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ABOUT ONE TYPE OF SEQUENCES THAT ARE NOT A SHAUDER BASIS IN HILBERT SPACES

Let H be a Hilbert space and a (not necessarily bounded) sequence of its elements $\{e_n\}_{n=1}^{\infty}$ has a bounded subsequence $\{e_{n_k}\}_{k=1}^{\infty}$ such that $|(e_{n_k}, e_{n_m})| \geqslant \alpha > 0$ for all sufficiently large $k, m \in N, k \neq m$. It is proved that such a sequence $\{e_n\}_{n=1}^{\infty}$ is not a basic sequence and thus is not a Schauder basis in H. Note that the results of this paper generalize and offer a short and more simple proof of some recent results obtained in this direction.

Keywords: Schauder basis, basic sequence, Hilbert space, orthonormal sequence and orthonormal basis, weakly convergent sequences.

Introduction

We begin by recalling some notions.

Definition 1 (see [1, 2, 3]). A sequence of vectors $\{x_n\}_{n=1}^{\infty}$ in an infinite-dimensional Banach space X is said to be a *Schauder basis for* X if to each vector x in the space there corresponds a unique sequence of scalars $\{\alpha_n\}_{n=1}^{\infty}$ such that

$$x = \alpha_1 \cdot x_1 + \ldots + \alpha_n \cdot x_n + \ldots .$$

Definition 2 (see [3]). A sequence of vectors $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is said to be a *basic sequence* if it is a Schauder basis for the closure of its linear span.

It is well known that every separable Hilbert space possesses an orthonormal Schauder bases, i. e. a Schauder basis $\{e_n\}_{n=1}^{\infty}$ for which $\|e_n\| = 1$ and $(e_n, e_m) = 0$ for every $n, m \in N, n \neq m$. Besides it, it can easily be shown that every sequence $\{e_n\}_{n=1}^{\infty}$ of elements in any Hilbert space with the properties $\|e_n\| = 1$ and $(e_n, e_m) = 0$ for every $n, m \in N, n \neq m$, is a basic sequence in this space. It is easy to see that the number 1 in this formulation can easily be replaced by the other number by retaining the mentioned property. Therefore, it is natural to ask what would happen if the number 0 is replaced by the another number. First result in this direction (that is known to us) is obtained in [4].

Theorem 1 (see [4]). *Let H be a Hilbert space and $\{x_n\}_{n=1}^{\infty}$ be a sequence of elements in H with the following properties:*
1) $\|x_n\| = 1$ *for all* $n \in N$; 2) $(x_n, x_m) = a$, $0 < |a| < 1$; $n, m \in N$, $n \neq m$.
Then $\{x_n\}_{n=1}^{\infty}$ is not a basic sequence in H.

The next result in this direction is the following.

Theorem 2 (see [5]). *Let H be a Hilbert space and $\{x_n\}_{n=1}^{\infty}$ be a complete sequence of elements in H with the following properties:*
1) $\|x_n\| = 1$ *for all* $n \in N$; 2) $(x_n, x_m) \geqslant a > 0$, $n, m \in N$, $n \neq m$.
Then $\{x_n\}_{n=1}^{\infty}$ is not a Schauder basis in H.

Some time later Sadybekov and Sarsenby [6], investigating the unconditional basicity of sequences, obtained the following result, where they require from the sequence to be almost normalized instead of being normalized.

Theorem 3 (see [6]). *Let H be a Hilbert space and $\{x_n\}_{n=1}^{\infty}$ be a complete, minimal and an almost normalized sequence of elements in H with the property $|(x_n, x_m)| \geqslant a > 0$ for all sufficiently large numbers n and m. Then $\{x_n\}_{n=1}^{\infty}$ is not an unconditional basis in H.*

In this note we state a result that generalizes all the mentioned results and, besides it, offers a short and simpler proof of these results.

§ 1. Main result and its proof

Theorem 4. *Let H be a Hilbert space and a bounded sequence of its elements $\{e_n\}_{n=1}^{\infty}$ satisfies $|(e_n, e_m)| \geqslant \alpha > 0$ for $n, m \in N, n \neq m$. Then $\{e_n\}_{n=1}^{\infty}$ is not a basic sequence (and thus a Schauder basis) in H.*

P r o o f. Assume the contrary: the sequence $\{e_n\}_{n=1}^{\infty}$ satisfies the conditions of the theorem and is a basic sequence.

Since $\{e_n\}_{n=1}^{\infty}$ is assumed to be bounded, it has a weakly convergent subsequence $\{e_{n_k}\}_{k=1}^{\infty}$ (see, for example, [7, p. 81]); let x_0 be its weak limit. It is known that every subsequence of a basic sequence is also a basic sequence (it follows, for example, from [3, Theorem 1.1']). Therefore, the subsequence $\{e_{n_k}\}_{k=1}^{\infty}$ is also a basic sequence. Hence, it has a biorthogonal system, i. e. a sequence of elements $\{b_k\}_{k=1}^{\infty}$ such that

$$(b_k, e_{n_m}) = \delta_{km}, \tag{1}$$

where δ_{km} is a Kronecker symbol. Here, passing to the limit as $m \to \infty$ and taking into account that x_0 is a weak limit of $\{e_{n_m}\}_{m=1}^{\infty}$, we obtain that

$$(b_k, x_0) = 0 \quad \forall k \in N. \tag{2}$$

By the condition of the theorem we have $|(e_{n_k}, e_{n_m})| \geqslant a > 0$ for all $k, m \in N, n \neq m$. Therefore, taking into account that x_0 is a weak limit of $\{e_{n_m}\}_{m=1}^{\infty}$, by passing to the limit at first as $k \to \infty$ and then passing to the limit as $m \to \infty$, we obtain that $|(x_0, x_0)| > 0$. This relation implies that $x_0 \neq \theta$.

Now, since the weak limit of a sequence of elements lies in the closure of its linear span (see, for example, [7, p. 81] or [1, p. 216]), x_0 must have a representation

$$x_0 = \alpha_1 \cdot e_{n_1} + \alpha_2 \cdot e_{n_2} + \ldots + \alpha_k \cdot e_{n_k} + \ldots . \tag{3}$$

We find from here that

$$(b_k, x_0) = \alpha_1 \cdot (b_k, e_{n_1}) + \alpha_2 \cdot (b_k, e_{n_2}) + \ldots + \alpha_k \cdot (b_k, e_{n_k}) + \ldots$$

for all $k \in N$. From here, by using (1) and (2), we obtain that

$$\alpha_k = 0 \quad \forall k \in N.$$

These relations and (3) imply that $x_0 = \theta$. But this contradicts to the fact that $x_0 \neq \theta$. The obtained contradiction shows that our assumption is false. The theorem is proved. □

§ 2. Concluding remarks

As was already mentioned, if a sequence is a basic sequence then every subsequence of this original sequence is also a basic sequence (it follows, for example, from [3, Theorem 1.1']). This observation and the proof of the theorem from the previous section imply that the following more general result holds true.

Theorem 5. *Let H be a Hilbert space and a (not necessarily bounded) sequence of its elements $\{e_n\}_{n=1}^{\infty}$ has a bounded subsequence $\{e_{n_k}\}_{k=1}^{\infty}$ such that $|(e_{n_k}, e_{n_m})| \geqslant \alpha > 0$ for all sufficiently large $k, m \in N, k \neq m$. Then $\{e_n\}_{n=1}^{\infty}$ is not a basic sequence (and thus a Schauder basis) in H.*

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Об одном классе последовательностей, не являющихся базисом Шаудера в гильбертовом пространстве

Ключевые слова: базис Шаудера, базисная последовательность, гильбертово пространство, ортонормированная последовательность и ортонормированный базис, слабо сходящиеся последовательности.

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Пусть H — гильбертово пространство и (необязательно ограниченная) последовательность $\{e_n\}_{n=1}^{\infty}$ его элементов содержит ограниченную подпоследовательность $\{e_{n_k}\}_{k=1}^{\infty}$ такую, что $|(e_{n_k}, e_{n_m})| \geqslant \alpha > 0$ для любых достаточно больших $k, m \in N, k \neq m$. Доказано, что такая последовательность $\{e_n\}_{n=1}^{\infty}$ не является базисной последовательностью и, следовательно, базисом Шаудера в пространстве H. Полученные результаты обобщают и предлагают короткое и более простое доказательство некоторых недавних результатов, полученных в этом направлении.

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