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UNLOCKING OF PREDICATE: APPLICATION TO CONSTRUCTING A NON-ANTICIPATING SELECTION

We consider an approach to constructing a non-anticipating selection of a multivalued mapping; such a problem arises in control theory under conditions of uncertainty. The approach is called “unlocking of predicate” and consists in the reduction of finding the truth set of a predicate to searching fixed points of some mappings. Unlocking of predicate gives an extra opportunity to analyze the truth set and to build its elements with desired properties. In this article, we outline how to build “unlocking mappings” for some general types of predicates: we give a formal definition of the predicate unlocking operation, the rules for the construction and calculation of “unlocking mappings” and their basic properties. As an illustration, we routinely construct two unlocking mappings for the predicate “be non-anticipating mapping” and then on this base we provide the expression for the greatest non-anticipating selection of a given multifunction.

Keywords: predicate unlocking, fixed points, nonanticipating mappings.

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Introduction

We consider an approach to constructing a non-anticipating selection of a multivalued mapping; such a problem arises in control theory under conditions of uncertainty. The approach is called “unlocking of predicate” and consists in the reduction of finding the truth set of a predicate to searching fixed points of some mappings. Unlocking of predicate gives an extra opportunity to analyze the truth set and to build its elements with desired properties.

This concept is used in many fields of mathematics: in differential equations and differential inclusions; in game theory, when studying the saddle points (see [1]) and the Nash equilibria (see [2,3]); in dynamic games, when constructing the stable (weakly invariant) sets (see [4,5]) and non-anticipating selections of multivalued mappings (see [6,7]). However, in all the above cases “unlocking mappings” are presented as a ready–made product: a method for constructing an “unlocking mapping” has remained beyond the consideration.

In this article, we outline how to build “unlocking mappings” for some general types of predicates: we give a formal definition of the predicate unlocking operation, the rules for the construction and calculation of “unlocking mappings” and their basic properties. As an illustration, we routinely construct two unlocking mappings for the predicate “be non-anticipating mapping” and then on this base we provide the expression for the greatest non-anticipating selection of a given multifunction. This work continues [8] where the procedure for the predicate "be Nash equilibrium" is presented.

§ 1. Notation and definitions

1. Hereinafter, we use the set–theoretic symbols (quantifiers, propositional bundles, $\varnothing$ for the empty set); $\overset{\text{def}}{=}$ for the equality by definition; $\overset{\text{def}}{\Leftrightarrow}$ for the equivalence by definition. We accept the axiom of choice. A set consisting of sets is called a family. By $\mathcal{P}(T)$ (by $\mathcal{P}'(T)$), we denote the family of all (all nonempty) subsets of an arbitrary set T; the family $\mathcal{P}(T)$ is also called Boolean of the set T. If A and B are non-empty sets, then B^A is the set of all functions from the set A to the set B (see [9]). If $f \in B^A$ and $C \in \mathcal{P}'(A)$, then $(f\,|\,C) \in B^C$ is the restriction of f to the set C: $(f\,|\,C)(x) \overset{\text{def}}{=} f(x)\ \forall x \in C$. We denote the image of the set $C \in \mathcal{P}(A)$ under the function f by $f(C)$: $f(C) \overset{\text{def}}{=} \{f(x) : x \in C\}$. When $f \in \mathcal{P}(B)^A$, f is called a multivalued function or multifunction

(m/f) from A in B. The term “mapping” means a function or m/f. In case $F \in \mathcal{P}'(B^A)$, we denote $(F\,|\,C) \overset{\mathrm{def}}{=} \{(f\,|\,C) : f \in F\}$. If $f \in B^A$, we denote by f^{-1} the m/f from B into A defined by the rule

$$f^{-1}(b) \overset{\mathrm{def}}{=} \begin{cases} \{a \in A \mid b = f(a)\}, & b \in f(A), \\ \varnothing, & b \notin f(A) \end{cases} \qquad \forall b \in B.$$

We call the m/f f^{-1} *inverse mapping*. If $f \in \mathcal{P}(B)^A$, i. e. f is a m/f, we define f^{-1} by

$$f^{-1}(b) \overset{\mathrm{def}}{=} \begin{cases} \{a \in A \mid b \in f(a)\}, & b \in \bigcup f(A), \\ \varnothing, & b \notin \bigcup f(A) \end{cases} \qquad \forall b \in B.$$

For any $f \in X^X$ we denote by $\mathbf{Fix}(f)$ the set of all fixed points of f: $\mathbf{Fix}(f) \overset{\mathrm{def}}{=} \{x \in X \mid f(x) = x\}$. In the case when f is a m/f, the set $\mathbf{Fix}(f)$ is defined by: $\mathbf{Fix}(f) \overset{\mathrm{def}}{=} \{x \in X \mid x \in f(x)\}$.

2. *A predicate* P on a non-empty set X is identified with the same name function from $\{0,1\}^X$. We say that $x \in X$ satisfies the predicate P and write it down by $P(x)$ iff $P(x) = 1$. The set of all $x \in X$ satisfying the predicate P is called *the set of truth* (of the predicate P). Following the definition of an inverse mapping, we denote this set by $P^{-1}(1)$. The set of all predicates on X is denoted by $\mathfrak{PR}(X)$. We denote by $\mathfrak{T}$ (by $\mathfrak{F}$) the predicate on X defined by: $\mathfrak{T}^{-1}(1) = X$ ($\mathfrak{F}^{-1}(0) = X$). Hence, for any $P \in \mathfrak{PR}(X)$, the equalities $P = \mathfrak{T}\&P = \mathfrak{F} \vee P$, where “&” (“$\vee$”) denotes logical “and” (“or”), are valid.

We call *unlocking of predicate* P the operation of constructing a mapping $\mathcal{F}_P \in \mathcal{P}(X)^X \cup X^X$ that satisfies the condition

$$\mathbf{Fix}(\mathcal{F}_P) = P^{-1}(1). \tag{1.1}$$

The mapping $\mathcal{F}_P$ with property (1.1), is called *unlocking mapping* (for the predicate P). Denote by $\mathfrak{UM}(P)$ the set of all unlocking m/f for the predicate P. Thus, $\mathfrak{UM}(P) \in \mathcal{P}'(\mathcal{P}(X)^X)$. The formal exclusion of functions (the set X^X) from $\mathfrak{UM}(P)$ is dummy, because every function f satisfying $\mathbf{Fix}(f) = P^{-1}(1)$ is represented by the m/f F_f in $\mathfrak{UM}(P)$: $F_f(x) \overset{\mathrm{def}}{=} \{f(x)\}\ \forall x \in X$. So, for a function f we write down $f \in \mathfrak{UM}(P)$ keeping in mind the inclusion $F_f \in \mathfrak{UM}(P)$.

3. For any set $X \neq \varnothing$ and a partial ordering relation $\preccurlyeq \in \mathcal{P}'(X \times X)$, we denote by $(X, \preccurlyeq)$ the corresponding partially ordered set (poset). A set $C \subset X$ is called a *chain* if it is totally ordered by $\preccurlyeq$: $(x \preccurlyeq y) \vee (y \preccurlyeq x)\ \forall x, y \in C$. In particular, $\varnothing$ is a chain. Following [10], we call a poset $(X, \preccurlyeq)$ a *chain–complete* poset if there exists the greatest lower bound $\inf C \in X$ for any chain $C \subset X$. In particular, every chain–complete poset $(X, \preccurlyeq)$ has the greatest element $\top \in X$ (the greatest lower bound of the empty chain), and, thus, it is not empty. For $Y \in \mathcal{P}(X)$, we denote by $\top_Y$ and $\bot_Y$ the greatest and the least elements of the set Y, respectively, if they exist. A poset is called a *complete lattice* iff any subset has the greatest and the least elements. So, any complete lattice is a chain–complete poset. Let $(X, \preccurlyeq)$ be a non-empty poset and $f \in X^X$. The function f is called *restrictive* if $f(x) \preccurlyeq x$ for every $x \in X$. The function f is called *isotone* if the implication $(x \preccurlyeq y) \Rightarrow (f(x) \preccurlyeq f(y))$ holds for all $x, y \in X$.

4. Denote the class of ordinals by $\mathbf{ORD}$. For a set X, we denote by $|X|$ the least ordinal that is equipotent to the set X (the cardinal of X). The relation of order (strict order) on the class of cardinals is denoted by $<=$ ($<$). For any set H, let $|H|^+ \in \mathbf{ORD}$ be the least ordinal among the ordinals η with the property $|H| < |\eta|$.

§ 2. Calculus of unlocking mappings

2.1. The order, restrictions and logical operations

1. Let X be a nonempty set. On the set $\mathcal{P}(X)^X$, we introduce the partial order $\preccurlyeq$, assuming that $(g \preccurlyeq f) \overset{\mathrm{def}}{\Leftrightarrow} (g(x) \subset f(x)\ \forall x \in X)\ \ \forall f,\ g \in \mathcal{P}(X)^X$ Then we have the equivalence $(g \preccurlyeq f) \Leftrightarrow (g^{-1} \preccurlyeq f^{-1})$. The poset $(\mathcal{P}(X)^X, \preccurlyeq)$ is a complete lattice. It is also easy to check that,

for any $P \in \mathfrak{PR}(X)$, the poset $(\mathfrak{UM}(P), \preccurlyeq)$ forms a complete sublattice (a subset being a complete lattice) in $(\mathcal{P}(X)^X, \preccurlyeq)$ and the equalities are true:

$$\top_{\mathfrak{UM}(P)}(x) = \begin{cases} X, & P(x), \\ X \setminus \{x\}, & \neg P(x), \end{cases} \qquad \bot_{\mathfrak{UM}(P)}(x) = \begin{cases} \{x\}, & P(x), \\ \varnothing, & \neg P(x). \end{cases}$$

In particular, for the predicates $\mathfrak{T}$, $\mathfrak{F}$, the relations $\top_{\mathfrak{UM}(\mathfrak{T})}(x) = X$, $\bot_{\mathfrak{UM}(\mathfrak{T})}(x) = \{x\}$, $\top_{\mathfrak{UM}(\mathfrak{F})}(x) = X \setminus \{x\}$, and $\bot_{\mathfrak{UM}(\mathfrak{F})(x)} = \varnothing$ are valid for all $x \in X$. By definition, we have $\top_{\mathfrak{UM}(P)} = \top^{-1}_{\mathfrak{UM}(P)}$, $\bot_{\mathfrak{UM}(P)} = \bot^{-1}_{\mathfrak{UM}(P)}$.

Lemma 1. *For all $f \in \mathcal{P}(X)^X$, the relations*

$$(f \preccurlyeq \top_{\mathfrak{UM}(P)}) \Rightarrow (\mathbf{Fix}(f) \subset P^{-1}(1)), \qquad (\bot_{\mathfrak{UM}(P)} \preccurlyeq f) \Rightarrow (P^{-1}(1) \subset \mathbf{Fix}(f))$$

are fulfilled. Consequently,

$$\mathfrak{UM}(P) = \{f \in \mathcal{P}(X)^X \mid \bot_{\mathfrak{UM}(P)} \preccurlyeq f \preccurlyeq \top_{\mathfrak{UM}(P)}\}, \qquad (f \in \mathfrak{UM}(P)) \Leftrightarrow (f^{-1} \in \mathfrak{UM}(P)).$$

2. For any $\phi \in \mathcal{P}(X)^X$ and $Y \in \mathcal{P}'(X)$, we denote by $[\phi \,|\, Y]$ the following mapping $[\phi \,|\, Y](y) \overset{\text{def}}{=} Y \cap \phi(y)$ $\forall y \in Y$. Recall that the restriction $(P \,|\, Y) \in \mathfrak{PR}(Y) \overset{\text{def}}{=} \{0,1\}^Y$ of $P \in \mathfrak{PR}(X)$ is defined by $(P \,|\, Y)(y) \overset{\text{def}}{=} P(y)$, $\forall y \in Y$.

Lemma 2. *For all $P \in \mathfrak{PR}(X)$, $Y \in \mathcal{P}'(X)$ the equalities $\mathfrak{UM}((P \,|\, Y)) = \{[\phi \,|\, Y] : \phi \in \mathfrak{UM}(P)\}$ are valid.*

3. The following lemma provides unlocking mappings for some expressions of propositional logic.

Lemma 3. *If $P, Q \in \mathfrak{PR}(X)$, then the equalities are valid:*

$$\mathfrak{UM}(\neg P) = \{f \in \mathcal{P}(X)^X \mid \exists g \in \mathfrak{UM}(P) : f(x) = X \setminus g(x)\ \forall x \in X\},$$
$$\mathfrak{UM}(P \& Q) = \{f \in \mathcal{P}(X)^X \mid \exists g \in \mathfrak{UM}(P) \exists q \in \mathfrak{UM}(Q) : f(x) = g(x) \cap q(x)\ \forall x \in X\},$$
$$\mathfrak{UM}(P \vee Q) = \{f \in \mathcal{P}(X)^X \mid \exists g \in \mathfrak{UM}(P) \exists q \in \mathfrak{UM}(Q) : f(x) = g(x) \cup q(x)\ \forall x \in X\}.$$

Using the above relations, one can construct unlocking mappings for a variety of other propositional logic expressions.

Corollary 1. *For all $P \in \mathfrak{PR}(X)$, $f \in \mathfrak{UM}(P)$, $T \in \mathfrak{UM}(\mathfrak{T})$, and $F \in \mathfrak{UM}(\mathfrak{F})$, the m/f f_T, $f_F \in \mathcal{P}(X)^X$ defined by $f_T(x) \overset{\text{def}}{=} T(x) \cap f(x)$, $f_F(x) \overset{\text{def}}{=} F(x) \cup f(x)$ $\forall x \in X$, are unlocking m/f for the predicate P: $f_T, f_F \in \mathfrak{UM}(P)$. In addition, we have the relations:*

$$\bot_{\mathfrak{UM}(P)}(x) = \bot_{\mathfrak{UM}(\mathfrak{T})}(x) \cap f(x) = \{x\} \cap f(x),$$
$$\top_{\mathfrak{UM}(P)}(x) = \top_{\mathfrak{UM}(\mathfrak{F})}(x) \cup f(x) = (X \setminus \{x\}) \cup f(x).$$

4. Lemma 4 is based on the corollary 1 and allows us to construct an unlocking function from a given unlocking m/f in the case when X is an ordered set. Let $(X, \leqslant)$ be a poset and the m/f $\mathbf{LE}_X \in \mathcal{P}(X)^X$ is defined by

$$\mathbf{LE}_X(x) \overset{\text{def}}{=} \{y \in X \mid y \leqslant x\}. \tag{2.1}$$

Notice that $\mathbf{LE}_X \in \mathfrak{UM}(\mathfrak{T})$.

Lemma 4. *Let $(X, \preccurlyeq)$ be a nonempty poset, $P \in \mathfrak{PR}(X)$, and $f \in \mathfrak{UM}(P)$. Let $G \in \mathcal{P}(X)^X$ be defined by*

$$G(x) \overset{\text{def}}{=} \mathbf{LE}_X(x) \cap f(x), \qquad x \in X, \tag{2.2}$$

$Y \overset{\text{def}}{=} \{y \in X \mid G(y) \neq \varnothing\}$, and the function $g \in Y^Y$ be defined by

$$g(x) \overset{\text{def}}{=} \begin{cases} \top_{G(x)}, & \exists \top_{G(x)}, \\ y \in G(x), & \neg\exists \top_{G(x)}, \end{cases} \qquad x \in Y.$$

Then g is restrictive on $(Y, \preccurlyeq)$ and $\mathbf{Fix}(g) = P^{-1}(1)$.

2.2. Unlocking the conjunction of predicates defined on a product

The conjunction of a set of predicates is an important particular case. Using this peculiarity, lemma 5 gives the construction of the corresponding unlocking m/f.

Let $\mathcal{I}$, $(X_\iota)_{\iota\in\mathcal{I}}$ be non-empty sets and

$$X \overset{\text{def}}{=} \prod_{\iota\in\mathcal{I}} X_\iota. \tag{2.3}$$

We call an element $x \in X$ a *tuple from* X (or simply a *tuple* if the set X is fixed) and denote the ι–th element of the tuple x by x_ι: $x_\iota \overset{\text{def}}{=} (x \,|\, \{\iota\}) \in X_\iota$. Denote by $(y, x_{-\iota})$ the tuple from X resulted from the tuple $x \in X$ by substituting the element $y \in X_\iota$ at the position of x_ι:

$$(y, x_{-\iota})_\jmath \overset{\text{def}}{=} \begin{cases} y, & \jmath = \iota, \\ x_\jmath, & \jmath \in \mathcal{I} \setminus \{\iota\} \end{cases} \qquad \forall x \in X \ \forall y \in X_\iota \ \forall \iota \in \mathcal{I}.$$

Let a family of predicates $P_\jmath \in \mathfrak{PR}(X)$, $\jmath \in \mathcal{J}$ on the product X be given. Let the predicate $P \in \mathfrak{PR}(X)$ have the form $P(x) \overset{\text{def}}{\Leftrightarrow} (P_\jmath(x) \ \forall \jmath \in \mathcal{J})$ $x \in X$. Let $|\mathcal{J}| <= |\mathcal{I}|$ and $q \in \mathcal{I}^{\mathcal{J}}$ be the corresponding injection of $\mathcal{J}$ into $\mathcal{I}$. Define the m/f $\mathcal{F}_P \in \mathcal{P}(X)^X$ as follows:

$$\mathcal{F}_P(x) \overset{\text{def}}{=} \prod_{\iota\in\mathcal{I}} \mathcal{B}_\iota(x) \qquad \forall x \in X, \tag{2.4}$$

where m/f $\mathcal{B}_\iota, \mathcal{B}_{\iota\jmath} \in \mathcal{P}(X_\iota)^X$ are defined by

$$\begin{gathered} \mathcal{B}_\iota(x) \overset{\text{def}}{=} \begin{cases} \mathcal{B}_{\iota q^{-1}(\iota)}(x)((y, x_{-\iota}))\}, & \iota \in q(\mathcal{J}), \\ X_\iota, & \iota \notin q(\mathcal{J}), \end{cases} \\ \mathcal{B}_{\iota\jmath}(x) \overset{\text{def}}{=} \{y \in X_\iota \mid P_\jmath((y, x_{-\iota}))\}, \quad x \in X, \ \iota \in \mathcal{I}, \ \jmath \in \mathcal{J}. \end{gathered} \tag{2.5}$$

Lemma 5. $\mathcal{F}_P \in \mathfrak{UM}(P)$.

§ 3. The greatest non-anticipating selection

In [6, 7], the representation of non-anticipating selections of a m/f as the set of fixpoins of a function (noted by Γ) is provided. In other words, the unlocking of predicate "be non-anticipating selection" is fulfilled. At the same time, the process of constructing the function Γ remained out of consideration. In this section, we carry out the process using constructions from [6, 7] and relations from the section 2.

3.1. Notation and definitions

Hereinafter, we fix $D \overset{\text{def}}{=} I \times X$, where I and X are non-empty sets. Select the set $\mathbf{C} \in \mathcal{P}'(X^I)$ whose elements are considered as "realizations of control actions". So, the sets I and X are analogues of time and state space respectively. Then we select and fix the sets Y and $\Omega \in \mathcal{P}'(Y^I)$. Elements of Ω are treated as "realizations of uncertainty factors". Let $\mathbf{M} \overset{\text{def}}{=} \mathcal{P}(\mathbf{C})^\Omega$ denote the set of all m/f from Ω into $\mathbf{C}$: $\alpha(\omega) \subset \mathbf{C}$ for any $\omega \in \Omega$, $\alpha \in \mathbf{M}$.

The partial order $\sqsubseteq$ on $\mathbf{M}$ is defined by

$$(\phi \sqsubseteq \psi) \overset{\text{def}}{\Leftrightarrow} (\phi(\omega) \subset \psi(\omega) \quad \forall \omega \in \Omega) \qquad \forall \phi, \psi \in \mathbf{M}.$$

One can verify that the poset $(\mathbf{M}, \sqsubseteq)$ is a complete lattice. For any $\phi, \psi \in \mathbf{M}$, we call m/f ϕ a *selection* of ψ iff $\phi \sqsubseteq \psi$.

Let $\mathcal{X} \in \mathcal{P}(I)$ be a non–empty set and $\Omega(\omega \,|\, A) \overset{\text{def}}{=} \{\nu \in \Omega \mid (\nu \,|\, A) = (\omega \,|\, A)\}$ $\forall \omega \in \Omega$, $\forall A \in \mathcal{X}$. A m/f $\phi \in \mathbf{M}$ is called $\mathcal{X}$*–non-anticipating*, iff

$$(\omega' \in \Omega(\omega \,|\, A)) \Rightarrow \big((\phi(\omega) \,|\, A) \subset (\phi(\omega') \,|\, A)\big) \qquad \forall A \in \mathcal{X}, \ \forall \omega, \omega' \in \Omega. \tag{3.1}$$

Remark 1. Due to the equivalence

$$(\omega' \in \Omega(\omega \,|\, A)) \Leftrightarrow (\omega \in \Omega(\omega' \,|\, A)) \Leftrightarrow ((\omega \,|\, A) = (\omega' \,|\, A)) \qquad \forall A \in \mathcal{X},\ \forall \omega, \omega' \in \Omega,$$

implications (3.1) are equivalent to the relations

$$((\omega \,|\, A) = (\omega' \,|\, A)) \Rightarrow \big((\phi(\omega) \,|\, A) = (\phi(\omega') \,|\, A)\big) \qquad \forall A \in \mathcal{X},\ \forall \omega, \omega' \in \Omega,$$

which are usually considered as the definition of non-anticipating property.

Fix the family $\mathcal{X}$ and a m/f $\mathcal{M} \in \mathbf{M}$. Our aim is to find the greatest in $(\mathbf{M}, \sqsubseteq)$ $\mathcal{X}$-non-anticipating selection of the m/f $\mathcal{M}$. So, we should find a m/f $\phi \in \mathbf{M}$, satisfying condition (3.1), the inequality $\phi \sqsubseteq \mathcal{M}$, and such that the relation $\beta \sqsubseteq \phi$ is valid for any $\beta \in \mathbf{M}$ satisfying (3.1) and the inequality $\beta \sqsubseteq \mathcal{M}$.

For the analysis of the problem above, we define the predicate $P_{na} \in \mathfrak{PR}(\mathbf{M})$ "be $\mathcal{X}$-non-anticipating mapping" by

$$P_{na}(\phi) \overset{\text{def}}{\Leftrightarrow} \big((\omega' \in \Omega(\omega \,|\, A)) \Rightarrow \big((\phi(\omega) \,|\, A) \subset (\phi(\omega') \,|\, A)\big)\ \forall A \in \mathcal{X}\ \forall \omega, \omega' \in \Omega\big) \qquad \forall \phi \in \mathbf{M}, \quad (3.2)$$

and introduce some new notation. For arbitrary $A \in \mathcal{X}$, $\Psi \subset \Omega$, $\omega \in \Omega$, $H \subset \mathbf{C}$, $h \in \mathbf{C}$, and $\phi \in \mathbf{M}$ we set

$$\Psi(\omega \,|\, A) \overset{\text{def}}{=} \{\nu \in \Psi \mid (\nu \,|\, A) = (\omega \,|\, A)\}, \qquad H(h \,|\, A) \overset{\text{def}}{=} \{f \in H \mid (f \,|\, A) = (h \,|\, A)\}, \qquad (3.3)$$

$$\Psi(-\omega \,|\, A) \overset{\text{def}}{=} \Psi(\omega \,|\, A) \setminus \{\omega\},$$

$$[\phi](\omega \,|\, A) \overset{\text{def}}{=} \bigcap_{\nu \in \Omega(\omega \,|\, A)} \big(\phi(\nu) \,|\, A\big), \qquad (3.4)$$

$$[\phi](-\omega \,|\, A) \overset{\text{def}}{=} \bigcap_{\nu \in \Omega(-\omega \,|\, A)} \big(\phi(\nu) \,|\, A\big). \qquad (3.5)$$

3.2. Unlocking the predicate "be $\mathcal{X}$-non-anticipating mapping"

It follows directly from definition (3.2) that P_{na} is the conjunction of the family $\{P_\omega \mid \omega \in \Omega\}$, where

$$P_\omega(\phi) \overset{\text{def}}{\Leftrightarrow} \big((\omega' \in \Omega(\omega \,|\, A)) \Rightarrow \big((\phi(\omega) \,|\, A) \subset (\phi(\omega') \,|\, A)\big)\ \forall A \in \mathcal{X}\big) \qquad \forall \omega \in \Omega,\ \forall \phi \in \mathbf{M}. \quad (3.6)$$

Then we transform (3.6) using notation (3.4).

Lemma 6.

$$P_\omega(\phi) \Leftrightarrow \big([\phi](\omega \,|\, A) = \big(\phi(\omega) \,|\, A\big) \quad \forall A \in \mathcal{X}\big) \qquad \forall \omega \in \Omega,\ \forall \phi \in \mathbf{M}. \qquad (3.7)$$

So, the predicate P_{na} has the form (see (3.2))

$$P_{na}(\phi) \Leftrightarrow (P_\omega(\phi)\ \forall \omega \in \Omega) \Leftrightarrow \big([\phi](\omega \,|\, A) = \big(\phi(\omega) \,|\, A\big) \quad \forall A \in \mathcal{X}\ \forall \omega \in \Omega\big) \qquad \forall \phi \in \mathbf{M}.$$

According to scheme (2.3)–(2.4), we represent $\mathbf{M}$ as the product of Ω copies of the set $\mathcal{P}(\mathbf{C})$. By the definitions the index set in the conjunction representing P_{na} coincides with the one in the product representing $\mathbf{M}$. Hence, the injection q in (2.5) can be chosen as the identity mapping. Then we have

$$\mathcal{I} \overset{\text{def}}{=} \Omega, \quad X_\iota \overset{\text{def}}{=} X_\omega \overset{\text{def}}{=} \mathcal{P}(\mathbf{C}), \quad \omega \in \Omega, \quad \mathbf{M} \overset{\text{def}}{=} X \overset{\text{def}}{=} \prod_{\iota \in \mathcal{I}} X_\iota \overset{\text{def}}{=} \prod_{\omega \in \Omega} \mathcal{P}(\mathbf{C}),$$

$$\mathcal{B}_\iota \overset{\text{def}}{=} \mathcal{B}_\omega \in \mathcal{P}(\mathcal{P}(\mathbf{C}))^{\mathbf{M}}, \quad \mathcal{F}_{P_{na}}(\phi) \overset{\text{def}}{=} \prod_{\omega \in \Omega} \mathcal{B}_\omega(\phi) \in \mathcal{P}(\mathcal{P}(\mathbf{C})) \overset{\text{def}}{=} \mathcal{P}(\mathbf{M})^{\mathbf{M}}.$$

We provide this list of "actors and performers" for the convenience of tracking scheme (2.3)–(2.4).

According to (2.5), (3.7) and notation (3.3)–(3.5), we construct the expression for $\mathcal{B}_\omega \in \mathcal{P}(\mathcal{P}(\mathbf{C}))^{\mathbf{M}}$ (recall that q is the identity map):

Lemma 7.

$$\mathcal{B}_\omega(\phi) = \mathcal{P}\left(\bigcap_{A\in\mathcal{X}} \bigcup_{\substack{h\in\mathbf{C}\\(h\,|\,A)\in[\phi](-\omega\,|\,A)}} \mathbf{C}(h\,|\,A)\right) \qquad \forall\omega\in\Omega,\ \forall\phi\in\mathbf{M}.$$

By lemma 5, the inclusion $\mathcal{F}_{P_{na}} \in \mathfrak{UM}(P_{na})$, where the mapping $\mathcal{F}_{P_{na}} \in \mathcal{P}(\mathbf{M})^{\mathbf{M}}$ (see (2.4)) has the form

$$\mathcal{F}_{P_{na}}(\phi) \overset{\mathrm{def}}{=} \prod_{\omega\in\Omega}\mathcal{P}\left(\bigcap_{A\in\mathcal{X}} \bigcup_{\substack{h\in\mathbf{C}\\(h\,|\,A)\in[\phi](-\omega\,|\,A)}} \mathbf{C}(h\,|\,A)\right) \qquad \forall\phi\in\mathbf{M}, \tag{3.8}$$

is true.

Formally speaking the unlocking operation for the predicate P_{na} is performed. But we need some steps to apply result (3.8) for solving the initial problem of constructing the greatest non-anticipating selection of the given m/f $\mathcal{M}$.

3.3. Design of the greatest $\mathcal{X}$-non-anticipating selection

We turn to the construction of the greatest $\mathcal{X}$-non-anticipating selection of $\mathcal{M}$. So, our aim is to find $\top_{(P_{na}\,|\,\mathbf{M}_\mathcal{M})^{-1}(1)}$, where $(P_{na}\,|\,\mathbf{M}_\mathcal{M}) \in \mathfrak{PR}(\mathbf{M}_\mathcal{M})$ is the restriction of the predicate P_{na} to the non-empty set $\mathbf{M}_\mathcal{M} \subset \mathbf{M}$, $\mathbf{M}_\mathcal{M} \overset{\mathrm{def}}{=} \{\phi\in\mathbf{M} \mid \phi \sqsubseteq \mathcal{M}\}$. By the inclusion $\mathcal{F}_{P_{na}} \in \mathfrak{UM}(P_{na})$, we should find the greatest element among fixpoints of (3.8) belonging the poset $(\mathbf{M}_\mathcal{M}, \sqsubseteq)$. One can verify that the poset $(\mathbf{M}_\mathcal{M}, \sqsubseteq)$ is also a complete lattice. Hence, it is a non-empty poset.

Using lemma 1, we construct from $\mathcal{F}_{P_{na}}$ an unlocking m/f $\mathcal{F}_{(P_{na}\,|\,\mathbf{M}_\mathcal{M})}$ for the predicate $(P_{na}\,|\,\mathbf{M}_\mathcal{M})$:

$$\mathcal{F}_{(P_{na}\,|\,\mathbf{M}_\mathcal{M})}(\phi) = [\mathcal{F}_{P_{na}}\,|\,\mathbf{M}_\mathcal{M}](\phi) = \mathbf{M}_\mathcal{M} \cap \mathcal{F}_{P_{na}}(\phi) = \prod_{\omega\in\Omega}\mathcal{P}\left(\bigcap_{A\in\mathcal{X}} \bigcup_{\substack{h\in\mathbf{C}\\(h\,|\,A)\in[\phi](-\omega\,|\,A)}} \mathcal{M}(\omega)(h\,|\,A)\right)$$

for all $\phi\in\mathbf{M}_\mathcal{M}$. Now we use lemma 4 for "narrowing" m/f $\mathcal{F}_{(P_{na}\,|\,\mathbf{M}_\mathcal{M})} \in \mathfrak{UM}((P_{na}\,|\,\mathbf{M}_\mathcal{M}))$. Note that the lemma is valid in our case: $(\mathbf{M}_\mathcal{M}, \sqsubseteq)$ is a nonempty poset. The m/f $\mathbf{LE}_{\mathbf{M}_\mathcal{M}}$ (see (2.1)) in this case is defined by

$$\mathbf{LE}_{\mathbf{M}_\mathcal{M}}(\alpha) = \prod_{\omega\in\Omega}\mathcal{P}(\alpha(\omega)), \qquad \alpha\in\mathbf{M}_\mathcal{M}. \tag{3.9}$$

Following (2.2) and (3.9), we construct m/f $\bar{G} \in \mathcal{P}(\mathbf{M}_\mathcal{M})^{\mathbf{M}_\mathcal{M}}$:

$$\bar{G}(\phi) \overset{\mathrm{def}}{=} \mathcal{F}_{(P_{na}\,|\,\mathbf{M}_\mathcal{M})}(\phi) \cap \mathbf{LE}_{\mathbf{M}_\mathcal{M}}(\phi) = \prod_{\omega\in\Omega}\mathcal{P}\left(\bigcap_{A\in\mathcal{X}} \bigcup_{\substack{h\in\mathbf{C}\\(h\,|\,A)\in[\phi](-\omega\,|\,A)}} \phi(\omega)(h\,|\,A)\right) \qquad \forall\phi\in\mathbf{M}_\mathcal{M}.$$

Due to inclusion $\varnothing\in\mathcal{P}(X)$ for any set X, the inequalities $\bar{G}(\phi)\neq\varnothing$, $\phi\in\mathbf{M}_\mathcal{M}$ hold. Consider the function $\gamma\in(\mathbf{M}_\mathcal{M})^{\mathbf{M}_\mathcal{M}}$ defined by the rule $\gamma(\psi) \overset{\mathrm{def}}{=} \sup_{(\mathbf{M}_\mathcal{M},\sqsubseteq)} \bar{G}(\psi)$ $\forall\psi\in\mathbf{M}_\mathcal{M}$. For the function, it follows that

$$\gamma(\phi) = \prod_{\omega\in\Omega}\bigcap_{A\in\mathcal{X}} \bigcup_{\substack{h\in\mathbf{C}\\(h\,|\,A)\in[\phi](-\omega\,|\,A)}} \phi(\omega)(h\,|\,A) \qquad \forall\phi\in\mathbf{M}_\mathcal{M}. \tag{3.10}$$

Equalities (3.10) imply that γ is isotone and the inclusions $\gamma(\phi)\in\bar{G}(\phi)$, $\forall\phi\in\mathbf{M}_\mathcal{M}$ are valid. Hence, for all $\phi\in\mathbf{M}_\mathcal{M}$, the equality $\gamma(\phi)=\top_{\bar{G}(\phi)}$ is fulfilled. Since $\bar{G}$ and γ satisfy lemma 4, we conclude that: γ is an isotone restrictive function and $\mathbf{Fix}(\gamma) = P_{na}^{-1}(1)$.

The properties of the function γ allows us to use the following theorem.

Theorem 1 (see [11]). *Let $(X, \preccurlyeq)$ be a chain-complete poset, $f \in X^X$ be a restrictive function on $(X, \preccurlyeq)$, and an ordinal α satisfy $|X|^+ \preccurlyeq \alpha$. Then* $\mathbf{Fix}(f) = \{f^\alpha(x) : x \in X\}$.

So, for any $\alpha \in \mathbf{ORD}$ such that $|\mathbf{M}_\mathcal{M}|^+ \preccurlyeq \alpha$, the equality

$$(P_{na} \,|\, \mathbf{M}_\mathcal{M})^{-1}(1) = \{\gamma^\alpha(\psi) : \psi \in \mathbf{M}_\mathcal{M}\}$$

is true. Here we have the expression for the set of all non-anticipating selections of m/f $\mathcal{M}$.

As γ is isotone and $\mathbf{M}_\mathcal{M}$ is a complete lattice, we can use the Tarski theorem [12, Theorem 1]: the set $\mathbf{Fix}(\gamma) = (P_{na} \,|\, \mathbf{M}_\mathcal{M})^{-1}(1)$ is a complete lattice in $(\mathbf{M}_\mathcal{M}, \sqsubseteq)$. Hence, there is the greatest non-anticipating selection $\top_{(P_{na} \,|\, \mathbf{M}_\mathcal{M})^{-1}(1)}$ in the poset $\mathbf{M}_\mathcal{M}$. Due to another result of Patrick and Radhia Cousot (see [13, Theorem 3.2]), it can be described in terms of transfinite iterations of γ starting at $\mathcal{M}$:

$$\top_{(P_{na} \,|\, \mathbf{M}_\mathcal{M})^{-1}(1)} = \top_{\mathbf{Fix}(\gamma)} = \gamma^\alpha(\top_{\mathbf{M}_\mathcal{M}}) = \gamma^\alpha(\mathcal{M}). \tag{3.11}$$

Thus, we have the desired expression for the greatest non-anticipating selection of the m/f $\mathcal{M}$.

3.4. Functions Γ and γ

Write down expression (3.10) in the coordinate form:

$$\gamma(\phi)(\omega) = \bigcap_{A \in \mathcal{X}} \bigcup_{\substack{h \in \mathbf{C} \\ (h \,|\, A) \in [\phi](\omega \,|\, A)}} \phi(\omega)(h \,|\, A) \qquad \forall \omega \in \Omega \ \forall \phi \in \mathbf{M}. \tag{3.12}$$

Eliminating notation (3.3), (3.4) from (3.12), we get the equality $\gamma(\phi)(\omega) = \Gamma(\phi)(\omega)$ $\forall \phi \in \mathbf{M}$ $\forall \omega \in \Omega$, where Γ is given in [6, sec. 4]:

$$\Gamma(\phi)(\omega) \overset{\text{def}}{=} \{f \in \phi(\omega) \mid \forall A \in \mathcal{X}\ \forall \omega' \in \Omega(\omega \,|\, A) \exists f' \in \phi(\omega') : (f \,|\, A) = (f' \,|\, A)\} \qquad \forall \omega \in \Omega \ \forall \phi \in \mathbf{M}.$$

Relation (3.11) generalizes the presentation [6, theorem 6.1], where $\alpha = \omega$ (the least infinite ordinal) is used. In our case, the bigger ordinal compensates the absence of topological requirements on Ω, $\mathbf{C}$, and $\mathbf{M}$.

§ 4. Conclusion

The main application of the technique appears to be the existence theorems (for an equilibrium, for an equation solution). When a predicate is defined on a poset, the greatest solution can be explicitly written down as a limit of iterations.

It is interesting to notice that the process (of unlocking of predicate) can also be used in the opposite direction: well known Fan's result on saddle points [14] prompted the author to look for one more fixed point theorem [15].

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Размыкание предиката: приложение к задаче построения неупреждающего селектора

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В работе разрабатывается метод, именуемый «размыкание предиката», сводящий задачу поиска множества истинности предиката к задаче поиска множества неподвижных точек некоторого (вообще говоря, многозначного) отображения. Предлагаемая техника дает дополнительные возможности анализа задач и построения решений путем систематического привлечения результатов теории неподвижных точек. Даны формальное определение операции размыкания предиката, способы построения и исчисления размыкающих отображений и их основные свойства. В случае когда область определения предиката частично упорядочена, указаны способы построения размыкающих функций, обладающих свойством сужаемости. Это позволило получить представления интересующих элементов решения в виде итерационных пределов. Предлагаемый подход в силу абстрактности имеет широкую сферу применения. Вместе с тем эффективность полученного решения зависит от специфики рассматриваемой задачи и выбранного варианта реализации метода. В качестве иллюстрации в работе рассмотрена процедура построения размыкающего отображения для предиката «быть неупреждающим селектором». На основе этого отображения получено выражение для наибольшего неупреждающего селектора заданной мультифункции.

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