MSC: 34D23, 34H15, 93D15

© ***V. A. Zaitsev***

GLOBAL ASYMPTOTIC STABILIZATION OF BILINEAR CONTROL SYSTEMS WITH PERIODIC COEFFICIENTS [1]

Sufficient conditions for uniform global asymptotic stabilization of the origin are obtained for bilinear control systems with periodic coefficients. The proof is based on the use of the Krasovsky theorem on global asymptotic stability of the origin for periodic systems. The stabilizing control function is feedback control constructed as the quadratic form of the phase variables and depends on time periodically.

Keywords: global asymptotic stability, stabilization, Lyapunov function, bilinear systems, periodic systems.

§ 1. Introduction

The present paper continues the research carried out in [Z.I, Z.II][2]. Consider a bilinear control system with periodic coefficients

$$\dot{y} = (F(t) + u_1 G_1(t) + \ldots + u_r G_r(t))y, \quad y \in \mathbb{R}^n, \quad t \in \mathbb{R}_+, \tag{1}$$

where F, G_k are C^l functions, $l \geqslant 0$, $F(t+T) \equiv F(t)$, $G_k(t+T) \equiv G_k(t)$, $t \in \mathbb{R}_+ := [0, +\infty)$, $k = \overline{1,r}$, $T > 0$. Sufficient conditions for uniform global asymptotic stabilization of the solution $y = 0$ of the system (1) were obtained in [Z.I, Z.II] for the case in which the drift $F(t)y$ of the system (1) is constant, i.e. $F(t)y \equiv Fy$. Then those conditions were obtained for systems *reducible by a Lyapunov transformation* to a system (1) with a constant drift, in particular, for systems (1) with a time-varying periodic drift $F(t)y$. In the present paper, we obtain new sufficient conditions for uniform global asymptotic stabilization of the origin of the system (1) with a periodic drift $F(t)y$. These conditions do not depend on any Lyapunov transformation. In contrast to results in [Z.II, § 5], these conditions are obtained directly rather than as a consequence of Theorems 8–10 [Z.II]. These conditions differ from those in [Z.II, § 5] and are more general. The proven results improve some of the results of [Z.I, Z.II]. They are a further generalization of the Jurdjevic–Quinn theorems on stabilization of the null solution by *damping control* to periodic systems.

By $\mathbb{R}^n$ denote the real n-dimensional vector space with the canonical basis $e_1 = \operatorname{col}(1, 0, \ldots, 0)$, $\ldots$, $e_n = \operatorname{col}(0, \ldots, 0, 1)$. Next, $|x| = \sqrt{x^\top x}$ is the norm in $\mathbb{R}^n$, $\top$ is the operation of transposition of a vector or a matrix, $O_\delta := \{x \in \mathbb{R}^n : |x| < \delta\}$, $M_{n,m}$ is the space of real $n \times m$ matrices, $M_n := M_{n,n}$, $I \in M_n$ is the identity matrix. We identify a quadratic form $V(t, y) = y^\top P(t) y$ with the symmetric matrix $P(t) \in M_n$ defining this form. The inequality $P(t) > 0$ (or $P(t) \leqslant 0$) for a matrix $P(t) = P^\top(t)$ means, by definition, that the quadratic form $V(t, y) = y^\top P(t) y$ is positive definite (respectively, negative semi-definite), i.e. $\exists\, \alpha > 0\ \forall\, t \in \mathbb{R}_+\ \forall\, y \in \mathbb{R}^n\ V(t, y) \geqslant \alpha |y|^2$ (respectively, $V(t, y) \leqslant 0$).

Proofs of the main results are based on the Krasovsky theorem on global asymptotic stability of the null solution of a system of differential equations with periodic coefficients. Let us formulate this theorem. Consider a system

$$\dot{y} = f(t, y), \quad y \in \mathbb{R}^n, \quad t \in \mathbb{R}_+, \tag{2}$$

where $f \in C(\mathbb{R}_+ \times \mathbb{R}^n)$, $f(t, 0) \equiv 0$, $t \in \mathbb{R}_+$, and f is semi-globally Lipschitz continuous in y, i.e. for any compact set $K \subset \mathbb{R}^n$ there exists a ℓ_K such that $|f(t, y') - f(t, y'')| < \ell_K |y' - y''|$, $\forall\, y', y'' \in K$,

[1]The research is supported by the Russian Foundation for Basic Research (projects no. 12–01–00195, 12–01–31077).

[2]Zaitsev V.A. A Generalization of Jurdjevic–Quinn Theorem and Stabilization of Bilinear Control Systems with Periodic Coefficients: I, II, *Differential Equations*, (to appear).

$t \in \mathbb{R}_+$. By $y(t, t_0, y_0)$ denote the solution of (2) with an initial condition $y(t_0) = y_0$. The solution $y = 0$ of (2) is said to be *asymptotically stable in the whole* [1, § 5] (or, in other words, *globally asymptotically stable* [2, I.2]) if

(a) it is Lyapunov stable, and

(b) it is globally attractive, i.e.

$$\forall\, \delta > 0 \quad \forall\, \eta > 0 \quad \forall\, t_0 \geqslant 0 \quad \exists\, T = T(\delta, \eta, t_0) > 0 \quad \forall\, y_0 \in O_\delta \quad \forall t \geqslant t_0 + T \quad |y(t, t_0, y_0)| < \eta.$$

The solution $y = 0$ of (2) is said to be *uniformly globally asymptotically stable* [2, I.2] if

(a) it is uniformly stable, i.e. $\forall\, \varepsilon > 0\; \exists\, \delta > 0\; \forall\, t_0 \geqslant 0\; \forall\, y_0 \in O_\delta\; \forall t \geqslant t_0\; |y(t, t_0, y_0)| < \varepsilon$;

(b) it is uniformly globally attractive, i.e.

$$\forall\, \delta > 0 \quad \forall\, \eta > 0 \quad \exists\, T = T(\delta, \eta) > 0 \quad \forall\, t_0 \geqslant 0 \quad \forall\, y_0 \in O_\delta \quad \forall t \geqslant t_0 + T \quad |y(t, t_0, y_0)| < \eta.$$

We say that a function γ *belongs to class* $\mathcal{K}_\infty$ [3, § 4.4] if $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, $\gamma(0) = 0$, it is continuous, strictly increasing, and $\gamma(s) \to \infty$ as $s \to \infty$. Let $V(t, y)$ be a C^1 function. We use the notation $\dot{V}(t, y) := \partial V(t, y)/\partial t + \nabla_y V(t, y) \cdot f(t, y)$ for the derivative of V along the trajectories of (2). Define $E(V) = \{(t, y) \in \mathbb{R}_+ \times \mathbb{R}^n : \dot{V}(t, y) = 0\}$ and let $M(V)$ be the union of all solutions $y(t)$, $t \in \mathbb{R}_+$ of (2) with the property that $(t, y(t)) \in E(V)$ for all $t \geqslant 0$. The set $M(V)$ is said to be *the largest positive invariant set of the system* (2) *relative to* $E(V)$. The following theorem of Krasovsky holds.

Theorem 1 (see [1, § 14], [2, II.1]). *Suppose that the system* (2) *is periodic, i.e.* $f(t + T, y) \equiv f(t, y)$, $T > 0$, $t \in \mathbb{R}_+$, $y \in \mathbb{R}^n$. *Suppose that there exists a* C^1 *function* $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ *having period* T *such that for some function* $\gamma \in \mathcal{K}_\infty$ *and for all* $t \in \mathbb{R}_+$, $y \in \mathbb{R}^n$ *the following conditions hold:*

1) $V(t, y) \geqslant \gamma(|y|)$; $V(t, 0) = 0$. 2) $\dot{V}(t, y) \leqslant 0$. 3) $M(V) = \{0\}$.

Then the solution $y = 0$ *of* (2) *is globally asymptotically stable.*

Remark 1. The properties of uniform and non-uniform global asymptotic stability are equivalent for time-varying periodic systems (in particular, for time-invariant systems). Therefore, Theorem 1 in fact asserts *uniform* global asymptotic stability of the null solution of (2).

§ 2. Main results

Consider the periodic system (1). Let

$$\dot{y} = F(t)y, \quad y \in \mathbb{R}^n, \quad t \in \mathbb{R}_+ \tag{3}$$

be the corresponding non-perturbed system, i.e. the system (1) with $u := (u_1, \ldots, u_r) = 0$. By $Y(t, s)$ denote the Cauchy matrix of the system (3), i.e. the solution of the matrix initial value problem $\dot{Y} = F(t)Y$, $Y(s) = I$. We define the operator $D_F^i G(t)$, $i \geqslant 0$ by the equalities

$$D_F^0 G(t) := G(t), \quad D_F G(t) := G(t)F(t) - F(t)G(t) + \dot{G}(t), \quad D_F^{i+1} G(t) := D_F(D_F^i G)(t).$$

We define the operator $W_F^i P(t)$, $i \geqslant 0$, where $P(t)$ is a symmetric matrix, by the equalities

$$W_F^0 P(t) := P(t), \quad W_F P(t) := F^\top(t)P(t) + P(t)F(t) + \dot{P}(t), \quad W_F^{i+1} P(t) = W_F(W_F^i P)(t).$$

Thus, for a quadratic form $V(t, y) = y^\top P(t)y$, the derivative of V along the trajectories of (3) is $\dot{V}(t, y) = y^\top R(t)y$, where $R(t) = W_F P(t)$.

Suppose C^{l+1} function $V_0(t, y) = y^\top P(t)y$, $t \in \mathbb{R}_+$ satisfies the following conditions:

$$1)\ P(t + T) \equiv P(t); \qquad 2)\ P(t) = P^\top(t) > 0; \qquad 3)\ W_F P(t) \leqslant 0. \tag{4}$$

Let us construct the control function with period T

$$\widehat{u}_k(t, y) = -y^\top \big(G_k^\top(t)P(t) + P(t)G_k(t)\big)y, \quad y \in \mathbb{R}^n, \quad t \in \mathbb{R}_+, \quad k = \overline{1, r}. \tag{5}$$

Substituting $\widehat{u}_k(t,y)$ into the system (1), we obtain the closed-loop system

$$\dot{y} = H(t,y) = F(t)y + \sum_{k=1}^{r} \widehat{u}_k(t,y)G_k(t)y, \quad y \in \mathbb{R}^n, \quad t \in \mathbb{R}_+ \tag{6}$$

whose right-hand side is T-periodic in t. Finding the derivative of $V_0(t,y)$ along the trajectories of (6), we get

$$\dot{V}_0(t,y) = y^\top (W_F P)(t)y - \sum_{k=1}^{r} \widehat{u}_k^2(t,y).$$

By using condition 3) in (4), we obtain the inequality $\dot{V}_0(t,y) \leqslant 0$, $t \geqslant 0$, $y \in \mathbb{R}^n$. Therefore conditions (4) imply that the function $V_0(t,y)$ satisfies conditions 1), 2) of Theorem 1 for the system (6). Consider the set $E(V_0) = \{(t,y) \in \mathbb{R}_+ \times \mathbb{R}^n : \dot{V}_0(t,y) = 0\}$. The set $E(V_0)$ coincides with the set

$$E_0(V_0) = \big\{(t,y) \in \mathbb{R}_+ \times \mathbb{R}^n : y^\top (W_F P)(t)y = 0 \;\&\; y^\top \big(G_k^\top(t)P(t) + P(t)G_k(t)\big)y = 0,\ k = \overline{1,r}\big\}.$$

Let $M(V_0)$ be the largest positive invariant set of the system (6) relative to $E(V_0)$. If $M(V_0) = \{0\}$, then, by Theorem 1, the null solution of the system (6) is uniformly globally asymptotically stable. Suppose that $\xi(t) = \xi(t,0,y_0)$, $t \in \mathbb{R}_+$ is a solution of the system (6) belonging to $M(V_0)$. Then

$$\xi^\top(t)(W_F P)(t)\xi(t) = 0, \quad \xi^\top(t)\Big(G_k^\top(t)P(t) + P(t)G_k(t)\Big)\xi(t) = 0 \quad \forall\, t \geqslant 0, \quad k = \overline{1,r}. \tag{7}$$

It follows that control (5) vanishes along the solution $\xi(t)$, i.e. $\widehat{u}_k(t,\xi(t)) \equiv 0$, $t \in \mathbb{R}_+$, $k = \overline{1,r}$. Hence, $\dot{\xi}(t) \equiv F(t)\xi(t)$, $t \in \mathbb{R}_+$. This means that $\xi(t)$ is a solution of the equation (3). Therefore, $\xi(t) = Y(t,0)y_0$. Substituting this equality into (7), we obtain the identities

$$y_0^\top Y^\top(t,0)(W_F P)(t)Y(t,0)y_0 \equiv 0, \quad t \in \mathbb{R}_+, \tag{8}$$

$$y_0^\top Y^\top(t,0)\Big(G_k^\top(t)P(t) + P(t)G_k(t)\Big)Y(t,0)y_0 \equiv 0, \quad t \in \mathbb{R}_+, \quad k = \overline{1,r}. \tag{9}$$

Let $M_0(V_0)$ be the largest positive invariant set of the system (3) relative to $E_0(V_0)$. This means that $M_0(V_0) \subset \mathbb{R}^n$ is the union of all semi-trajectories $y(t)$, $t \in \mathbb{R}_+$ of the system (3) with the property that $(t,y(t)) \in E_0(V_0)$ for all $t \geqslant 0$. In that case we obtain that $\xi(t) \in M_0(V_0)$, $t \in \mathbb{R}_+$. It follows that $M(V_0) \subset M_0(V_0)$. Therefore if $M_0(V_0) = \{0\}$, then $M(V_0) = \{0\}$. The condition $M_0(V_0) = \{0\}$ means that the equalities (8), (9) hold only if $y_0 = 0$. Thus, the following theorem holds.

Theorem 2. *Suppose that there exists a quadratic form* $V_0(t,y) = y^\top P(t)y$, $P(t) = P^\top(t)$ *satisfying conditions* (4) *such that the following condition holds:*

(A) *the identities* (8), (9) *hold only if* $y_0 = 0$.

Then the null solution of the system (1) *is uniformly globally asymptotically stabilizable by the* T*-periodic control* (5).

Remark 2. We proved a similar statement in [Z.I, Theorem 5]. In that theorem, the matrix F was constant and we required existence of a *time-invariant* quadratic form satisfying (4). In this theorem, the conditions are weaker. The matrices F and P may be T-periodic. Thus, Theorem 2 generalizes Theorem 5 [Z.1].

The following theorem generalizes Theorem 6 [Z.I] to systems with a periodic drift $F(t)y$. In what follows, we assume that the smoothness $l = \infty$ unless otherwise specified. We write the following identities:

$$y_0^\top Y^\top(t,0)\left(W_F^{i+1}P\right)(t)Y(t,0)y_0 \equiv 0, \quad t \in \mathbb{R}_+, \tag{10}$$

$$y_0^\top Y^\top(t,0)\left(W_F^i\big(G_k^\top(t)P(t) + P(t)G_k(t)\big)\right)Y(t,0)y_0 \equiv 0, \quad t \in \mathbb{R}_+, \tag{11}$$

$$y_0^\top Y^\top(t,0)\left(\left(D_F^i G_k(t)\right)^\top P(t) + P(t)\left(D_F^i G_k(t)\right)\right)Y(t,0)y_0 \equiv 0, \quad t \in \mathbb{R}_+. \tag{12}$$

Theorem 3. *Suppose that there exists a quadratic form* $V_0(t,y) = y^\top P(t)y$, $P(t) = P^\top(t)$, *satisfying conditions* (4) *such that one of the following conditions holds:*

(*a*) *the identities* (10), (11) *hold for all* $k = \overline{1,r}$, $i \in \mathbb{N}$ *only if* $y_0 = 0$;

(*b*) *the identities* (10), (12) *hold for all* $k = \overline{1,r}$, $i \in \mathbb{N}$ *only if* $y_0 = 0$;

(*c*) *there exists a* $\nu \geqslant 0$ *such that the identities* (8), (12) *hold for all* $k = \overline{1,r}$, $i = \overline{0,\nu}$ *only if* $y_0 = 0$.

Then the T*-periodic control* (5) *uniformly globally asymptotically stabilizes the null solution of the system* (1).

P r o o f. Let us construct a quadratic form $V_0(t,y) = y^\top P(t)y$ satisfying conditions (4). Let us prove that $M_0(V_0) = \{0\}$. Suppose $\xi(t) \in M_0(V_0)$, $t \in \mathbb{R}_+$. Then $\xi(t)$ is a solution of the equation (3) and the equalities (7) hold. We denote $y_0 = \xi(0)$, then $\xi(t) = Y(t,0)y_0$. Differentiating (7) i times, where $i \in \mathbb{N}$, we get

$$\xi^\top(t)(W_F^{i+1}P)(t)\xi(t) \equiv 0, \qquad t \in \mathbb{R}_+, \tag{13}$$

$$\xi^\top(t)\left(W_F^i\big(G_k^\top(t)P(t) + P(t)G_k(t)\big)\right)\xi(t) \equiv 0, \quad t \in \mathbb{R}_+, \quad k = \overline{1,r}. \tag{14}$$

Substituting $\xi(t) = Y(t,0)y_0$ into (13), (14), we obtain the identities (10), (11), $k = \overline{1,r}$, $i \in \mathbb{N}$. If the condition (a) of Theorem 3 is satisfied, then $y_0 = 0$. In that case $\xi(t) \equiv 0$ and hence $M_0(V_0) = \{0\}$ as required.

Next, consider the function $w(t,y) = y^\top(W_F P)(t)y$. We denote $S(V_0) = \{(t,y) \in \mathbb{R}_+ \times \mathbb{R}^n : w(t,y) = 0\}$. Since $w(t,y) \leqslant 0$ $\forall t \in \mathbb{R}_+$, $\forall y \in \mathbb{R}^n$, we see that every point $(\widehat{t},\widehat{y}) \in S(V_0)$ is a maximum point for the function $w(t,y)$. Therefore, $\partial w(\widehat{t},\widehat{y})/\partial t = 0$ and $\nabla_y w(\widehat{t},\widehat{y}) = 0$ for any point $(\widehat{t},\widehat{y}) \in S(V_0)$ such that $\widehat{t} > 0$. By (7), we have $(s,\xi(s)) \in S(V_0)$ for any $s > 0$. Therefore, $\nabla_y w(t,y)\big|_{(s,\xi(s))} = 0$. We have $\nabla_y w(t,y) = 2(W_F P)(t)y$. Hence $(W_F P)(s)\xi(s) = 0$ for all $s > 0$. By continuity, this equality holds for $s = 0$. Thus,

$$\left(F^\top(t)P(t) + P(t)F(t) + \dot{P}(t)\right)\xi(t) \equiv 0, \quad t \in \mathbb{R}_+. \tag{15}$$

The equality (15) implies that

$$\begin{gathered}\xi^\top(t)\left(G_k^\top(t)\big[F^\top(t)P(t) + P(t)F(t) + \dot{P}(t)\big] + \right.\\ \left. + \big[F^\top(t)P(t) + P(t)F(t) + \dot{P}(t)\big]G_k(t)\right)\xi(t) \equiv 0, \quad t \in \mathbb{R}_+\end{gathered} \tag{16}$$

for all $k = \overline{1,r}$. Let us prove by induction that the identity

$$\xi^\top(t)\left(\left(D_F^i G_k(t)\right)^\top P(t) + P(t)\left(D_F^i G_k(t)\right)\right)\xi(t) \equiv 0, \quad t \in \mathbb{R}_+ \tag{17}$$

holds for all $i \in \mathbb{N}$, $k = \overline{1,r}$. Next, for simplicity, we omit $t \in \mathbb{R}_+$; let prime and dot denote the derivative. The equality (14) has the following form for $i = 1$:

$$\xi^\top\Big[F^\top(G_k^\top P + PG_k) + (G_k^\top P + PG_k)F + (\dot{G}_k^\top P + P\dot{G}_k) + (G_k^\top\dot{P} + \dot{P}G_k)\Big]\xi \equiv 0. \tag{18}$$

Subtracting (16) from (18), we get

$$\xi^\top\Big[(G_kF - FG_k + \dot{G}_k)^\top P + P(G_kF - FG_k + \dot{G}_k)\Big]\xi \equiv 0. \tag{19}$$

The identity (19) is nothing else than the identity (17) for $i = 1$. Therefore, the basis of induction is proved. Let the identity (17) hold for $i = s$. Differentiating (17) for $i = s$, we obtain

$$\begin{gathered}\xi^\top\Big[F^\top\big\{(D_F^sG_k)^\top P + P(D_F^sG_k)\big\} + \big\{(D_F^sG_k)^\top P + P(D_F^sG_k)\big\}F + \\ + \left((D_F^sG_k)^\top\right)' P + P(D_F^sG_k)' + (D_F^sG_k)^\top P' + P'(D_F^sG_k)\Big]\xi \equiv 0.\end{gathered} \tag{20}$$

The equality (15) implies that

$$\xi^\top\Big[(D_F^s G_k)^\top\big\{F^\top P + PF + \dot P\big\} + \big\{F^\top P + PF + \dot P\big\}(D_F^s G_k)\Big]\xi \equiv 0, \quad t\in\mathbb{R}_+ \tag{21}$$

for all $k=\overline{1,r}$. Subtracting (21) from (20), we get

$$\xi^\top\Big[\big\{(D_F^s G_k)F - F(D_F^s G_k) + (D_F^s G_k)'\big\}^\top P + P\big\{(D_F^s G_k)F - F(D_F^s G_k) + (D_F^s G_k)'\big\}\Big]\xi \equiv 0. \tag{22}$$

The identity (22) coincides with (17) for $i = s+1$. By induction, (17) holds for all $i\in\mathbb{N}$, $k=\overline{1,r}$. Substituting $\xi(t) = Y(t,0)y_0$ into (17), we obtain the identities (12). Thus, the identities (10), (12) hold for all $i\in\mathbb{N}$, $k=\overline{1,r}$. Therefore, if the condition (b) of Theorem 3 is satisfied, then $y_0 = 0$. In that case $\xi(t)\equiv 0$ and hence $M_0(V_0) = \{0\}$ as required.

Finally, substituting $\xi(t) = Y(t,0)y_0$ into (7) and (17), we obtain that the identities (8), (12) hold for all $k=\overline{1,r}$, $i=\overline{0,\nu}$ for every $\nu\geqslant 0$. Thus, if the condition (c) of Theorem 3 is satisfied, then $y_0=0$. Hence, $\xi(t)\equiv 0$ and therefore $M_0(V_0)=\{0\}$. The theorem is proved.

Remark 3. The conditions (a) and (b) of Theorem 3 are equivalent if the conditions (4) are satisfied. The condition (c) is stronger than (b) if the condition (4) are satisfied. Therefore, Theorem 3 under the conditions (c) is weaker than under the condition (b) or (a). On the other hand, being under the condition (c) it is sufficient to require C^l-smoothness of F, G_k and C^{l+1}-smoothness of V_0 rather than C^∞, where $l\geqslant 0$ is a sufficiently large number such that $\nu\leqslant l$.

Let us construct the following subspaces in $\mathbb{R}^n$ and M_n for every $t\in\mathbb{R}_+$:

$$Z^\nu(t,y) = \operatorname{span}\{(D_F^i G_k)(t)y,\ k=\overline{1,r},\ i=\overline{0,\nu}\},\quad \nu\geqslant 0,\quad y\in\mathbb{R}^n,$$
$$\mathcal{Z}^\nu(t) = \operatorname{span}\{(D_F^i G_k)(t),\ k=\overline{1,r},\ i=\overline{0,\nu}\},\quad \nu\geqslant 0,$$
$$K^\nu(t,y) = \operatorname{span}\{F(t)y, (D_F^i G_k)(t)y,\ k=\overline{1,r},\ i=\overline{0,\nu}\},\quad \nu\geqslant 0,\quad y\in\mathbb{R}^n,$$
$$\mathcal{K}^\nu(t) = \operatorname{span}\{F(t), (D_F^i G_k)(t),\ k=\overline{1,r},\ i=\overline{0,\nu}\},\quad \nu\geqslant 0.$$

The following inclusions hold for all $t\geqslant 0$, $y\in\mathbb{R}^n$:

$$\begin{array}{ccccccccc} \mathcal{Z}^0(t) & \subset & \mathcal{Z}^1(t) & \subset & \ldots & \subset & \mathcal{Z}^\infty(t) & & \\ \cap & & \cap & & & & \cap & & \\ \mathcal{K}^0(t) & \subset & \mathcal{K}^1(t) & \subset & \ldots & \subset & \mathcal{K}^\infty(t) & \subset & M_n, \end{array}$$

$$\begin{array}{ccccccccc} Z^0(t,y) & \subset & Z^1(t,y) & \subset & \ldots & \subset & Z^\infty(t,y) & & \\ \cap & & \cap & & & & \cap & & \\ K^0(t,y) & \subset & K^1(t,y) & \subset & \ldots & \subset & K^\infty(t,y) & \subset & \mathbb{R}^n. \end{array}$$

Lemma 1. 1. *Suppose that there exists a quadratic form* $V_0(t,y) = y^\top P(t)y$, $P(t) = P^\top(t)$, *satisfying conditions* (4). *Then the distribution* Z^∞ *has dimension less than* n *along any semi-trajectory* $\xi(t)$, $t\geqslant 0$ *of the system* (3) *belonging to* $M_0(V_0)$, *i.e.*

$$\dim Z^\infty\big(t,\xi(t)\big) < n\quad \forall\, t\in\mathbb{R}_+. \tag{23}$$

2. *Suppose that the stronger condition holds, namely, suppose that there exists a time-invariant quadratic form* $V_0(y) = y^\top Py$, $P = P^\top > 0$ *such that* $W_F P(t) := F^\top(t)P + PF(t)\leqslant 0$. *Then the distribution* K^∞ *has dimension less than* n *along any semi-trajectory* $\xi(t)$, $t\geqslant 0$ *of the system* (3) *belonging to* $M_0(V_0)$, *i.e.*

$$\dim K^\infty\big(t,\xi(t)\big) < n\quad \forall\, t\in\mathbb{R}_+. \tag{24}$$

P r o o f. Consider a solution $\xi(t) = \xi(t,0,y_0)$ of the system (3) such that $\xi(t)\in M_0(V_0)$ $\forall\, t\geqslant 0$. If $y_0 = 0$, then $\xi(t)\equiv 0$. Thus $K^\infty\big(t,\xi(t)\big) = Z^\infty\big(t,\xi(t)\big) = \{0\}$ $\forall\, t\in\mathbb{R}_+$. The assertion is obvious. Suppose $y_0\neq 0$. Then $\forall\, t\geqslant 0$ $\xi(t)\neq 0$, by uniqueness of solution, and the equalities (7) hold. We have $\nabla_y V_0(t,y) = 2P(t)y$. Therefore the second equality in (7) implies

$$\big\langle \nabla_y V_0\big(t,\xi(t)\big), G_k(t)\xi(t)\big\rangle \equiv 0,\quad t\in\mathbb{R}_+,\quad k=\overline{1,r}, \tag{25}$$

[we denote by $\langle a,b\rangle$ the inner product of $a,b\in\mathbb{R}^n$ here]. Next, the proof of Theorem 3 implies the equalities (17) for all $i\in\mathbb{N}$, $k=\overline{1,r}$. The equalities (17) imply that

$$\big\langle \nabla_y V_0\big(t,\xi(t)\big), (D_F^i G_k)(t)\xi(t)\big\rangle \equiv 0, \quad t\in\mathbb{R}_+, \quad k=\overline{1,r}, \quad i\in\mathbb{N}. \tag{26}$$

The equalities (25) and (26) imply that the vector $\nabla_y V_0\big(t,\xi(t)\big)$ is orthogonal to the space $Z^\infty\big(t,\xi(t)\big)$ for any $t\in\mathbb{R}_+$. Since $\xi(t)\neq 0$ for all $t\geqslant 0$ and $P(t)>0$, we have $P(t)\xi(t)\neq 0$ for all $t\geqslant 0$. Therefore, $\nabla_y V_0\big(t,\xi(t)\big)\neq 0\ \forall\, t\geqslant 0$. This implies (23).

If the matrix of the quadratic form is constant, i.e. $P(t)\equiv P$, then $\nabla_y V_0(y)=2Py$. The first equality in (7) has the form

$$\xi^\top(t)\big(F^\top(t)P+PF(t)\big)\xi(t)\equiv 0, \quad t\in\mathbb{R}_+.$$

This is equivalent to

$$\big\langle \nabla_y V_0(\xi(t)), F(t)\xi(t)\big\rangle \equiv 0, \quad t\in\mathbb{R}_+. \tag{27}$$

The equalities (25), (26), and (27) imply that the vector $\nabla_y V_0(\xi(t))$ is orthogonal to the space $K^\infty\big(t,\xi(t)\big)$ for any $t\in\mathbb{R}_+$. Since $\nabla_y V_0(\xi(t))\neq 0\ \forall\, t\geqslant 0$, this implies (24). The lemma is proved.

Lemma 1 obviously implies the following theorems generalizing Theorem 7 [Z.I] to systems (1) with a periodic drift.

Theorem 4. *Suppose that there exists a quadratic form $V_0(t,y)=y^\top P(t)y$, $P(t)=P^\top(t)$, satisfying conditions* (4), *and the following condition holds:*

$$\exists\, t_0\in\mathbb{R}_+ \quad \forall\, y\in\mathbb{R}^n\setminus\{0\} \quad \exists\,\nu\geqslant 0 \quad \dim Z^\nu(t_0,y)=n. \tag{28}$$

Then the T-periodic control (5) *uniformly globally asymptotically stabilizes the null solution of the system* (1).

P r o o f. By Theorem 2, it is sufficient to prove that $M_0(V_0)=\{0\}$. Consider a solution $\xi(t)=\xi(t,0,y_0)$ of the system (3) starting from the point $\xi(0)=y_0\neq 0$ such that $\xi(t)\in M_0(V_0)$ $\forall\, t\geqslant 0$. Hence $\xi(t)\neq 0\ \forall\, t\geqslant 0$, and in particular, $\xi(t_0)\neq 0$. By the condition of the theorem, for the point $y_1=\xi(t_0)$ there exists a $\nu\geqslant 0$ such that $\dim Z^\nu\big(t_0,\xi(t_0)\big)=n$. This contradicts condition (23). The theorem is proved.

Theorem 5. *Suppose that there exists a time-invariant quadratic form $V_0(y)=y^\top Py$, $P=P^\top>0$ such that $W_F P(t)\leqslant 0$, and the following condition holds:*

$$\exists\, t_0\in\mathbb{R}_+ \quad \forall\, y\in\mathbb{R}^n\setminus\{0\} \quad \exists\,\nu\geqslant 0 \quad \dim K^\nu(t_0,y)=n. \tag{29}$$

Then the T-periodic control (5) *uniformly globally asymptotically stabilizes the null solution of the system* (1).

The proof of Theorem 5 is similar to the proof of Theorem 4. The proof of Theorem 4 is identical to the proof of Theorem 4 [Z.I]. The first condition in Theorem 5 is stronger than in Theorem 4, but the second one is weaker. The first condition in Theorem 5 is satisfied, for example, if $F(t)$ is a skew-symmetric matrix. In that case one can set $P=I$. Theorems 4 and 5 imply obvious corollaries.

Corollary 1. *Suppose that there exists a quadratic form $V_0(t,y)=y^\top P(t)y$, $P(t)=P^\top(t)$, satisfying conditions* (4), *and the following condition holds:*

$$\exists\, t_0\in\mathbb{R}_+ \quad \exists\,\nu\geqslant 0 \quad \mathcal{Z}^\nu(t_0)=M_n. \tag{30}$$

Then the T-periodic control (5) *uniformly globally asymptotically stabilizes the null solution of the system* (1).

Corollary 2. *Suppose that there exists a time-invariant quadratic form* $V_0(y) = y^\top Py$, $P = P^\top > 0$ *such that* $W_F P(t) \leqslant 0$, *and the following condition holds:*

$$\exists\, t_0 \in \mathbb{R}_+ \quad \exists\, \nu \geqslant 0 \quad \mathcal{K}^\nu(t_0) = M_n. \tag{31}$$

Then the T*-periodic control* (5) *uniformly globally asymptotically stabilizes the null solution of the system* (1).

Remark 4. It is sufficient to require C^l-smoothness of F, G_k and C^{l+1}-smoothness of V_0 in Theorems 4, 5 and in Corollaries 1, 2, where $l \geqslant 0$ is a sufficiently large number such that $\nu \leqslant l$.

Next, we find in which case there exists a quadratic form satisfying conditions (4). The following statement holds. It is valuable in itself.

Theorem 6. *The following conditions are equivalent.*
1. *The system* (3) *with a* T*-periodic* C^l *matrix* $F(t)$ *is Lyapunov stable.*
2. *There exists a* T*-periodic* C^{l+1} *matrix* $P(t) = P^\top(t) > 0$ *such that* $(W_F P)(t) \leqslant 0$.

The implication $2 \Rightarrow 1$ obviously follows from First Lyapunov Theorem. The implication $1 \Rightarrow 2$ is proved below. The matrix $P(t)$ is constructed from the Cauchy matrix of the system (3). Thus Theorems 2, 3, 4, 6 and Corollary 1 imply the following assertion.

Corollary 3. *Suppose that the* T*-periodic system* (3) *is Lyapunov stable and one of the following conditions holds:* (A) *of Theorem* 2, *or* (a), (b) *or* (c) *of Theorem* 3, *or* (28), *or* (30). *Then the* T*-periodic control* (5) *uniformly globally asymptotically stabilizes the null solution of the system* (1). *Here the matrix* $P(t)$ *is constructed using Theorem* 6.

Theorem 5 and Corollary 2 imply, in particular, the following assertion.

Corollary 4. *Let the matrix* $F(t)$ *of the* T*-periodic system* (1) *be skew-symmetric, and let condition* (29) *or* (31) *hold. Then the* T*-periodic control* (5), *where* $P(t) = I$, *uniformly globally asymptotically stabilizes the null solution of the system* (1).

In Corollaries 3 and 4, it is sufficient to require the same C^l-smoothness of F, G_k as is required in the corresponding assertions from which Corollaries 3 and 4 follow.

Remark 5. The assertions proved in this section are more general than Corollaries 10–13 of section 5 [Z.II] obtained for the same system (1). The conditions and/or the control function in the assertions [Z.II, Sect. 5] depend on a Lyapunov matrix $L(t)$ transforming a system (1) with a time-varying periodic drift $F(t)y$ to a system (1) with a time-invariant drift. The conditions and the control function obtained here in Theorems 3, 4, and 5 and in Corollaries 1, 2, 3, and 4 do not depend on any Lyapunov transformation. Furthermore, Theorem 3 does not impose the same rigid condition 2) of Theorem 8 [Z.II] as Corollary 10 [Z.II] does. Therefore, Theorem 3 is stronger than Corollary 10 [Z.II]. Theorems 4 and 5 and Corollary 11 [Z.II] overlap, but do not follow from one another. The formulations of Theorems 4 and 5 are simpler and clearer than the formulation of Corollary 11 [Z.II]. Theorem 6 and Corollaries 3 and 4 are new.

§ 3. Consequence for consistent systems

Let us introduce the following notation. Let $\otimes$ be the tensor (Kronecker) product of matrices; let $\operatorname{vec} : M_n \to \mathbb{R}^{n^2}$ be the mapping that "expands" each matrix $H = \{h_{ij}\}$, $i, j = \overline{1, n}$, over rows into the column vector $\operatorname{vec} H = \operatorname{col}(h_{11}, \ldots, h_{1n}, \ldots, h_{n1}, \ldots, h_{nn}) \in \mathbb{R}^{n^2}$. For the system (1), we construct a linear control system

$$\dot z = R(t)z + T(t)v, \quad z \in \mathbb{R}^N, \quad v \in \mathbb{R}^r, \quad t \in \mathbb{R}_+. \tag{32}$$

Here $N = n^2$, $R(t) = F(t) \otimes I - I \otimes F^\top(t) \in M_N$, $I \in M_n$; $T(t) = [\operatorname{vec} G_1(t), \ldots, \operatorname{vec} G_r(t)] \in M_{N,r}$. We say that the system (32) is *the large system*. The definition of *consistency* for an arbitrary time-varying system (1) was introduced in [4]. Earlier [5], this notion was introduced for the system $\dot{y} = \big(F(t) + G(t)U(t)H(t)\big)y$, which is a special case of the system (1).

The system (1) is *consistent* on interval $[\alpha, \beta]$ if and only if the large system (32) is completely controllable on $[\alpha, \beta]$ [6, Proposition 2].

Suppose that the coefficients of the system (1) (and consequently, of the system (32)) are C^l functions. Let C_R be the operator that takes each matrix $T(t) \in M_{N,r}$ to the matrix $C_R T(t) = \dot{T}(t) - R(t)T(t) \in M_{N,r}$. Set $C_R^0 T(t) := T(t)$, $C_R^{i+1} T := C_R(C_R^i T)$. It is known [7, § 20] that if there exists a point t_0 on $[\alpha, \beta]$ at which the rank of the matrix $\Lambda(t) = [C_R^0 T(t), C_R^1 T(t), \ldots, C_R^{N-1} T(t)]$ is equal to N, then the system (32) is completely controllable on $[\alpha, \beta]$. If the coefficients of the system (32) are analytic, then the converse also holds [8].

We apply the mapping vec^{-1} to the columns of the matrix $C_R T(t)$. Then the kth column turns into the matrix $\dot{G}_k(t) - (F(t)G_k(t) - G_k(t)F(t))$, which is nothing else than the matrix $D_F G_k(t)$. Applying the mapping vec^{-1} to the kth column of the matrix $C_R^i T(t)$, we obtain the matrix $D_F^i G_k(t)$. Thus, the condition $\operatorname{rank} \Lambda(t_0) = N$ is equivalent to

$$\operatorname{span} \left\{ D_F^i G_k(t_0),\ k = \overline{1, r},\ i = \overline{0, n^2 - 1} \right\} = M_n. \tag{33}$$

Therefore the following proposition holds.

Proposition 1. *Suppose that the coefficients of the system* (1) *are analytic. Then the system* (1) *is consistent on* $[\alpha, \beta]$ *if and only if there exists a point* $t_0 \in [t_\alpha, t_\beta]$ *such that the equality* (33) *is satisfied.*

It follows that, for a system (1) that is consistent on some interval $[\alpha, \beta]$ and has analytic coefficients, the condition (30) holds for $\nu = n^2 - 1$.

Now assume that the coefficients of the system (1) (and consequently, of the system (32)) are T-periodic. If an arbitrary T-periodic system (32) is completely controllable on some $[\alpha, \beta]$, then it is completely controllable on any interval of length NT. Therefore, one does not need to specify the interval of complete controllability (and, accordingly, of consistency) for periodic systems. Proposition 1 and Corollary 3 imply the following assertion.

Corollary 5. *Suppose that the coefficients of the system* (1) *are analytic and T-periodic, and the following conditions hold:*

(a) *the system* (1) *is consistent;*

(b) *the system* (3) *is Lyapunov stable.*

Then the T-periodic control (5) *uniformly globally asymptotically stabilizes the null solution of the system* (1). *Here the matrix $P(t)$ is constructed using Theorem* 6.

Corollary 5 and Corollary 16 [Z.II] coincide. The difference is in the construction of stabilizing control. In Corollary 16 [Z.II], we additionally need to construct a Lyapunov transformation and the resulting control function is, generally speaking, $2T$-periodic.

§ 4. Proof of Theorem 6

We use the following notation in this section. Let $\mathbb{C}^n = \{x = \operatorname{col}(x_1, \ldots, x_n) : x_j \in \mathbb{C}, j = \overline{1, n}\}$ be the complex n-dimensional vector space with the norm $|x| = \sqrt{x^* x}$; let $*$ be the Hermitian conjugation of a vector or a matrix, i.e. $x^* = \overline{x}^\top$; let $M_{n,m}(\mathbb{C})$ be the space of the complex $n \times m$ matrices and $M_n(\mathbb{C}) := M_{n,n}(\mathbb{C})$; let $|A| = \max_{|x|=1} |Ax|$ be the norm in $M_n(\mathbb{C})$.

Suppose $R \in M_n(\mathbb{C})$ is a Hermitian matrix, i.e. $R = R^*$. It defines the quadratic form $V(x) = x^* R x$ on $\mathbb{C}^n$. For each $x \in \mathbb{C}^n$ we have $V(x) \in \mathbb{R}$. Indeed, let $R = P + iS$, $x = p + iq$, $P, S \in M_n$, $p, q \in \mathbb{R}^n$. The equality $R = R^*$ implies that $P^\top = P$, $S^\top = -S$. It follows that

$$V(x) = x^* R x = (p^\top - iq^\top)(P + iS)(p + iq) = p^\top P p + q^\top P q - 2p^\top S q. \tag{34}$$

The equality (34) implies, in particular, that if $x \in \mathbb{R}^n$, then

$$x^* R x = x^* (\operatorname{Re} R) x. \tag{35}$$

We identify the quadratic form $V(x) = x^* R x$ with the Hermitian matrix $R = R^*$ defining this form. The inequality $R > 0$ ($R \leqslant 0$) means for the matrix $R = R^*$ that the quadratic form $V(x) = x^* R x$ is positive definite (respectively, negative semi-definite), i.e. $\forall\, x \in \mathbb{C}^n \setminus \{0\}$ $V(x) > 0$ (respectively, $V(x) \leqslant 0$).

When all eigenvalues of $A \in M_n(\mathbb{C})$ have a negative real part, A is called a *Hurwitz matrix*. The following assertion holds (see, e.g., [9, Theorems 12.3.3, 13.1.1]).

Lemma 2. *Let a matrix $A \in M_n(\mathbb{C})$ be a Hurwitz matrix. Then for any positive definite Hermitian matrix $H \in M_n(\mathbb{C})$ there exists a positive definite Hermitian matrix $R \in M_n(\mathbb{C})$ such that*

$$A^* R + R A = -H. \tag{36}$$

The solution R of the Lyapunov equation (36) is unique. It is defined by the formula

$$R = \int_0^\infty \exp(A^* t) H \exp(At)\, dt. \tag{37}$$

The improper integral (37) converges. One can verify that the matrix R is Hermitian and positive definite.

Consider a system of differential equations

$$\dot{x} = Ax, \quad x \in \mathbb{C}^n. \tag{38}$$

Lemma 3. *Let the system* (38) *be Lyapunov stable. Then there exists a Hermitian matrix $Q = Q^* > 0$ such that $A^* Q + QA \leqslant 0$.*

This proposition was proved in Lemma 2 [Z.II] for a real system. Let us prove this Lemma for the complex system (38).

P r o o f. Let the system (38) be Lyapunov stable. Then there exists a matrix $S \in M_n(\mathbb{C})$ such that $A = S^{-1} J S$, where $J = \operatorname{diag}\{J_1, J_2\}$; here $J_1 \in M_\alpha(\mathbb{C})$, $\operatorname{Re} \lambda_j(J_1) = 0$, $j = \overline{1, \alpha}$ and the elementary divisors corresponding to the eigenvalues λ_j of the matrix J_1 are linear, i.e. $J_1 = \operatorname{diag}\{\delta_1 i, \ldots, \delta_\alpha i\}$, $\delta_j \in \mathbb{R}$, $j = \overline{1, \alpha}$; $J_2 \in M_{n-\alpha}(\mathbb{C})$, $\operatorname{Re} \lambda_j(J_2) < 0$, $j = \overline{1, n-\alpha}$. By structure of the matrix J_1, we have $J_1^* + J_1 = 0 \in M_\alpha(\mathbb{C})$.

Let $H \in M_{n-\alpha}(\mathbb{C})$ be an arbitrary Hermitian positive definite matrix, i.e. $H = H^* > 0$. Set $T = \operatorname{diag}\{0, H\} \in M_n(\mathbb{C})$, where $0 \in M_\alpha$. Then $T^* = T \geqslant 0$. Set $W = S^* T S$. Then we have $W^* = W \geqslant 0$ also. Since the matrix J_2 is a Hurwitz matrix, we see from Lemma 2 that there exists a matrix $R \in M_{n-\alpha}(\mathbb{C})$, $R^* = R > 0$ such that $J_2^* R + R J_2 = -H < 0$. Set $L = \operatorname{diag}\{I, R\} \in M_n(\mathbb{C})$, where $I \in M_\alpha$. Then $L^* = L > 0$. Set $Q = S^* L S$. Then we have $Q^* = Q > 0$ also. Thus,

$$\begin{aligned} A^* Q + QA = (S^{-1} J S)^* S^* L S + S^* L S (S^{-1} J S) = S^* (J^* L + L J) S = \\ = S^* \operatorname{diag}\{J_1^* + J_1, J_2^* R + R J_2\} S = S^* \operatorname{diag}\{0, -H\} S = S^* (-T) S = -W \leqslant 0. \end{aligned}$$

The lemma is proved.

P r o o f o f T h e o r e m 6. Let us prove the implication $1 \Rightarrow 2$. Let us construct the monodromy matrix $Y(T, 0)$ of the system (3). Let us find any logarithm of the matrix $Y(T, 0)$ and set

$$A = \frac{1}{T} \ln Y(T, 0). \tag{39}$$

Then $A \in M_n(\mathbb{C})$ and $\exp(AT) = Y(T, 0)$. Set

$$L(t) = \exp(At) Y(0, t), \quad t \in \mathbb{R}_+. \tag{40}$$

Since $F(t)$ is a C^l function, it follows that $Y(0,t)$ is a C^{l+1} function; hence, $L(t)$ is a C^{l+1} function. Matrix $L(t)$ is complex, T-periodic. It satisfies the equation

$$\dot{L}(t) = AL(t) - L(t)F(t), \quad t \in \mathbb{R}_+. \tag{41}$$

In addition, $L(t)$ is a Lyapunov matrix, i.e. $\sup_{t\in\mathbb{R}_+}\big(|L(t)| + |L^{-1}(t)| + |\dot{L}(t)|\big) < \infty$. Let us apply the Lyapunov transformation $x = L(t)y$ to the system (3). The system (3) turns into the system

$$\dot{x} = Ax, \quad x \in \mathbb{C}^n, \quad A \in M_n(\mathbb{C}).$$

This system is Lyapunov stable, because the Lyapunov transformation preserves the property of stability. By Lemma 3, there exists a matrix $Q = Q^* > 0$ such that

$$A^*Q + QA \leqslant 0. \tag{42}$$

Let us construct the matrix

$$R(t) = L^*(t)QL(t). \tag{43}$$

Then $R \in C^{l+1}\big(\mathbb{R}_+, M_n(\mathbb{C})\big)$, $R(t+T) \equiv R(t)$, $R(t) = R^*(t) > 0$, $t \in \mathbb{R}_+$. We write $R(t)$ in the form $R(t) = P(t) + iS(t)$, $P(t), S(t) \in M_n$. Here

$$P(t) = \big(R(t) + \overline{R(t)}\big)/2, \qquad S(t) = (-i)\big(R(t) - \overline{R(t)}\big)/2. \tag{44}$$

Then $P \in C^{l+1}\big(\mathbb{R}_+, M_n)\big)$, $P(t+T) \equiv P(t)$, $P(t) = P^\top(t) > 0$, $t \in \mathbb{R}_+$. Let us show that $W_F P(t) \leqslant 0$. We have $W_F R(t) = W_F P(t) + iW_F S(t)$. It follows from the equality (35) that for any $y \in \mathbb{R}^n$ the equality

$$y^\top (W_F R)(t)y = y^\top (W_F P)(t)y \tag{45}$$

holds. On the other hand, using (41), (43), and omitting $t \in \mathbb{R}_+$ for simplicity, we get

$$\begin{aligned} W_F R = F^\top R + RF + \dot{R} = F^*R + RF + \dot{L}^*QL + L^*Q\dot{L} = \\ = F^*L^*QL + L^*QLF + (AL - LF)^*QL + L^*Q(AL - LF) = L^*(A^*Q + QA)L. \end{aligned} \tag{46}$$

Hence, equality (46) and inequality (42) imply that for any $y \in \mathbb{C}^n$ the inequality

$$y^*(W_F R)(t)y = y^*L^*(t)\big(A^*Q + QA\big)L(t)y \leqslant 0$$

holds. By (45), for any $y \in \mathbb{R}^n$, we have

$$y^\top (W_F P)(t)y = y^\top (W_F R)(t)y = y^*(W_F R)(t)y \leqslant 0.$$

Thus, the matrix $P(t)$ is the desired symmetric matrix. For constructing this matrix, we used formulas (39), (40), (42), (43), (44). Theorem 6 is proved.

REFERENCES

1. Krasovskii N.N. *Nekotorye zadachi teorii ustoichivosti dvizheniya* (Some problems of the theory of stability of motion), Moscow: Fizmatgiz, 1959, 221 p.
2. Rouche N., Habets P., Laloy M. *Stability theory by Liapunov's direct method*, New York–Heidelberg–Berlin: Springer–Verlag, 1977, 396 p.
3. Khalil H. *Nonlinear Systems. Third Edition*, NJ: Prentice Hall, Upper Saddle River, 2002, 750 p.
4. Zaitsev V.A., Tonkov Ye.L. Attainability, consistency and the rotation method by V.M. Millionshchikov, *Russian Mathematics*, 1999, vol. 43, no. 2, pp. 42–52.
5. Popova S.N., Tonkov E.L. Control over the Lyapunov exponents of consistent systems. I, *Differential Equations*, 1994, vol. 30, no. 10, pp. 1556–1564.
6. Zaitsev V.A. Consistent Systems and Pole Assignment: I, *Differential Equations*, 2012, vol. 48, no. 1, pp. 120–135.
7. Krasovskii N.N. *Teoriya upravleniya dvizheniem* (Control Theory of Motion), Moscow: Nauka, 1968.

8. Chang A. An algebraic characterization of controllability, *IEEE Transactions on Automatic Control*, 1965, vol. AC–10, no. 1, pp. 112–113.
9. Lancaster P., Tismenetsky M. *The Theory of Matrices. Second Edition with Applications*, Academic Press, 1985, 570 p.

Received 25.03.2012

Zaitsev Vasili Aleksandrovich, Candidate of Physics and Mathematics, Associate Professor, Department of Differential Equations, Udmurt State University, ul. Universitetskaya, 1, Izhevsk, 426034, Russia.
E-mail: verba@udm.ru

В. А. Зайцев
Глобальная асимптотическая стабилизация билинейных управляемых систем с периодическими коэффициентами

Ключевые слова: глобальная асимптотическая устойчивость, стабилизация, функция Ляпунова, билинейные системы, периодические системы.

УДК 517.977 + 517.925.51

Для билинейной управляемой системы с периодическими коэффициентами получены достаточные условия равномерной глобальной асимптотической стабилизации нулевого решения. Доказательство основано на применении теоремы Красовского об асимптотической устойчивости в целом нулевого решения для периодических систем. Стабилизирующее управление построено по принципу обратной связи. Оно имеет вид квадратичной формы от фазовой переменной и является периодическим по времени.

Поступила в редакцию 25.03.2012

Зайцев Василий Александрович, к. ф.-м. н., доцент, кафедра дифференциальных уравнений, Удмуртский государственный университет, 426034, Россия, г. Ижевск, ул. Университетская, 1.
E-mail: verba@udm.ru