_-} - E\right)^{-1}.$$

Therefore, the Green function of the HN Hamiltonian has the form

$$G(n-n', E) = \frac{1}{2(t\pm\gamma)e^{ip_\pm} - E}\left(e^{ip_+|n-n'|}\theta(n-n') + e^{ip_-|n-n'|}(1-\theta(n-n'))\right), \tag{A.6}$$

where $e^{ip_\pm}$ are determined from (A.3) and (A.4),

$$\theta(n) = \begin{cases} 1, & n \geqslant 0, \\ 0, & n < 0. \end{cases}$$

Since the equation $E^2 = 4(t^2 - \gamma^2)$ describes the singular point of the Green function (A.6), the following condition must be met:

$$2(t \pm \gamma)e^{ip_\pm} - E = \pm\sqrt{E^2 - 4(t^2 - \gamma^2)} \neq 0. \tag{A.7}$$

The inequality $|e^{ip_+}| < 1$ implies, according to (A.6), the exponential decrease of the Green function. In the case of $|e^{ip_+}| > 1$, the wave functions increase. Thus, the equation

$$|e^{ip_+}| = r, \tag{A.8}$$

for $r = 1$, defines the boundary between localized and resonant (decaying) (see [13, 14]) states. We rewrite the equality (A.8) in terms of the system parameters:

$$e^{ip_+} = \sqrt{\frac{t-\gamma}{t+\gamma}}(w \pm \sqrt{w^2 - 1}), \tag{A.9}$$

where $w = E/(2\sqrt{t^2 - \gamma^2})$. The equation (A.8), due to (A.9), takes the form

$$e^{ip_+} = re^{i\varphi} \tag{A.10}$$

where φ is real. Since the (two-valued) function $z = w \pm \sqrt{t^2 - \gamma^2}$ is inverse to the Zhukovsky function $w = \frac{1}{2}(z + 1/z)$, then, from (A.10), we get

$$\frac{E}{\sqrt{t^2 - \gamma^2}} = \sqrt{\frac{t+\gamma}{t-\gamma}} re^{i\varphi} + \sqrt{\frac{t-\gamma}{t+\gamma}} \cdot \frac{1}{r} e^{-i\varphi},$$

whence $E = \operatorname{Re} E + i \operatorname{Im} E$, where

$$\operatorname{Re} E = \left(r(t+\gamma) + \frac{1}{r}(t-\gamma)\right)\cos\varphi, \quad \operatorname{Im} E = \left(r(t+\gamma) - \frac{1}{r}(t-\gamma)\right)\sin\varphi.$$

Note that the following equality is valid:

$$\left(\frac{\operatorname{Re} E}{r(t+\gamma) + r^{-1}(t-\gamma)}\right)^2 + \left(\frac{\operatorname{Im} E}{r(t+\gamma) - r^{-1}(t-\gamma)}\right)^2 = 1,$$

which, for $r = 1$, describes the boundary between localized and resonant states in the parameter space (see (3.1) in the main text).

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Ю. П. Чубурин, Т. С. Тинюкова

Спектральные свойства и неэрмитов скин-эффект в модели Хатано–Нельсона

Ключевые слова: модель Хатано–Нельсона, собственные значения, собственные функции, неэрмитов скин-эффект.

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В настоящее время продолжают активно изучаться неэрмитовы топологические системы. В данной статье в строгом подходе изучена одна из ключевых неэрмитовых систем — модель Хатано–Нельсона H. Найдена функция Грина для этого гамильтониана. С помощью функции Грина аналитически получены собственные значения и собственные функции H для конечных и полубесконечных цепей, а также для бесконечной цепи с локальным потенциалом. Обсуждается неэрмитов скин-эффект для упомянутых выше моделей. Также описана граница между локализованными и резонансными состояниями (при нулевой энергии — это граница между неэрмитовыми топологическими фазами).

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