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ON $\mathcal{L}$-INJECTIVE MODULES

Let $\mathcal{M} = \{(M, N, f, Q) \mid M, N, Q \in R\text{-Mod},\ N \leq M,\ f \in \mathrm{Hom}_R(N, Q)\}$ and let $\mathcal{L}$ be a nonempty subclass of $\mathcal{M}$. Jirásko introduced the concept of $\mathcal{L}$-injective module as a generalization of injective module as follows: a module Q is said to be $\mathcal{L}$-injective if for each $(B, A, f, Q) \in \mathcal{L}$ there exists a homomorphism $g\colon B \to Q$ such that $g(a) = f(a)$, for all $a \in A$. The aim of this paper is to study $\mathcal{L}$-injective modules and some related concepts. Some characterizations of $\mathcal{L}$-injective modules are given. We present a version of Baer's criterion for $\mathcal{L}$-injectivity. The concepts of $\mathcal{L}$-M-injective module and s-$\mathcal{L}$-M-injective module are introduced as generalizations of M-injective modules and give some results about them. Our version of the generalized Fuchs criterion is given. We obtain conditions under which the class of $\mathcal{L}$-injective modules is closed under direct sums. Finally, we introduce and study the concept of $\sum$-$\mathcal{L}$-injectivity as a generalization of $\sum$-injectivity and $\sum$-τ-injectivity.

Keywords: injective module, generalized Fuchs criterion, hereditary torsion theory, t-dense, preradical, natural class.

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Introduction

Throughout this article, unless otherwise specified, R will denote an associative ring with non-zero identity, and all modules are left unital R-modules. By a class of modules we mean a non-empty class of modules. The class of all left R-modules is denoted by R-Mod and by $\Re$ we mean the set $\{(M, N) \mid N \leq M,\ M \in R\text{-Mod}\}$, where $N \leq M$ is a notation which means that N is a submodule of M. Given a family of modules $\{M_i\}_{i \in I}$, for each $j \in I$, $\pi_j\colon \bigoplus_{i \in I} M_i \to M_j$ denotes the canonical projection homomorphism. Let M be a module and let Y be a subset of M. The left annihilator of Y in R will be denoted by $l_R(Y)$, i.e., $l_R(Y) = \{r \in R \mid ry = 0,\ \forall y \in Y\}$. Given $a \in M$, let $(Y : a)$ denote the set $\{r \in R \mid ra \in Y\}$, and let $ann_R(a) := (0 : a)$. The right annihilator of a subset I of R in M will be denoted by $r_M(I)$, i.e., $r_M(I) = \{m \in M \mid rm = 0,\ \forall r \in I\}$. The class $\{I \mid I \text{ is a left ideal of } R \text{ such that } ann_R(m) \subseteq I, \text{ for some } m \in M\}$ will be denoted by $\Omega(M)$.

An R-module M is said to be injective if, for any module B, every homomorphism $f\colon A \to M$, where A is any submodule of B, extends to a homomorphism $g\colon B \to M$ [3]. The notation $g \restriction A = f$ means that g is an extension of f. Let M and N be modules. Recall that N is said to be M-injective if every homomorphism from a submodule of M to N extends to a homomorphism from M to N [2]. A module M is said to be quasi-injective if M is M-injective. The injective envelope of a module M will be denoted by $E(M)$.

Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. A submodule B of a module A is said to be τ-dense in A if A/B is τ-torsion (i.e., $A/B \in \mathcal{T}$). A submodule A of a module B is said to be τ-essential in B if it is τ-dense and essential in B. A torsion theory τ is said to be Noetherian if for every ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ of left ideals of R with $I_\infty = \bigcup_{j=1}^{\infty} I_j$ a τ-dense left ideal in R, there exists a positive integer n such that I_n is τ-dense in R. A module M is said to be τ-injective if every homomorphism from a τ-dense submodule of B to M extends to a homomorphism from B to M, where B is any module [8]. Let M be an R-module. A τ-injective envelope (or τ-injective hull) of M is a τ-injective module E which is a τ-essential extension of M [6]. Every R-module M has a τ-injective envelope and it is unique up to isomorphism [8]. We use the notation $E_\tau(M)$ to denote an τ-injective envelope of M. A τ-injective module E is said to be $\sum$-τ-injective if $E^{(A)}$ is τ-injective for any index set A; E is said to be countably $\sum$-τ-injective if $E^{(C)}$ is τ-injective for any countable index set C. Let E and M be modules. Then E is said to be τ-M-injective if any homomorphism from a τ-dense submodule of M to E extends to a homomorphism from M to E. A module E is said to be τ-quasi-injective if E is τ-E-injective.

Let $\mathcal{M} = \{(M, N, f, Q) \mid M, N, Q \in R\text{-Mod}, N \leq M, f \in \mathrm{Hom}_R(N, Q)\}$ and consider the following conditions on $\mathcal{L}$ that will be useful later, where $\mathcal{L}$ always denotes a nonempty subclass of $\mathcal{M}$:

(α) $(M, N, f, Q) \in \mathcal{L}$, $(M, N', f', Q) \in \mathcal{M}$ and $(M, N, f, Q) \preceq (M, N', f', Q)$ implies inclusion $(M, N', f', Q) \in \mathcal{L}$, where $\preceq$ is a partial order on $\mathcal{M}$ defined by:

$$(M, N, f, Q) \preceq (M', N', f', Q') \Longleftrightarrow M = M', \;\; N \subseteq N', \;\; Q = Q', \;\; f' \restriction N = f;$$

(β) $(M, N, f, A) \in \mathcal{L}$, $i : A \to B$ implies $(M, N, if, B) \in \mathcal{L}$, where i is an inclusion homomorphism;

(γ) $(M, N, f, A) \in \mathcal{L}$, $g : A \to B$ an isomorphism, implies $(M, N, gf, B) \in \mathcal{L}$;

(δ) $(M, N, f, A) \in \mathcal{L}$, $g : A \to B$ a homomorphism, implies $(M, N, gf, B) \in \mathcal{L}$;

(λ) $(M, N, f, A) \in \mathcal{L}$, $g : A \to B$ a split epimorphism, implies $(M, N, gf, B) \in \mathcal{L}$;

(μ) $(M, N, f, Q) \in \mathcal{L}$, implies $(R, (N : x), f_x, Q) \in \mathcal{L}\, \forall x \in M$, where $f_x : (N : x) \to Q$ is a homomorphism defined by $f_x(r) = f(rx)$ $\forall r \in (N : x)$.

Jirásko in [14] introduced the concepts of $\mathcal{L}$-injective module as a generalization of injective module as follows: a module Q is said to be $\mathcal{L}$-injective if for each $(B, A, f, Q) \in \mathcal{L}$, there exists a homomorphism $g : B \to Q$ such that $(g \restriction A) = f$. An $\mathcal{L}$-injective module E is said to be an $\mathcal{L}$-injective envelope (or $\mathcal{L}$-injective hull) of a module M if there is no proper $\mathcal{L}$-injective submodule of E containing M [14]. If a module M has an $\mathcal{L}$-injective envelope and it is unique up to isomorphic then we will use the notation $E_{\mathcal{L}}(M)$ to denote an $\mathcal{L}$-injective envelope of M. Clearly, injective module and all its generalizations are special cases of $\mathcal{L}$-injectivity.

The aim of this article is to study $\mathcal{L}$-injectivity and some related concepts.

In Section 1, we give some characterizations of $\mathcal{L}$-injective modules. For example, in Theorem 1 we give a version of Baer's criterion for $\mathcal{L}$-injectivity. Also, in Theorem 2 we extend a characterization due to [20, Theorem 2, p. 8] of $\mathcal{L}$-injective modules over commutative Noetherian rings.

In Section 2, we introduce the concepts of $\mathcal{L}$-M-injective module and s-$\mathcal{L}$-M-injective module as generalizations of M-injective modules and give some results on them. For examples, in Theorem 3 we prove that if $\mathcal{L}$ is a nonempty subclass of $\mathcal{M}$ satisfying conditions (α), (β), and (γ) and $M, Q \in R$-Mod such that M satisfies condition ($E_{\mathcal{L}}$), then Q is $\mathcal{L}$-M-injective if and only if $f(M) \leq Q$, for all $f \in \mathrm{Hom}_R(E_{\mathcal{L}}(M), E_{\mathcal{L}}(Q))$ with $(M, L, f{\restriction}L, Q) \in \mathcal{L}$ where $L = \{m \in M \mid f(m) \in Q\} = M \bigcap f^{-1}(Q)$. Also, in Proposition 2 we generalize [6, Proposition 14.12, p. 66], [5, Proposition 1, p. 1954] and Fuchs's result in [12]. Moreover, our version of the generalized Fuchs criterion is given in Proposition 3 in which we prove that if $\mathcal{L}$ is a nonempty subclass of $\mathcal{M}$ satisfying conditions (α) and (μ) and $M, Q \in R$-Mod such that M satisfies condition ($\mathcal{L}$), then a module Q is s-$\mathcal{L}$-M-injective if and only if for each $(R, I, f, Q) \in \mathcal{L}$ with $\ker(f) \in \Omega(M)$, there exists an element $x \in Q$ such that $f(a) = ax$ $\forall a \in I$.

In Section 3, we study direct sums of $\mathcal{L}$-injective modules. In Proposition 4 we prove that for any family $\{E_\alpha\}_{\alpha \in A}$ of $\mathcal{L}$-injective modules, where A is an infinite index set, if $\mathcal{L}$ satisfies conditions (α), (μ), and (δ) and $\bigoplus_{\alpha \in C} E_\alpha$ is an $\mathcal{L}$-injective module for any countable subset C of A, then $\bigoplus_{\alpha \in A} E_\alpha$ is an $\mathcal{L}$-injective module. In Theorem 4, we prove that for any nonempty subclass $\mathcal{L}$ of $\mathcal{M}$ which satisfies conditions (α) and (δ) and for any nonempty class $\mathcal{K}$ of modules closed under isomorphic copies and $\mathcal{L}$-injective hulls, if the direct sum of any family $\{E_i\}_{i \in N}$ of $\mathcal{L}$-injective R-modules in $\mathcal{K}$ is $\mathcal{L}$-injective, then every ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ of left ideals of R in $H_{\mathcal{K}}(R)$ with $I_\infty = \bigcup_{j=1}^{\infty} I_j$ being s-$\mathcal{L}$-dense in R terminates. Also, in Theorem 5 we generalize results in [17, p. 643] and [8, Proposition 5.3.5, p. 165] in which we prove that for any nonempty subclass $\mathcal{L}$ of $\mathcal{M}$ which satisfies conditions (α), (μ), (δ), and (I) and for any nonempty class $\mathcal{K}$ of modules closed under isomorphic copies and submodules, if every ascending chain $J_1 \subseteq J_2 \subseteq \ldots$ of left ideals of R, where $(J_{i+1}/J_i) \in \mathcal{K}$ $\forall i \in \mathbb{N}$ and $J_\infty = \bigcup_{i=1}^{\infty} J_i$ is s-$\mathcal{L}$-dense in R, terminates, then every direct sum of $\mathcal{L}$-injective modules in $\mathcal{K}$ is $\mathcal{L}$-injective.

Finally, in Section 4, we introduce the concept of $\sum$-$\mathcal{L}$-injectivity as a generalization of $\sum$-injectivity and $\sum$-τ-injectivity and prove Theorem 6 in which we generalize Faith's result [11, Proposition 3, p. 184] and [6, Theorem 16.16, p. 98].

§ 1. Some Characterizations of $\mathcal{L}$-Injective Modules

One well-known result concerning injective modules states that an R-module M is injective if and only if every homomorphism from a left ideal of R to M extends to a homomorphism from R to M if and only if for each left ideal I of R and every $f \in \mathrm{Hom}_R(I, M)$ there is an $m \in M$ such that $f(r) = rm$ $\forall r \in I$. This is known as Baer's condition [3]. Baer's result shows that the left ideals of R form a test set for injectivity.

The following theorem gives a version of Baer's criterion for $\mathcal{L}$-injectivity.

Theorem 1 (**Generalized Baer's Criterion**). *Consider the following three conditions for an R-module M:*

(1) *M is $\mathcal{L}$-injective;*

(2) *for every $(R, I, f, M) \in \mathcal{L}$, there exists an R-homomorphism $g \in \mathrm{Hom}_R(R, M)$ such that $g(a) = f(a)$, for all $a \in I$;*

(3) *for each $(R, I, f, M) \in \mathcal{L}$, there exists an element $m \in M$ such that $f(r) = rm$, $\forall r \in I$.*

Then (2) *and* (3) *are equivalent and* (1) *implies* (2)*. Moreover, if $\mathcal{L}$ satisfies conditions* (α) *and* (μ)*, then all the three conditions are equivalent.*

P r o o f. (1) $\Rightarrow$ (2) and (2) $\Leftrightarrow$ (3) are obvious.

(2) $\Rightarrow$ (1). Let $\mathcal{L}$ satisfy conditions (α) and (μ) and let $(B, A, f, M) \in \mathcal{L}$. Let $S = \{(C, \varphi) \mid A \leq C \leq B\ ,\ \varphi \in \mathrm{Hom}_R(C, M)$ such that $(\varphi{\restriction}A) = f\ \}$. Define on S a partial order $\preceq$ by

$$(C_1, \varphi_1) \preceq (C_2, \varphi_2) \Longleftrightarrow C_1 \leq C_2 \text{ and } (\varphi_2 \restriction C_1) = \varphi_1.$$

Clearly, $S \neq \emptyset$ since $(A, f) \in S$. Furthermore, one can show that S is inductive in the following manner. Let $F = \{(A_i, f_i) \mid i \in I\}$ be an ascending chain in S. Let $A_\infty = \cup_{i \in I} A_i$. Then for any $a \in A_\infty$ there is a $j \in I$ such that $a \in A_j$, and so we can define $f_\infty : A_\infty \to M$, by $f_\infty(a) = f_j(a)$. It is straightforward to check that f_∞ is well defined and (A_∞, f_∞) is an upper bound for F in S. Then by Zorn's Lemma, S has a maximal element, say (B', g'). We will prove that $B' = B$.

Suppose that there exists $x \in B \setminus B'$. It is clear that $(B, A, f, M) \preceq (B, B', g', M)$. Since $(B, A, f, M) \in \mathcal{L}$ and $\mathcal{L}$ satisfies condition (α), it follows that $(B, B', g', M) \in \mathcal{L}$. Since $\mathcal{L}$ satisfies condition (μ), we have $(R, (B' : x), g'_x, M) \in \mathcal{L}$. By hypothesis, there exists a homomorphism $g : R \to M$ such that $g(r) = g'_x(r) = g'(rx)$, $\forall r \in (B' : x)$. Define $\psi : B' + Rx \to M$ by $\psi(b + rx) = g'(b) + g(r)$, $\forall b \in B'$, $\forall r \in R$. It is clear that ψ is a well-defined homomorphism and $(B', g') \preceq (B' + Rx, \psi)$. Since $(B' + Rx, \psi) \in S$ and $B' \subsetneq B' + Rx$, we have a contradiction to maximality of (B', g') in S. Hence $B' = B$ and this means that there exists a homomorphism $g' : B \to M$ such that $(g'{\restriction}A) = f$. Thus M is $\mathcal{L}$-injective.

Now we will introduce the concept of P-filter as follows.

Definition 1. Let $\Re = \{(M, N) \mid N \leq M,\ M \in R\text{-Mod}\}$ and let ρ be a nonempty subclass of $\Re$. We say that ρ is a P-filter if ρ satisfies the following conditions:

(i) if $(M, N) \in \rho$ and $N \leq K \leq M$, then $(M, K) \in \rho$;

(ii) for all $M \in R$-Mod, $(M, M) \in \rho$;

(iii) if $(M, N) \in \rho$, then $(R, (N : x)) \in \rho$, $\forall x \in M$.

Example 1. All of the following subclasses of $\Re$ are P-filters.

(1) $\rho_{\mathcal{T}} = \{(M, N) \in \Re \mid N \leq M$ such that $M/N \in \mathcal{T},\ M \in R\text{-Mod}\}$, where $\mathcal{T}$ is a nonempty class of modules closed under submodules and homomorphic images.

(2) $\rho_\infty = \Re = \{(M, N) \mid N \leq M,\ M \in R\text{-Mod}\}$.

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(3) $\rho_\tau = \{(M,N) \in \Re \mid N$ is τ-dense in M, $M \in R$-Mod$\}$, where τ is a hereditary torsion theory.

(4) $\rho_r = \{(M,N) \in \Re \mid N \leq M$ such that $r(M/N) = M/N$, $M \in R$-Mod$\}$, where r is a left exact preradical.

(5) $\rho_{max} = \{(M,N) \in \Re \mid N$ is a maximal submodule in M or $N = M$, $M \in R$-Mod$\}$.

(6) $\rho_e = \{(M,N) \in \Re \mid N \leq_e M$, $M \in R$-Mod$\}$.

It is clear that the P-filters from (2) to (5) are special cases of P-filter in (1). Also, if ρ is a P-filter then the subclass $\rho_R = \{(R,I) \in \rho \mid I$ is a left ideal of $R\}$ of $\Re$ is also P-filter.

Notations 1. We will fix the following notations.

– For any two P-filters ρ_1 and ρ_2, we will denote by $\mathcal{L}_{(\rho_1,\rho_2)}$ the subclass $\mathcal{L}_{(\rho_1,\rho_2)} = \{(M,N,f,Q) \in \mathcal{M} \mid M,N,Q \in R$-Mod, $(M,N) \in \rho_1$ and $f \in \mathrm{Hom}_R(N,Q)$ such that $(M, \ker(f)) \in \rho_2\}$.

– For any two nonempty classes of modules $\mathcal{T}$ and $\mathcal{F}$, we will denote by $\mathcal{L}_{(\mathcal{T},\mathcal{F})}$ the subclass $\mathcal{L}_{(\mathcal{T},\mathcal{F})} = \{(M,N,f,Q) \in \mathcal{M} \mid M,N,Q \in R$-Mod, $N \leq M$ such that $M/N \in \mathcal{T}$ and $f \in \mathrm{Hom}_R(N,Q)$ with $M/\ker(f) \in \mathcal{F}\}$. It is clear that $\mathcal{L}_{(\mathcal{T},\mathcal{F})} = \mathcal{L}_{(\rho_\mathcal{T},\rho_\mathcal{F})}$, if $\mathcal{T}$ and $\mathcal{F}$ are closed under submodules and homomorphic images.

– For any two preradicals r and s, we will denote by $\mathcal{L}_{(r,s)}$ the subclass $\mathcal{L}_{(r,s)} = \{(M,N,f,Q) \in \mathcal{M} \mid M,N,Q \in R$-Mod, $N \leq M$ such that $r(M/N) = M/N$ and $f \in \mathrm{Hom}_R(N,Q)$ with $s(M/\ker(f)) = M/\ker(f)\}$. It is clear that $\mathcal{L}_{(r,s)} = \mathcal{L}_{(\rho_r,\rho_s)}$, if r and s are left exact preradicals.

– For any torsion theory τ, we will denote by $\mathcal{L}_\tau$ the subclass $\mathcal{L}_\tau = \{(M,N,f,Q) \in \mathcal{M} \mid M,N,Q \in R$-Mod, N is a τ-dense in M and $f \in \mathrm{Hom}_R(N,Q)\}$. It is clear that $\mathcal{L}_\tau = \mathcal{L}_{(\rho_\tau,\rho_\infty)}$, if τ is a hereditary torsion theory.

Lemma 1. *Let ρ_1 and ρ_2 be two P-filters. Then $\mathcal{L}_{(\rho_1,\rho_2)}$ satisfies conditions (α), (δ), and (μ).*

P r o o f. It is obvious.

The following corollary is a generalization of Baer's result in [3], [19, Proposition 2.1, p. 201], [14, Baer's Lemma 2.2, p. 628] and [4, Theorem 2.4, p. 319].

Corollary 1. *Let ρ_1 and ρ_2 be two P-filters. Then the following conditions are equivalent for R-module M:*

(1) *M is $\mathcal{L}_{(\rho_1,\rho_2)}$-injective;*

(2) *for every $(R,I,f,M) \in \mathcal{L}_{(\rho_1,\rho_2)}$ there exists an R-homomorphism $g \in \mathrm{Hom}_R(R,M)$ such that $g(a) = f(a)$, for all $a \in I$;*

(3) *for each $(R,I,f,M) \in \mathcal{L}_{(\rho_1,\rho_2)}$ there exists an element $m \in M$ such that $f(r) = rm$, $\forall r \in I$.*

P r o o f. By Lemma 1 and Theorem 1.

The following characterization of $\mathcal{L}$-injectivity is a generalization of [18, Proposition 1.4, p. 3] and [8, Proposition 2.1.3, p. 53].

Proposition 1. *Consider the following three conditions for R-module M:*

(1) *Q is $\mathcal{L}$-injective;*

(2) *for every $(M,N,f,Q) \in \mathcal{L}$ with $N \leq_e M$, the homomorphism f extends to a homomorphism from M to Q;*

(3) *for every $(R,I,f,Q) \in \mathcal{L}$ with $I \leq_e R$, the homomorphism f extends to a homomorphism from R to Q.*

Then (1) *implies* (2), (2) *implies* (3) *and, if $\mathcal{L}$ satisfies conditions (α) and (μ), then* (3) *implies* (1).

P r o o f. (1) $\Rightarrow$ (2) and (2) $\Leftrightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1). Let $\mathcal{L}$ satisfy (α) and (μ) and let $(R, I, f, Q) \in \mathcal{L}$. Let I^c be a complement left ideal of I in R and let $C = I \oplus I^c$. Thus, by [1, Proposition 5.21, p. 75], $C \leq_e R$. Define $g : C = I \oplus I^c \to Q$ by $g(a + b) = f(a)$, $\forall a \in I$ and $\forall b \in I^c$. It is clear that g is a well-defined homomorphism and $(R, I, f, Q) \preceq (R, C, g, Q)$. Since $\mathcal{L}$ satisfies condition (α), $(R, C, g, Q) \in \mathcal{L}$. By hypothesis, there exists a homomorphism $h : R \to Q$ such that $(h \restriction C) = g$. Thus $(h \restriction I) = (g \restriction I) = f$ and this implies that Q is $\mathcal{L}$-injective, by Theorem 1.

In the following theorem we extend a characterization due to [20, Theorem 2, p. 8] of $\mathcal{L}$-injective modules over commutative Noetherian rings.

Theorem 2. *Let R be a commutative Noetherian ring, let M be an R-module and suppose that $\mathcal{L}$ satisfies conditions (α) and (μ). Then M is $\mathcal{L}$-injective if and only if for every $(R, I, f, M) \in \mathcal{L}$, where I is a prime ideal of R, the homomorphism f extends to a homomorphism from R to M.*

P r o o f. ($\Longrightarrow$) This is obvious.

($\Longleftarrow$) Let $(B, A, f, M) \in \mathcal{L}$ and let $S = \{(C, \varphi) \mid A \leq C \leq B,\ \varphi \in \mathrm{Hom}_R(C, M)$ such that $(\varphi \restriction A) = f\}$. Define on S a partial order $\preceq$ by

$$(C_1, \varphi_1) \preceq (C_2, \varphi_2) \Longleftrightarrow C_1 \leq C_2 \text{ and } (\varphi_2 \restriction C_1) = \varphi_1.$$

As in the proof of Theorem 1, we can prove that S has a maximal element, say (B', g'). We will prove that $B' = B$. Suppose that there exists an $x \in B \setminus B'$. By [20, Theorem 1, p. 8], there exists an element $r_0 \in R$ such that $(B' : r_0x)$ is a prime ideal in R and $r_0x \notin B'$. It is clear that $(B, A, f, M) \preceq (B, B', g', M)$. Since $(B, A, f, M) \in \mathcal{L}$ and $\mathcal{L}$ satisfies condition (α), it follows that $(B, B', g', M) \in \mathcal{L}$. Since $\mathcal{L}$ satisfies condition (μ), it follows that $(R, (B' : b), g'_b, M) \in \mathcal{L}$, $\forall b \in B$. Put $y = r_0x$, thus $y \in B \setminus B'$ and hence $(R, (B' : y), g'_y, M) \in \mathcal{L}$. By hypothesis, there exists a homomorphism $g : R \to M$ such that $g(r) = g'_y(r) = g'(ry)$, $\forall r \in (B' : y)$. Define $\psi : B' + Ry \to M$ by $\psi(b + ry) = g'(b) + g(r)$, $\forall b \in B'$, $\forall r \in R$. As in the proof of Theorem 1, we can prove that ψ is a well-defined homomorphism and $(B', g') \preceq (B' + Ry, \psi)$. Since $(B' + Ry, \psi) \in S$ and $B' \subsetneqq B' + Ry$, we have a contradiction to maximality of (B', g') in S. Hence $B' = B$ and this mean that there exists a homomorphism $g' : B \to M$ such that $(g' \restriction A) = f$. Thus M is $\mathcal{L}$-injective.

Corollary 2. *Let ρ_1 and ρ_2 be two P-filters, let R be a commutative Noetherian ring and let M be an R-module. Then M is $\mathcal{L}_{(\rho_1, \rho_2)}$-injective if and only if for every $(R, I, f, M) \in \mathcal{L}_{(\rho_1, \rho_2)}$, where I is a prime ideal of R, the homomorphism f extends to a homomorphism from R to M.*

P r o o f. By Lemma 1 and Theorem 2.

Corollary 3 (**see [20, Theorem 2, p. 8]**). *Let R be a commutative Noetherian ring, let M be an R-module. Then M is injective if and only if every homomorphism $f : I \to M$, where I is a prime ideal of R, can be extended to a homomorphism from R to M.*

P r o o f. By taking the two P-filters $\rho_1 = \rho_2 = \Re$ and applying Corollary 2.

§ 2. $\mathcal{L}$-M-Injectivity and s-$\mathcal{L}$-M-Injectivity

In this section, we introduce the concepts of $\mathcal{L}$-M-injective modules and s-$\mathcal{L}$-M-injective modules as generalizations of M-injective modules and give some results about them.

Definition 2. Let $M, Q \in R$-Mod. A module Q is said to be $\mathcal{L}$-M-injective if for every $(M, N, f, Q) \in \mathcal{L}$ the homomorphism f extends to a homomorphism from M to Q. A module Q is said to be $\mathcal{L}$-quasi-injective if Q is $\mathcal{L}$-Q-injective.

Let $M, Q \in R$-Mod, it is well-known that a module Q is M-injective if and only if $f(M) \leq Q$, for every homomorphism $f : E(M) \to E(Q)$ [16, Lemma 1.13, p. 7].

For an analogous result for $\mathcal{L}$-M-injectivity we first fix the following condition.

$(E_{\mathcal{L}})$: Let $\mathcal{L}$ be a subclass of $\mathcal{M}$. Then a module M satisfies condition $(E_{\mathcal{L}})$ if M has an $\mathcal{L}$-injective envelope which is unique up to M-isomorphism and $(E_{\mathcal{L}}(M), N, f, Q) \in \mathcal{L}$ whenever $(M, N, f, Q) \in \mathcal{L}$.

The next theorem is the first main result of this section in which we give a generalization of [16, Lemma 1.13, p. 7] and [7, Theorem 2.1, p. 34].

Theorem 3. *Let $M, Q \in R$-Mod and let $\mathcal{L}$ satisfy conditions (α), (β), and (γ). Consider the following two conditions.*

(1) Q is $\mathcal{L}$-M-injective.

(2) $f(M) \leq Q$, for all $f \in \operatorname{Hom}_R(E_{\mathcal{L}}(M), E_{\mathcal{L}}(Q))$ with $(M, L, f{\restriction}L, Q) \in \mathcal{L}$, where $L = \{m \in M \mid f(m) \in Q\} = M \bigcap f^{-1}(Q)$.

Then (1) implies (2) and, if M satisfies condition $(E_{\mathcal{L}})$, then (2) implies (1).

P r o o f. (1) $\Rightarrow$ (2). Let $f \in \operatorname{Hom}_R(E_{\mathcal{L}}(M), E_{\mathcal{L}}(Q))$ with $(M, L, f{\restriction}L, Q) \in \mathcal{L}$, where $L = \{m \in M \mid f(m) \in Q\} = M \bigcap f^{-1}(Q)$. Define $g : L \to Q$ by $g(a) = f(a)$, $\forall a \in L$ (i. e., $g = f{\restriction}L$). It is clear that g is a homomorphism and $(M, L, g, Q) \in \mathcal{L}$. By $\mathcal{L}$-M-injectivity of Q, there exists a homomorphism $h : M \to Q$ such that $(h{\restriction}L) = g$. Since $Q \bigcap (f\text{-}h)(M) = 0$ and Q is an essential submodule of $E_{\mathcal{L}}(Q)$ (by [14, Theorem 1.19, p. 627]), it follows that $(f - h)(M) = 0$ and this implies that $f(M) = h(M) \leq Q$.

(2) $\Rightarrow$ (1). Let M satisfy condition $(E_{\mathcal{L}})$ and let $(M, N, f, Q) \in \mathcal{L}$, thus $(E_{\mathcal{L}}(M), N, f, Q) \in \mathcal{L}$. Since $\mathcal{L}$ satisfies condition (β), it follows that $(E_{\mathcal{L}}(M), N, if, E_{\mathcal{L}}(Q)) \in \mathcal{L}$, where i is the inclusion mapping from Q into $E_{\mathcal{L}}(Q)$. By $\mathcal{L}$-injectivity of $E_{\mathcal{L}}(Q)$, there exists a homomorphism $h\colon E_{\mathcal{L}}(M) \to E_{\mathcal{L}}(Q)$ such that $h(n) = f(n)$ $\forall n \in N$. Let $L = \{m \in M \mid h(m) \in Q\}$. We will prove that $(M, L, g, Q) \in \mathcal{L}$, where $g = h \restriction L$. Let $x \in N$, thus $h(x) = f(x) \in Q$ and hence $x \in L$. Thus $N \leq L$ and $(g \restriction N) = f$. Thus $(M, N, f, Q) \preceq (M, L, g, Q)$. Since $\mathcal{L}$ satisfies condition (α), it follows that $(M, L, g, Q) \in \mathcal{L}$. By hypothesis, we have $h(M) \leq Q$ and hence $h' = h{\restriction} M : M \to Q$ is such that $(h' \restriction N) = f$. Thus Q is an $\mathcal{L}$-M-injective module.

Corollary 4. *Let $M, Q \in R$-Mod and let ρ_1 and ρ_2 be two P-filters. If M satisfies condition $(E_{\mathcal{L}_{(\rho_1,\rho_2)}})$, then the following two conditions are equivalent:*

(1) Q is $\mathcal{L}_{(\rho_1,\rho_2)}$-$M$-injective;

(2) $f(M) \leq Q$, for all $f \in \operatorname{Hom}_R(E_{\mathcal{L}_{(\rho_1,\rho_2)}}(M), E_{\mathcal{L}_{(\rho_1,\rho_2)}}(Q))$ with $(M, L, f{\restriction}L, Q) \in \mathcal{L}$, where $L = \{m \in M \mid f(m) \in Q\} = M \bigcap f^{-1}(Q)$.

P r o o f. By Lemma 1 and Theorem 3.

Let $M, Q \in R$-Mod and let τ be any hereditary torsion theory. A module Q is s-τ-M-injective if for any $N \leq M$ every homomorphism from a τ-dense submodule of N to Q extends to a homomorphism from N to Q [6, Definition 14.6, p. 65].

As a generalization of s-τ-M-injectivity and hence of M-injectivity we introduce the concept of s-$\mathcal{L}$-M-injectivity as follows.

Definition 3. Let $M, Q \in R$-Mod. A module Q is said to be s-$\mathcal{L}$-M-injective if Q is $\mathcal{L}$-N-injective, for all $N \leq M$. A module Q is said to be s-$\mathcal{L}$-quasi-injective if Q is s-$\mathcal{L}$-Q-injective.

Fuchs in [12] has obtained a condition similar to Baer's Criterion that characterizes quasi-injective modules, Bland in [5] has generalized that to s-τ-quasi-injective modules, and Charalambides in [6] has generalized that to s-τ-M-injective modules.

Our next aim is to generalize Fuchs's condition once again in order to characterize s-$\mathcal{L}$-M-injective modules. We begin with the following condition.

($\mathcal{L}$): Let $\mathcal{L}$ be a subclass of $\mathcal{M}$ and let M be a module. Then M satisfies condition ($\mathcal{L}$) if for every $(B, A, f, Q) \in \mathcal{L}$ we have $(Rm, (A : x)m, f_{(x,m)}, Q) \in \mathcal{L}$, for all $m \in M$ and $x \in B$ with $ann_R(m) \subseteq (\ker(f) : x)$, where $f_{(x,m)} : (A : x)m \to Q$ is a well-defined homomorphism defined by $f_{(x,m)}(rm) = f(rx)$, for all $r \in (A : x)$.

A subclass $\mathcal{L}$ of $\mathcal{M}$ is said to be full subclass if every R-module satisfies condition ($\mathcal{L}$).

Example 2. All of the following subclasses of $\mathcal{M}$ are full subclasses.

(1) $\mathcal{L}_{(T,F)}$, where T and F are nonempty classes of modules closed under submodules and homomorphic images.

(2) $\mathcal{L} = \mathcal{M}$.

(3) $\mathcal{L}_\tau$, where τ is a hereditary torsion theory.

(4) $\mathcal{L}_{(\rho,\sigma)}$, where ρ and σ are left exact preradicals.

In following proposition, we generalize [6, Proposition 14.12, p. 66], [5, Proposition 1, p. 1954] and Fuchs's result in [12], and it is necessary for our version of the Generalized Fuchs criterion.

Proposition 2. *Consider the following statements, where $M, Q \in R$-Mod:*

(1) Q is s-$\mathcal{L}$-M-injective;

(2) if $m \in M$ with $(Rm, K, f, Q) \in \mathcal{L}$, then the homomorphism f extends to a homomorphism from Rm to Q;

(3) if $K \leq N$ are modules, not necessarily submodules of M such that $(N, K, f, Q) \in \mathcal{L}$ and $\Omega(N) \subseteq \Omega(M)$, then the homomorphism f extends to a homomorphism from N to Q.

Then (1) implies (2) and (3) implies (1). Moreover, if $\mathcal{L}$ satisfies condition (α) and M satisfies condition ($\mathcal{L}$), then all above statements are equivalent.

P r o o f. (1) $\Rightarrow$ (2). Let $m \in M$ with $(Rm, K, f, Q) \in \mathcal{L}$. Thus Q is $\mathcal{L}$-Rm-injective, since Q is s-$\mathcal{L}$-M-injective and hence there exists a homomorphism $g : Rm \to Q$ such that $(g{\restriction}K) = f$.

(2) $\Rightarrow$ (3). Let $\mathcal{L}$ satisfy condition (α) and M satisfy condition ($\mathcal{L}$). Let $K \leq N$ be modules, not necessarily submodules of M with $(N, K, f, Q) \in \mathcal{L}$ and $\Omega(N) \subseteq \Omega(M)$. Let $S = \{(C, \varphi) \mid K \leq C \leq N,\ \varphi \in \mathrm{Hom}_R(C, M)$ such that $(\varphi \restriction K) = f\}$. Define on S a partial order $\preceq$ by

$$(C_1, \varphi_1) \preceq (C_2, \varphi_2) \Longleftrightarrow C_1 \leq C_2 \text{ and } (\varphi_2 \restriction C_1) = \varphi_1.$$

As in the proof of Theorem 1, we can prove that S has a maximal element, say (X, h). It suffices to show that $X = N$. Suppose that there exists an $n \in N \setminus X$. It is clear that $(N, K, f, Q) \preceq (N, X, h, Q)$. Since $(N, K, f, Q) \in \mathcal{L}$ and $\mathcal{L}$ satisfies condition (α), it follows that $(N, X, h, Q) \in \mathcal{L}$. Since $ann_R(n) \in \Omega(N)$ and $\Omega(N) \subseteq \Omega(M)$ (by assumption), we have $ann_R(n) \in \Omega(M)$ and this implies that there exists an $m \in M$ such that $ann_R(m) \subseteq ann_R(n)$. Since $ann_R(n) \subseteq (\ker(h) : n)$, we obtain $ann_R(m) \subseteq (\ker(h) : n)$. Since $m \in M$ and $n \in N \setminus X$ such that $ann_R(m) \subseteq (\ker(h) : n)$ and since M satisfies condition ($\mathcal{L}$), we get $(Rm, (X : n)m, h_{(n,m)}, Q) \in \mathcal{L}$. By hypothesis, there exists a homomorphism $\varphi^* : Rm \to Q$ such that $\varphi^*(am) = h_{(n,m)}(am)$ for all $am \in (X : n)m$. Define $h^* : X + Rn \to Q$ by $h^*(x + rn) = = h(x) + \varphi^*(rm)$, $\forall x \in X$ and $\forall r \in R$. Clearly, h^* is a well-defined homomorphism. For all $a \in K$ we have $h^*(a) = h^*(a + 0.n) = h(a) + \varphi^*(0.m) = h(a) = f(a)$ and hence $(h^*{\restriction}K) = f$. Since $K \leq X + Rn \leq N$, it follows that $(X + Rn, h^*) \in S$. Since $(h^* \restriction X) = h$ and $X \leq X + Rn \leq N$, we have $(X, h) \preceq (X + Rn, h^*)$. Since $n \in X + Rn$ and $n \notin X$, it follows that $X \subsetneqq X + Rn$ and this contradicts the maximality of (X, h) in S. Thus $X = N$ and this implies that there exists a homomorphism $h : N \to Q$ such that $(h{\restriction}K) = f$.

(3) $\Rightarrow$ (1). Let $N \le M$ with $(N, K, f, Q) \in \mathcal{L}$. Let $I \in \Omega(N)$, thus there exists an element $n \in N$ such that $ann_R(n) \subseteq I$ and hence there exists an element $n \in M$ such that $ann_R(n) \subseteq I$ and this implies that $I \in \Omega(M)$ and so $\Omega(N) \subseteq \Omega(M)$. By hypothesis, there exists a homomorphism $g : N \to Q$ such that $(g{\restriction}K) = f$. Thus Q is $\mathcal{L}$-N-injective module, for all $N \le M$ and this implies that Q is s-$\mathcal{L}$-M-injective.

There follows the last main result of this section in which we generalize [6, Proposition 14.13, p. 68], [5, Proposition 2, p. 1955] and [12, Lemma 2, p. 542]. It is our version of generalized Fuchs criterion.

Proposition 3 (**Generalized Fuchs criterion**). *Consider the following conditions, where M, $Q \in R$-Mod:*

(1) *Q is s-$\mathcal{L}$-M-injective;*

(2) *for each $(R, I, f, Q) \in \mathcal{L}$ with $\ker(f) \in \Omega(M)$, the homomorphism f extends to a homomorphism from R to Q;*

(3) *for each $(R, I, f, Q) \in \mathcal{L}$ with $\ker(f) \in \Omega(M)$, there exists an element $x \in Q$ such that $f(a) = ax$ $\forall a \in I$.*

Then (2) $\Leftrightarrow$ (3) and if M satisfies condition ($\mathcal{L}$) then (1) implies (2). Moreover, if $\mathcal{L}$ satisfies conditions (α) and (μ), then (2) implies (1).

P r o o f. (2) $\Leftrightarrow$ (3). This is obvious.

(1) $\Rightarrow$ (2). Let M satisfy condition ($\mathcal{L}$) and let $(R, I, f, Q) \in \mathcal{L}$ with $\ker(f) \in \Omega(M)$. Thus there exists an element $m \in M$ such that $ann_R(m) \subseteq \ker(f)$. Since $\ker(f) = (\ker(f) : 1)$, where 1 is the identity element of R, we have $ann_R(m) \subseteq (\ker(f) : 1)$. Since M satisfies condition ($\mathcal{L}$), we get $(Rm, (I : 1)m, f_{(1,m)}, Q) \in \mathcal{L}$ and hence $(Rm, Im, f_{(1,m)}, Q) \in \mathcal{L}$. Since Q is s-$\mathcal{L}$-M-injective, it follows from Proposition 2 that there exists a homomorphism $h : Rm \to Q$ such that $h \circ i_2 = f_{(1,m)}$, where i_2 is the inclusion mapping from Im into Rm. Define $v_1 : I \to Im$ by $v_1(a) = am$, $\forall a \in I$, and define $v_2 : R \to Rm$ by $v_2(r) = rm$, $\forall r \in R$. It is clear that v_1 and v_2 are homomorphisms and for all $a \in I$ we have $(v_2 \circ i_1)(a) = (i_2 \circ v_1)(a)$, where i_1 is the inclusion mapping from I into R. Define $g : R \to Q$ by $g(r) = (h \circ v_2)(r)$, $\forall r \in R$. It is clear that g is a homomorphism and for all $a \in I$ we have that $(g \circ i_1)(a) = f_{(1,m)}(v_1(a)) = f_{(1,m)}(am) = f(a.1) = f(a)$. Thus there exists a homomorphism $g : R \to Q$ such that $(g{\restriction}I) = f$.

(2) $\Rightarrow$ (1). Let $\mathcal{L}$ satisfy conditions (α) and (μ). Let $K \le N \le M$ such that $(N, K, f, Q) \in \mathcal{L}$ and let $S = \{(C, \varphi) \mid K \le C \le N,\ \varphi \in \mathrm{Hom}_R(C, M)$ such that $(\varphi{\restriction}K) = f\}$. Define on S a partial order $\preceq$ by

$$(C_1, \varphi_1) \preceq (C_2, \varphi_2) \Longleftrightarrow C_1 \le C_2 \text{ and } (\varphi_2 \restriction C_1) = \varphi_1.$$

As in the proof of Theorem 1, we can prove that S has a maximal element, say (X, h). It suffices to show that $X = N$. Suppose that there exists an $n \in N \setminus X$. It is clear that $(N, K, f, Q) \preceq (N, X, h, Q)$. Since $(N, K, f, Q) \in \mathcal{L}$ and $\mathcal{L}$ satisfies condition (α), we have $(N, X, h, Q) \in \mathcal{L}$. Since $\mathcal{L}$ satisfies condition (μ) and $n \in N \setminus X$, we get $(R, (X : n), h_n, Q) \in \mathcal{L}$. Since $(0 : n) \subseteq \ker(h_n)$ and $n \in M$, it follows that $\ker(h_n) \in \Omega(M)$. By hypothesis, there exists a homomorphism $\varphi^* : R \to Q$ such that $(\varphi^*{\restriction}(X : n)) = h_n$. Define $h^* : X + Rn \to Q$ by $h^*(x + rn) = h(x) + \varphi^*(r)$, $\forall x \in X$, $\forall r \in R$. We can prove that h^* is a well-defined homomorphism, $(X, h) \preceq (X + Rn, h^*)$ and $(X + Rn, h^*) \in S$. Since $n \in X + Rn$ and $n \notin X$, it follows that $X \subsetneqq X + Rn$ and this contradicts the maximality of (X, h) in S. Thus $X = N$ and this implies that there exists a homomorphism $h : N \to Q$ such that $(h{\restriction}K) = f$. Thus Q is $\mathcal{L}$-N-injective module for all $N \le M$ and hence Q is s-$\mathcal{L}$-M-injective R-module.

§ 3. Direct Sums of $\mathcal{L}$-Injective Modules

The direct sums of $\mathcal{L}$-injective modules is not $\mathcal{L}$-injective, in general. For example: let $\{T_i\}_{i \in I}$ be a family of rings with unit and let $R = \prod_{i \in I} T_i$ be the ring product of the family $\{T_i\}_{i \in I}$, where addition and multiplication are defined componentwise. Let $A = \sqcup_{i\in I} T_i$ be the direct sum of T_i, $\forall i \in I$. If each ${}_{T_i}T_i$ is injective, $\forall\, i \in I$ and I is infinite, then ${}_RA$ is a direct sum of injective modules, but ${}_RA$ is not itself injective, by [15, p. 140]. Hence we have that ${}_RA$ is a direct sum of $\mathcal{L}$-injective modules, but ${}_RA$ is not itself $\mathcal{L}$-injective where $\mathcal{L} = \mathcal{M}$.

Further we study conditions under which the class of $\mathcal{L}$-injective modules is closed under direct sums.

Let $\{E_\alpha\}_{\alpha\in A}$ be a family of modules and let $E = \bigoplus_{\alpha\in A} E_\alpha$. For any $x = (x_\alpha)_{\alpha\in A} \in E$, we define the support of x as the set $\{\alpha \in A|\ x_\alpha \neq 0\}$ and denote it by $\operatorname{supp}(x)$. For any $X \subseteq E$, we define $\operatorname{supp}(X)$ as the set $\bigcup\limits_{x\in X} \operatorname{supp}(x) = \{\alpha \in A \mid (\exists x \in X)\, x_\alpha \neq 0\}$.

The following condition will be useful later.

(F): Let $\{E_\alpha\}_{\alpha\in A}$ be a family of modules, where A is an infinite index set and let $\mathcal{L}$ be a subclass of $\mathcal{M}$. We say that $\mathcal{L}$ satisfies condition (F) for a family $\{E_\alpha\}_{\alpha\in A}$, if for any $(R, I, f, \bigoplus_{\alpha\in A} E_\alpha) \in \mathcal{L}$ the set $\operatorname{supp}(\operatorname{im}(f))$ is finite.

Lemma 2. *Suppose that A is any index set, C is any countable subset of A, and $\{E_\alpha\}_{\alpha\in A}$ is any family of modules. Define $\pi_C : \bigoplus_{\alpha\in A} E_\alpha \to \bigoplus_{\alpha\in C} E_\alpha$ by $\pi_C(x) = x_C$, for all $x \in \bigoplus_{\alpha\in A} E_\alpha$ where $\pi_\alpha(x_C) = \pi_\alpha(\pi_C(x)) = \begin{cases} \pi_\alpha(x), & \text{if } \alpha \in C, \\ 0, & \text{if } \alpha \notin C, \end{cases}$ $\forall \alpha \in A$, where π_α is the αth projection homomorphism. Then π_C is a well-defined homomorphism and if $x \in \bigoplus_{\alpha\in C} E_\alpha$, then $\pi_C(x) = x$.*

P r o o f. An easy check.

Lemma 3. *Let $\{M_i\}_{i\in I}$ be any family of modules. If M_i is $\mathcal{L}$-injective, $\forall\, i \in I$ and $\mathcal{L}$ satisfies condition (λ), then $\prod_{i\in I} M_i$ is $\mathcal{L}$-injective.*

P r o o f. This is obvious.

The next corollary immediately follows from Lemma 3.

Corollary 5. *Let $\mathcal{L}$ satisfy condition (λ) and let $\{M_i\}_{i\in I}$ be any family of $\mathcal{L}$-injective modules. If I is a finite set, then $\bigoplus_{i\in I} M_i$ is $\mathcal{L}$-injective.*

Lemma 4. *Let $\mathcal{L}$ satisfy conditions (α), (μ), and (δ) and let $\{E_\alpha\}_{\alpha\in A}$ be any family of $\mathcal{L}$-injective modules, where A is an infinite index set. If $\mathcal{L}$ satisfies condition (F) for a family $\{E_\alpha\}_{\alpha\in A}$, then $\bigoplus_{\alpha\in A} E_\alpha$ is an $\mathcal{L}$-injective module.*

P r o o f. Suppose that $\mathcal{L}$ satisfies condition (F) for the family $\{E_\alpha\}_{\alpha\in A}$ and let $(R, I, f, \bigoplus_{\alpha\in A} E_\alpha) \in \mathcal{L}$. Thus $\operatorname{supp}(\operatorname{im}(f))$ is finite and this implies that $f(I) \subseteq \bigoplus_{\alpha\in F} E_\alpha$, where F is a finite subset of A. Since E_α is $\mathcal{L}$-injective, $\forall \alpha \in F$, it follows from Corollary 5 that $\bigoplus_{\alpha\in F} E_\alpha$ is $\mathcal{L}$-injective. Define $\pi_F : \bigoplus_{\alpha\in A} E_\alpha \to \bigoplus_{\alpha\in F} E_\alpha$ by $\pi_F(x) = x_F$, for all $x \in \bigoplus_{\alpha\in A} E_\alpha$, where $\pi_\alpha(x_F) = \pi_\alpha(\pi_F(x)) = \begin{cases} \pi_\alpha(x), & \text{if } \alpha \in F, \\ 0, & \text{if } \alpha \notin F, \end{cases}$ $\forall \alpha \in A$, where π_α is the αth projection homomorphism. By Lemma 2, it follows that π_F is a well-defined homomorphism. Since $(R, I, f, \bigoplus_{\alpha\in A} E_\alpha) \in \mathcal{L}$ and $\mathcal{L}$ satisfies condition (δ), we have $(R, I, \pi_F \circ f, \bigoplus_{\alpha\in F} E_\alpha) \in \mathcal{L}$. By $\mathcal{L}$-injectivity of $\bigoplus_{\alpha\in F} E_\alpha$, there exists a homomorphism $g : R \to \bigoplus_{\alpha\in F} E_\alpha$ such that $g(a) = (\pi_F \circ f)(a)$, $\forall a \in I$. Put $g' = i_1 \circ g : R \to \bigoplus_{\alpha\in A} E_\alpha$, where $i_1 : \bigoplus_{\alpha\in F} E_\alpha \to \bigoplus_{\alpha\in A} E_\alpha$ is the inclusion homomorphism. Then for each $a \in I$ we have $g'(a) = \pi_F(f(a))$. Since $f(I) \subseteq \bigoplus_{\alpha\in F} E_\alpha$, we have $f(a) \in \bigoplus_{\alpha\in F} E_\alpha$, $\forall a \in I$. Thus, by Lemma 2, it follows that $\pi_F(f(a)) = f(a)$, $\forall a \in I$ and hence $g'(a) = f(a)$, $\forall a \in I$. Since $\mathcal{L}$ satisfies conditions (α) and (μ), it follows from Theorem 1 that $\bigoplus_{\alpha\in A} E_\alpha$ is $\mathcal{L}$-injective.

The following proposition generalizes Proposition 8.13 in [13, p. 83].

Proposition 4. *Let $\mathcal{L}$ satisfy conditions (α), (μ), and (δ) and let $\{E_\alpha\}_{\alpha\in A}$ be any family of $\mathcal{L}$-injective modules, where A is an infinite index set. If $\bigoplus_{\alpha\in C} E_\alpha$ is an $\mathcal{L}$-injective module for any countable subset C of A, then $\bigoplus_{\alpha\in A} E_\alpha$ is an $\mathcal{L}$-injective module.*

Proof. Let $\pi_\beta : \bigoplus_{\alpha\in A} E_\alpha \to E_\beta$ be the natural projection homomorphism. Assume that $\bigoplus_{\alpha\in A} E_\alpha$ is not $\mathcal{L}$-injective, thus by Lemma 4 there exists $(R, I, f, \bigoplus_{\alpha\in A} E_\alpha) \in \mathcal{L}$ such that $\mathrm{supp}(\mathrm{im}(f))$ is infinite. Since $\mathrm{supp}(\mathrm{im}(f))$ is an infinite set, it follows that $\mathrm{supp}(\mathrm{im}(f))$ contains a countable infinite subset, say C. For any $\alpha \in C$, we have $\alpha \in \mathrm{supp}(\mathrm{im}(f))$ and this implies that there exists an $x \in \mathrm{im}(f)$ such that $x_\alpha \neq 0$. Thus for any $\alpha \in C$ we have $\pi_\alpha(\mathrm{im}(f)) \neq 0$. Define $\pi_C : \bigoplus_{\alpha\in A} E_\alpha \to \bigoplus_{\alpha\in C} E_\alpha$ as in Lemma 2. Note that $C = \mathrm{supp}(\mathrm{im}(\pi_C \circ f))$. Since $(R, I, f, \bigoplus_{\alpha\in A} E_\alpha) \in \mathcal{L}$ and $\mathcal{L}$ satisfies condition (γ), it follows that $(R, I, \pi_C \circ f, \bigoplus_{\alpha\in C} E_\alpha) \in \mathcal{L}$. Since C is a countable subset of A, it follows from the hypothesis that $\bigoplus_{\alpha\in C} E_\alpha$ is $\mathcal{L}$-injective. By Theorem 1, there exists an element $y \in \bigoplus_{\alpha\in C} E_\alpha$ such that $(\pi_C \circ f)(a) = ay$, $\forall a \in I$. Let $\alpha \in \mathrm{supp}(\mathrm{im}(\pi_C \circ f))$, thus there is an $r \in I$ such that $\pi_\alpha((\pi_C \circ f)(r)) \neq 0$. Hence $\pi_\alpha(ry) \neq 0$ and this implies that $\pi_\alpha(y) \neq 0$. Thus $\alpha \in \mathrm{supp}(y)$ and hence $\mathrm{supp}(\mathrm{im}(\pi_C \circ f)) \subseteq \mathrm{supp}(y)$. Since $C = \mathrm{supp}(\mathrm{im}(\pi_C \circ f))$, we have $C \subseteq \mathrm{supp}(y)$ and this is a contradiction, since $\mathrm{supp}(y)$ is finite (because $y \in \bigoplus_{\alpha\in C} E_\alpha$) and C is infinite. Thus $\bigoplus_{\alpha\in A} E_\alpha$ is an $\mathcal{L}$-injective module.

By Proposition 4 and Lemma 1 we can prove the following corollary.

Corollary 6. *Let ρ_1 and ρ_2 be two P-filters and let $\{E_\alpha\}_{\alpha\in A}$ be any family of modules, where A is an infinite index set. If $\bigoplus_{\alpha\in C} E_\alpha$ is an $\mathcal{L}_{(\rho_1,\rho_2)}$-injective module for any countable subset C of A, then $\bigoplus_{\alpha\in A} E_\alpha$ is an $\mathcal{L}_{(\rho_1,\rho_2)}$-injective module.*

Now we can state the following result, found in [13, Proposition 8.13, p. 83] as a corollary.

Corollary 7. *Let $\{E_\alpha\}_{\alpha\in A}$ be any family of τ-injective modules, where A is an infinite index set. If $\bigoplus_{\alpha\in C} E_\alpha$ is a τ-injective module for any countable subset C of A, then $\bigoplus_{\alpha\in A} E_\alpha$ is a τ-injective module.*

Proof. By taking the two P-filters $\rho_1 = \rho_\tau$ and $\rho_2 = \Re$ and applying Corollary 6.

Since the class of $\mathcal{L}$-injective modules is closed under isomorphism, when $\mathcal{L}$ satisfies (γ), it follows from Proposition 4 that we have the next corollary.

Corollary 8. *Consider the following three conditions, where $\mathcal{K}$ is a nonempty class of R-modules.*
(1) *Every direct sum of $\mathcal{L}$-injective R-modules in $\mathcal{K}$ is $\mathcal{L}$-injective.*
(2) *Every countable direct sum of $\mathcal{L}$-injective R-modules in $\mathcal{K}$ is $\mathcal{L}$-injective.*
(3) *For any family $\{E_i\}_{i\in N}$ of $\mathcal{L}$-injective R-modules in $\mathcal{K}$, $\bigoplus_{i\in N} E_i$ is $\mathcal{L}$-injective.*
Then (1) implies (2) and (2) implies (3), and if $\mathcal{L}$ satisfies conditions (α), (μ), and (δ), then (2) implies (1). Moreover, if $\mathcal{L}$ satisfies condition (γ), then (3) implies (2).

Definition 4. A submodule N of a module M is said to be *strongly* $\mathcal{L}$-dense in M (shortly, s-$\mathcal{L}$-dense) if $(M, N, I_N, N) \in \mathcal{L}$, where I_N is the identity homomorphism from N into N.

The following lemmas are clear.

Lemma 5. *If $N \le K \le M$ are modules such that N is s-$\mathcal{L}$-dense in M and $\mathcal{L}$ satisfies conditions (α) and (β), then K is s-$\mathcal{L}$-dense in M.*

Lemma 6. *Let ρ be any P-filter. Then $(M, N) \in \rho$ if and only if N is s-$\mathcal{L}_{(\rho,\infty)}$-dense in M.*

Following [10, p. 21], for any module M, denote by $H_{\mathcal{K}}(M)$ the set of left submodules N of M such that $(M/N) \in \mathcal{K}$, where $\mathcal{K}$ is any nonempty class of modules (i. e., $H_{\mathcal{K}}(M) = \{N \leq M \mid (M/N) \in \mathcal{K}\}$). In particular, $H_{\mathcal{K}}(R) = \{I \leq R \mid (R/I) \in \mathcal{K}\}$.

The following theorem is the first main result of this section.

Theorem 4. *Let $\mathcal{L}$ satisfy conditions (α) and (δ) and let $\mathcal{K}$ be any nonempty class of modules closed under isomorphic copies and $\mathcal{L}$-injective hulls. If the direct sum of any family $\{E_i\}_{i\in N}$ of $\mathcal{L}$-injective R-modules in $\mathcal{K}$ is $\mathcal{L}$-injective, then every ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ of left ideals of R in $H_{\mathcal{K}}(R)$ with $I_\infty = \bigcup_{j=1}^{\infty} I_j$ s-$\mathcal{L}$-dense in R, terminates.*

P r o o f. Let $I_1 \subseteq I_2 \subseteq \ldots$ be any ascending chain of left ideals of R in $H_{\mathcal{K}}(R)$ with $I_\infty = \bigcup_{j=1}^{\infty} I_j$ being a s-$\mathcal{L}$-dense left ideal in R. Thus $(R/I_j) \in \mathcal{K}$ $\forall j \in \mathbb{N}$. Since $\mathcal{L}$ satisfies conditions (α), (β), and (γ), it follows from [14, Theorem 1.12, p. 625] that every R-module M has an $\mathcal{L}$-injective hull which is unique up to M-isomorphism. Let $E_{\mathcal{L}}(R/I_j)$ be the $\mathcal{L}$-injective hull of R/I_j, $\forall j \in \mathbb{N}$. Since $\mathcal{K}$ is closed under $\mathcal{L}$-injective hulls, it follows that $E_{\mathcal{L}}(R/I_j) \in \mathcal{K}$, $\forall j \in \mathbb{N}$. Define $f : I_\infty = \bigcup_{j=1}^{\infty} I_j \to \bigoplus_{j=1}^{\infty} E_{\mathcal{L}}(R/I_j)$ by $f(r) = (r + I_j)_{j\in\mathbb{N}}$, for $r \in I_\infty$. Note that f is a well-defined mapping: for any $r \in I_\infty$, let n be the smallest positive integer such that $r \in I_n$. Since $I_n \subseteq I_{n+k}$, $\forall k \in \mathbb{N}$, we have $r \in I_{n+k}$ $\forall k \in \mathbb{N}$ and so $r + I_{n+k} = 0$, $\forall k \in \mathbb{N}$. Thus $(r + I_j)_{j\in\mathbb{N}} = (r + I_1, r + I_2, \ldots, r + I_{n-1}, 0, 0, \ldots) \in \bigoplus_{j=1}^{\infty} E_{\mathcal{L}}(R/I_j)$. Thus $f(I) \subseteq \bigoplus_{j=1}^{\infty} E_{\mathcal{L}}(R/I_j)$ and hence f is a well-defined mapping. It is clear that f is a homomorphism. Since I_∞ is a s-$\mathcal{L}$-dense left ideal in R, it follows that $(R, I_\infty, I_{I_\infty}, I_\infty) \in \mathcal{L}$. Since $\mathcal{L}$ satisfies condition (δ), we have $(R, I_\infty, f, \bigoplus_{j=1}^{\infty} E_{\mathcal{L}}(R/I_j)) \in \mathcal{L}$. Since $E_{\mathcal{L}}(R/I_j)$ is an $\mathcal{L}$-injective R-module in $\mathcal{K}$, $\forall j \in \mathbb{N}$, it follows from the hypothesis that $\bigoplus_{j=1}^{\infty} E_{\mathcal{L}}(R/I_j)$ is an $\mathcal{L}$-injective R-module. Thus, by Theorem 1, there exists an element $x \in \bigoplus_{j=1}^{\infty} E_{\mathcal{L}}(R/I_j)$ such that $f(r) = rx$ $\forall r \in I_\infty$. Since $x \in \bigoplus_{j=1}^{\infty} E_{\mathcal{L}}(R/I_j)$, we have $x = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$, for some $n \in \mathbb{N}$, and hence $(r + I_j)_{j\in N} = (rx_1, rx_2, \ldots, rx_n, 0, 0, \ldots)$ and this implies that $r + I_{n+k} = 0$, $\forall k \geq 1$ and $\forall r \in I_\infty$, Thus, $r \in I_{n+k}$, $\forall k \geq 1$ and $\forall r \in I_\infty$, and so $I_\infty = \bigcup_{j=1}^{\infty} I_j \subseteq I_{n+k}$, $\forall k \geq 1$. Since $I_{n+k} \subseteq I_\infty$, it follows that $I_\infty = I_{n+k}, \forall k \geq 1$, $I_t = I_{t+j}$, $\forall j \in \mathbb{N}$. Therefore the ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ terminates.

Now we will state the condition (I) on $\mathcal{L}$ as follows:

(I) : $(R, J, f, Q) \in \mathcal{L}$ implies that J is s-$\mathcal{L}$-dense in R. That is, $(R, J, f, Q) \in \mathcal{L}$ implies $(R, J, I_J, J) \in \mathcal{L}$.

Proposition 5. *Consider the following two conditions, where $\mathcal{K}$ is a nonempty class of R-modules.*

(1) *Every ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ of left ideals of R in $H_{\mathcal{K}}(R)$ with $I_\infty = \bigcup_{j=1}^{\infty} I_j$ s-$\mathcal{L}$-dense in R, terminates.*

(2) *The following conditions hold:*

(*a*) *$H_{\mathcal{K}}(R)$ has ACC on s-$\mathcal{L}$-dense left ideals in R;*

(*b*) *for every ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ of left ideals of R in $H_{\mathcal{K}}(R)$, where $I_\infty = \bigcup_{j=1}^{\infty} I_j$ is s-$\mathcal{L}$-dense in R, there exists a positive integer n such that I_n is s-$\mathcal{L}$-dense in R.*

If $\mathcal{L}$ satisfies conditions (α) and (β), then (1) *and* (2) *are equivalent.*

P r o o f. This is obvious.

Now we will give the second main result of this section.

Theorem 5. *Let $\mathcal{L}$ satisfy conditions (α), (μ), (δ), and (I) and let $\mathcal{K}$ be any nonempty class of modules closed under isomorphic copies and submodules. If every ascending chain $J_1 \subseteq J_2 \subseteq \ldots$ of left ideals of R, where $(J_{i+1}/J_i) \in \mathcal{K}$, $\forall i \in \mathbb{N}$ and $J_\infty = \bigcup_{i=1}^{\infty} J_i$ is s-$\mathcal{L}$-dense in R, terminates, then every direct sum of $\mathcal{L}$-injective modules in $\mathcal{K}$ is $\mathcal{L}$-injective.*

P r o o f. Let $\{E_i\}_{i\in\mathbb{N}}$ be any family of $\mathcal{L}$-injective modules in $\mathcal{K}$ and let $(R, J, f, \bigoplus_{i\in\mathbb{N}} E_i) \in \mathcal{L}$. For any $n \in \mathbb{N}$, put $J_n = \{x \in J \mid f(x) \in \bigoplus_{i=1}^{n} E_i\} = f^{-1}(\bigoplus_{i=1}^{n} E_i)$. It is clear that $J_1 \subseteq J_2 \subseteq \ldots$. Also, we have $J_\infty = \bigcup_{n\in\mathbb{N}} J_n = \bigcup_{n\in\mathbb{N}}(f^{-1}(\bigoplus_{i=1}^{n} E_i)) = f^{-1}(\bigcup_{n\in\mathbb{N}}(\bigoplus_{i=1}^{n} E_i) = f^{-1}(\bigoplus_{i=1}^{\infty} E_i)$. Since $(R, J, f, \bigoplus_{i\in\mathbb{N}} E_i) \in \mathcal{L}$ and $\mathcal{L}$ satisfies condition (I), it follows that $J = \bigcup_{i\in\mathbb{N}} J_i$ is s-$\mathcal{L}$-dense in R. For all $n \in \mathbb{N}$, define $\alpha_n : J_{n+1}/J_n \to \bigoplus_{i=1}^{n+1} E_i / \bigoplus_{i=1}^{n} E_i$ by $\alpha_n(x + J_n) = f(x) + (\bigoplus_{i=1}^{n} E_i)$, $\forall x \in I_{n+1}$. Then α_n is a well-defined monomorphism, since $J_n = f^{-1}(\bigoplus_{i=1}^{n} E_i)$. Since $(\bigoplus_{i=1}^{n+1} E_i / \bigoplus_{i=1}^{n} E_i) \simeq E_{n+1} \in \mathcal{K}$ and $\mathcal{K}$ is closed under isomorphic copies, we have $(\bigoplus_{i=1}^{n+1} E_i / \bigoplus_{i=1}^{n} E_i) \in \mathcal{K}$. Since $\mathrm{im}(\alpha_n) \le (\bigoplus_{i=1}^{n+1} E_i / \bigoplus_{i=1}^{n} E_i) \in \mathcal{K}$, and $\mathcal{K}$ is closed under submodules, it follows that $\mathrm{im}(\alpha_n) \in \mathcal{K}$. Since $(J_{n+1}/J_n) \simeq \mathrm{im}(\alpha_n)$ and $\mathcal{K}$ is closed under isomorphic copies, we obtain $(J_{n+1}/J_n) \in \mathcal{K}$. Thus we have the following ascending chain $J_1 \subseteq J_2 \subseteq \ldots$ of left ideals of R such that $(J_{i+1}/J_i) \in \mathcal{K}$, $\forall i \in \mathbb{N}$ and $J_\infty = \bigcup_{i=1}^{\infty} J_i$ is s-$\mathcal{L}$-dense in R. By hypothesis, there exists a positive integer n such that $J_n = J_{n+i}$, $\forall i \in \mathbb{N}$. Thus $J = J_\infty = \bigcup_{i=1}^{\infty} J_i = J_n$. This implies that $f(J) \subseteq \bigoplus_{i=1}^{n} E_i$. Thus $\mathrm{supp}(\mathrm{im}(f))$ is finite and hence $\mathcal{L}$ satisfies condition (F) for a family $\{E_i\}_{i\in\mathbb{N}}$. Thus by Lemma 4 we see that $\bigoplus_{i\in\mathbb{N}} E_i$ is an $\mathcal{L}$-injective module. Thus for any family $\{E_i\}_{i\in N}$ of $\mathcal{L}$-injective R-modules in $\mathcal{K}$, we have $\bigoplus_{i\in N} E_i$ is $\mathcal{L}$-injective. Since $\mathcal{L}$ satisfies conditions (α), (μ), and (δ), it follows from Corollary 8, that every direct sum of $\mathcal{L}$-injective modules in $\mathcal{K}$ is $\mathcal{L}$-injective.

A nonempty class $\mathcal{K}$ of modules is said to be a natural class if it is closed under submodules, arbitrary direct sums and injective hulls [9]. Examples of natural classes include R-Mod, any hereditary torsionfree classes, and stable hereditary torsion classes.

Now we can state the following result, found in [17, p. 643] as a corollary.

Corollary 9. *Let $\mathcal{K}$ be a natural class of modules closed under isomorphic copies. Then the following statements are equivalent:*

(1) *every direct sum of injective modules in $\mathcal{K}$ is injective;*

(2) *$H_{\mathcal{K}}(R)$ has ACC.*

P r o o f. (1) ⇒ (2). By taking $\mathcal{L} = \mathcal{M}$ and applying Lemma 1, Lemma 6 and Theorem 4.

(2) ⇒ (1). By taking $\mathcal{L} = \mathcal{M}$ and applying [17, Lemma 7, p. 637] and Theorem 5.

Corollary 10. *Let ρ be any P-filter and let $\mathcal{K}$ be any nonempty class of modules closed under isomorphic copies and submodules. If every ascending chain $J_1 \subseteq J_2 \subseteq \ldots$ of left ideals of R such that $(J_{i+1}/J_i) \in \mathcal{K}$, $\forall i \in \mathbb{N}$ and $J_\infty = \bigcup_{i=1}^{\infty} J_i$ is s-$\mathcal{L}_{(\rho,\infty)}$-dense in R terminates, then every direct sum of $\mathcal{L}_{(\rho,\infty)}$-injective modules in $\mathcal{K}$ is $\mathcal{L}_{(\rho,\infty)}$-injective.*

P r o o f. By Lemma 1, Lemma 6 and Theorem 5.

Let τ be a hereditary torsion theory. A nonempty class $\mathcal{K}$ of modules is said to be τ-natural class if $\mathcal{K}$ is closed under submodules, isomorphic copies, arbitrary direct sums and τ-injective hulls [8, p. 163].

Corollary 11 (**see [8, Proposition 5.3.5, p. 165]**). *Let $\mathcal{K}$ be a τ-natural and suppose that every ascending chain $J_1 \subseteq J_2 \subseteq \ldots$ of left ideals of R such that $(J_{i+1}/J_i) \in \mathcal{K}$, $\forall i \in \mathbb{N}$ and $J_\infty = \bigcup_{i=1}^{\infty} J_i$ is τ-dense in R terminates. Then every direct sum of τ-injective modules in $\mathcal{K}$ is τ-injective.*

P r o o f. Take $\rho = \rho_\tau$ and apply Corollary 10.

The following corollary, in which we give conditions under which the class of $\mathcal{L}$-injective modules is closed under direct sums, is one of the main aims of this section.

Corollary 12. *Consider the following three conditions:*

(1) *the class of $\mathcal{L}$-injective R-modules is closed under direct sums;*

(2) *every ascending chain* $I_1 \subseteq I_2 \subseteq \ldots$ *of left ideals of* R*, where* $I_\infty = \bigcup_{j=1}^{\infty} I_j$ *is* s-$\mathcal{L}$*-dense in* R*, terminates;*

(3) *the following conditions hold:*

(a) *every ascending chain* $I_1 \subseteq I_2 \subseteq \ldots$ *of* s-$\mathcal{L}$*-dense left ideals of* R *terminates;*

(b) *for every ascending chain* $I_1 \subseteq I_2 \subseteq \ldots$ *of left ideals of* R*, where* $I_\infty = \bigcup_{j=1}^{\infty} I_j$ *is* s-$\mathcal{L}$*-dense in* R*, there exists a positive integer* n *such that* I_n *is* s-$\mathcal{L}$*-dense in* R*.*

If $\mathcal{L}$ *satisfies conditions* (α) *and* (δ)*, then* (1) *implies* (2)*. Also,* (2) *implies* (3b) *and if* $\mathcal{L}$ *satisfies conditions* (α) *and* (β)*, then* (2) *implies* (3a)*. Moreover, if* $\mathcal{L}$ *satisfies conditions* (α)*,* (μ)*,* (δ)*, and* (I)*, then all above three conditions are equivalent.*

P r o o f. By taking $\mathcal{K} = R$-Mod and applying Theorem 4 and Proposition 5.

Corollary 13. *Let* ρ *be any* P*-filter. Then the following statements are equivalent.*

(1) *The class of* $\mathcal{L}_{(\rho,\rho_\infty)}$*-injective* R*-modules is closed under direct sums.*

(2) *Every ascending chain* $I_1 \subseteq I_2 \subseteq \ldots$ *of left ideals of* R*, where* $I_\infty = \bigcup_{j=1}^{\infty} I_j$ *is* s-$\mathcal{L}_{(\rho,\rho_\infty)}$*-dense in* R*, terminates.*

(3) *The following conditions hold.*

(a) *Every ascending chain* $I_1 \subseteq I_2 \subseteq \ldots$ *of* s-$\mathcal{L}_{(\rho,\rho_\infty)}$*-dense left ideals of* R *terminates.*

(b) *For every ascending chain* $I_1 \subseteq I_2 \subseteq \ldots$ *of left ideals of* R*, where* $I_\infty = \bigcup_{j=1}^{\infty} I_j$ *is* s-$\mathcal{L}_{(\rho,\rho_\infty)}$*-dense in* R*, there exists a positive integer* n *such that* I_n *is* s-$\mathcal{L}_{(\rho,\rho_\infty)}$*-dense in* R*.*

(4) *For any family* $\{E_i\}_{i\in N}$ *of* $\mathcal{L}_{(\rho,\rho_\infty)}$*-injective* R*-modules,* $\bigoplus_{i\in N} E_i$ *is* $\mathcal{L}_{(\rho,\rho_\infty)}$*-injective.*

P r o o f. By Lemma 1 and Lemma 6, it follows that $\mathcal{L}_{(\rho,\rho_\infty)}$ satisfies conditions (α), (μ), (δ), and (I). Thus, by Corollary 12 and Corollary 8, we have the equivalence of above four statements.

Corollary 14 (**see [8, Theorem 2.3.8, p. 73]**). *The following statements are equivalent:*

(1) R *has* ACC *on* τ*-dense left ideals and* τ *is Noetherian;*

(2) *the class of* τ*-injective* R*-modules is closed under direct sums;*

(3) *the class of* τ*-injective* R*-modules is closed under countable direct sums.*

P r o o f. Take $\rho = \rho_\tau$ and apply Corollary 13.

§ 4. $\sum$-$\mathcal{L}$-injective modules

Carl Faith in [11] introduced the concepts of $\sum$-injectivity and countably $\sum$-injectivity as follows. An injective module E is said to be $\sum$-injective if $E^{(A)}$ is injective for any index set A; E is said to be countably $\sum$-injective in case $E^{(C)}$ is injective for any countable index set C. Faith in [11] proved that an injective R-module E is $\sum$-injective if and only if R satisfies ACC on the E-annihilator left ideals if and only if E is countably $\sum$-injective. Charalambides in [6] introduced the concept of $\sum$-τ-injectivity and generalized Faith's result.

In this section, we introduce the concept of $\sum$-$\mathcal{L}$-injectivity as a general case of $\sum$-injectivity and $\sum$-τ-injectivity and prove the result (Theorem 6) in which we generalize Faith's result [11, Proposition 3, p. 184] and [6, Theorem 16.16, p. 98].

We start this section with the following definition of a $\sum$-$\mathcal{L}$-injective module.

Definition 5. Let E be an $\mathcal{L}$-injective module. We say that E is $\sum$-$\mathcal{L}$-injective if $E^{(A)}$ is $\mathcal{L}$-injective for any index set A. On the other hand, if $E^{(C)}$ is $\mathcal{L}$-injective for any countable index set C, we say that E is countably $\sum$-$\mathcal{L}$-injective.

The following corollary is a special case of Corollary 8, by taking $\mathcal{K} = \{E\}$.

Corollary 15. *Consider the following conditions.*
(1) *E is $\sum$-$\mathcal{L}$-injective.*
(2) *E is countably $\sum$-$\mathcal{L}$-injective.*
(3) *$E^{(\mathbb{N})}$ is $\mathcal{L}$-injective.*
Then: (1) implies (2) and (2) implies (3). If $\mathcal{L}$ satisfies conditions (α), (μ), and (δ), then (2) implies (1). Moreover, if $\mathcal{L}$ satisfies condition (γ), then (3) implies (2).

The next corollary is immediately follows from Lemma 1 and Corollary 15.

Corollary 16. *Let ρ_1 and ρ_2 be any two P-filters. Then the following conditions are equivalent for a module E.*
(1) *E is $\sum$-$\mathcal{L}_{(\rho_1,\rho_2)}$-injective.*
(2) *E is countably $\sum$-$\mathcal{L}_{(\rho_1,\rho_2)}$-injective.*
(3) *$E^{(\mathbb{N})}$ is $\mathcal{L}_{(\rho_1,\rho_2)}$-injective.*

Let E be a module. A left ideal I of R is said to be an E-annihilator if there is $N \subseteq E$ such that $I = (0 : N) = \{r \in R \mid rN = 0\}$ (i. e., I is the annihilator of a subset of E).

The following theorem is the main result of this section in which we generalize [6, Theorem 16.16, p. 98] and [11, Proposition 3, p. 184].

Theorem 6. *Consider the following three conditions for an $\mathcal{L}$-injective module E:*
(1) *E is countably $\sum$-$\mathcal{L}$-injective;*
(2) *every ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ of E-annihilators in R, where $I_\infty = \bigcup_{j=1}^{\infty} I_j$ is s-$\mathcal{L}$-dense in R, terminates;*
(3) *The following conditions hold.*
(*a*) *Every ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ of E-annihilators in R, where I_j is s-$\mathcal{L}$-dense in R $\forall j \in \mathbb{N}$, terminates.*
(*b*) *For every ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ of E-annihilators in R, where $I_\infty = \bigcup_{j=1}^{\infty} I_j$ is s-$\mathcal{L}$-dense in R, there exists a positive integer n such that I_n is s-$\mathcal{L}$-dense in R.*
Then: if $\mathcal{L}$ satisfies condition (δ), then (1) implies (2). Also, (2) implies (3b) and if $\mathcal{L}$ satisfies conditions (α) and (β), then (2) implies (3a). Moreover, if $\mathcal{L}$ satisfies conditions (α), (μ), (β), and (I), then (3) implies (1).

P r o o f. (1) $\Rightarrow$ (2). Let $\mathcal{L}$ satisfy condition (δ). Assume that (2) does not hold. Then there exist E-annihilators $I_1, I_2, \ldots$ in R such that $I_1 \subsetneqq I_2 \subsetneqq \ldots$ and $I_\infty = \bigcup_{j=1}^{\infty} I_j$ is s-$\mathcal{L}$-dense in R. Hence we have the following descending chain $r_E(I_1) \supsetneqq r_E(I_2) \supsetneqq \ldots$. For every $n \in \mathbb{N}$, choose $x_n \in r_E(I_n) - r_E(I_{n+1})$, thus $x = (x_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$. Define $f : I_\infty \to E^{\mathbb{N}}$ by $f(a) = ax$, $\forall a \in I_\infty$. It is clear that f is a homomorphism. For a fixed $a \in I_\infty$ let n be the smallest positive integer such that $a \in I_n$. Then, for every $k \geq 0$, $a \in I_n \subseteq I_{n+k}$. Since $x_{n+k} \in r_E(I_{n+k})$, we have $ax_{n+k} = 0$, $\forall k \geq 0$. Hence $ax \in E^{(\mathbb{N})}$. Thus f is a homomorphism from I_∞ into $E^{(\mathbb{N})}$. Since I_∞ is s-$\mathcal{L}$-dense in R, it follows that $(R, I_\infty, I_{I_\infty}, I_\infty) \in \mathcal{L}$. Since $\mathcal{L}$ satisfies condition (δ), we get $(R, I_\infty, f, E^{(\mathbb{N})}) \in \mathcal{L}$. Since $E^{(\mathbb{N})}$ is $\mathcal{L}$-injective, it follows from Theorem 1 that there exists an element $y \in E^{(\mathbb{N})}$ such that $f(a) = ay$, $\forall a \in I_\infty$. Since $y \in E^{(\mathbb{N})}$, we have $y = (y_1, y_2, \ldots, y_t, 0, 0, \ldots)$, for some $t \in \mathbb{N}$. Since $ax = f(a) = ay$, $\forall a \in I_\infty$, it follows that $(ax_1, ax_2, \ldots) = (ay_1, ay_2, \ldots, ay_t, 0, 0, \ldots)$ and this implies that $ax_{t+1} = 0$, $\forall a \in I_\infty$ and hence $x_{t+1} \in r_E(I_\infty)$. Since $I_{t+2} \subsetneqq I_\infty$, we have $r_E(I_\infty) \subseteq r_E(I_{t+2})$ and so $x_{t+1} \in r_E(I_{t+2})$. This contradicts the fact that $x_{t+1} \in r_E(I_{t+1}) - r_E(I_{t+2})$.

(2) $\Rightarrow$ (3*b*). Let $I_1 \subseteq I_2 \subseteq \ldots$ be any ascending chain of E-annihilators in R, where $I_\infty = \bigcup_{j=1}^{\infty} I_j$ is s-$\mathcal{L}$-dense in R. By hypothesis, there exists a positive integer n such that $I_n = I_{n+k}$, $\forall k \in \mathbb{N}$ and so $I_n = I_\infty$. Hence I_n is s-$\mathcal{L}$-dense in R.

(2) $\Rightarrow$ (3*a*). Let $\mathcal{L}$ satisfy conditions (α) and (β) and let $I_1 \subseteq I_2 \subseteq \ldots$ be any ascending chain of E-annihilators in R such that I_j are s-$\mathcal{L}$-dense left ideals of R. Since $I_1 \subseteq I_\infty$ and $\mathcal{L}$ satisfies conditions (α) and (β), we have from Lemma 5 that I_∞ is a s-$\mathcal{L}$-dense left ideal of R. By hypothesis, the chain $I_1 \subseteq I_2 \subseteq \ldots$ terminates.

(3) ⇒ (1). Let $\mathcal{L}$ satisfy conditions (α), (μ), (β), and (I) and let $(R, J, f, E^{(\mathbb{N})}) \in \mathcal{L}$. Since E is $\mathcal{L}$-injective, we have from Lemma 3 that $E^{\mathbb{N}}$ is $\mathcal{L}$-injective. Since $E^{(\mathbb{N})}$ is a submodule of $E^{\mathbb{N}}$, it follows that $g = i \circ f : J \to E^{\mathbb{N}}$ is a homomorphism, where $i : E^{(\mathbb{N})} \to E^{\mathbb{N}}$ is the inclusion homomorphism. Since $\mathcal{L}$ satisfies condition (β), we have $(R, J, i \circ f, E^{\mathbb{N}}) \in \mathcal{L}$. Thus, by Theorem 1, there is an element $x = (x_1, x_2, \ldots) \in E^{\mathbb{N}}$ such that $g(a) = ax$, $\forall a \in J$. Thus $f(a) = g(a) = ax$, $\forall a \in J$. Let $X = \{x_1, x_2, \ldots\}$ and $X_k = = X \setminus \{x_1, x_2, \ldots, x_k\} = \{x_{k+1}, x_{k+2}, \ldots\}$ for all $k \geq 1$. Thus we have the following descending chain of subsets of X : $X \supseteq X_1 \supseteq X_2 \supseteq \ldots$; this yields an ascending chain of E-annihilators in R: $l_R(X) \subseteq l_R(X_1) \subseteq l_R(X_2) \subseteq \ldots$. Let $J_{k+1} = l_R(X_k)$, for all $k \geq 0$, where $X_0 = X$ and $J_\infty = \bigcup_{i=1}^{\infty} J_i$. Since $f(J) \subseteq E^{(\mathbb{N})}$, it follows that, for any $a \in J$, either $ax_k = 0$, $\forall k \in \mathbb{N}$, or there is a largest integer $n \in \mathbb{N}$ such that $ax_n \neq 0$. If there is a largest integer $n \in \mathbb{N}$ such that $ax_n \neq 0$, then $ax_{n+k} = 0$, $\forall k \geq 1$. Therefore, $a \in l_R(X_n) = J_{n+1} \subseteq J_\infty$. Thus for any $a \in J$, we have $a \in J_\infty$, and this implies that $J \subseteq J_\infty$. Since $(R, J, f, E^{(\mathbb{N})}) \in \mathcal{L}$ and $\mathcal{L}$ satisfies condition (I), it follows that J is s-$\mathcal{L}$-dense left ideal in R. Since $J \subseteq J_\infty$ and $\mathcal{L}$ satisfies conditions (α) and (β), we have from Lemma 5 that J_∞ is s-$\mathcal{L}$-dense left ideal in R. Thus we have the following ascending chain $J_1 \subseteq J_2 \subseteq \ldots$ of E-annihilators in R such that J_∞ is s-$\mathcal{L}$-dense left ideal in R. By applying condition $(3b)$, there is an $s \in \mathbb{N}$ such that J_s is s-$\mathcal{L}$-dense left ideal in R. Since $J_s \subseteq J_{s+k}$, $\forall k \in \mathbb{N}$ and $\mathcal{L}$ satisfies conditions (α) and (β), it follows from Lemma 5 that J_{s+k} is s-$\mathcal{L}$-dense left ideal in R, $\forall k \in \mathbb{N}$. Thus we have the following ascending chain $J_s \subseteq J_{s+1} \subseteq \ldots$ of E-annihilators in R such that J_{s+k} is s-$\mathcal{L}$-dense left ideal in R $\forall k \in \mathbb{N}$. By applying condition $(3a)$, the chain $J_s \subseteq J_{s+1} \subseteq \ldots$ becomes stationary at a left ideal of R, say $J_t = l_R(X_{t-1})$ and so $J_t = J_\infty$. Thus for any $a \in J$, we have $ax_{t+k} = 0$, $\forall k \geq 0$ and then $a(0, 0, \ldots, 0, x_t, x_{t+1}, \ldots) = 0$. Take $y = (x_1, x_2, \ldots, x_{t-1}, 0, 0, \ldots)$. It is clear that $y \in E^{(\mathbb{N})}$ and for any $a \in J$, then $f(a) = ax = ax - a(0, 0, \ldots, 0, x_t, x_{t+1}, 0, 0, \ldots) = a(x_1, x_2, \ldots, x_{t-1}, 0, 0, \ldots) = ay$. Thus for every $(R, J, f, E^{(\mathbb{N})}) \in \mathcal{L}$ there exists an element $y \in E^{(\mathbb{N})}$ such that $f(a) = ay$, $\forall a \in J$. Since $\mathcal{L}$ satisfies conditions (α) and (μ), it follows from Theorem 1 that $E^{(\mathbb{N})}$ is $\mathcal{L}$-injective. Since $\mathcal{L}$ satisfies condition (γ), it follows from Corollary 15 that E is countably $\sum$-$\mathcal{L}$-injective.

Corollary 17. *Let ρ be any P-filter. Then the following conditions are equivalent.*

(1) *E is countably $\sum$-$\mathcal{L}_{(\rho,\infty)}$-injective.*

(2) *Every ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ of E-annihilators in R, where $I_\infty = \bigcup_{j=1}^{\infty} I_j$ is s-$\mathcal{L}_{(\rho,\infty)}$-dense left ideal in R, terminates.*

(3) *The following conditions hold.*

(a) *Every ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ of E-annihilators in R, where I_j is s-$\mathcal{L}_{(\rho,\infty)}$-dense left ideals of R $\forall j \in \mathbb{N}$, terminates.*

(b) *For every ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ of E-annihilators in R, where $I_\infty = \bigcup_{j=1}^{\infty} I_j$ is s-$\mathcal{L}_{(\rho,\infty)}$-dense left ideal in R, there exists a positive integer n such that I_n is s-$\mathcal{L}_{(\rho,\infty)}$-dense in R.*

(4) *E is $\sum$-$\mathcal{L}_{(\rho,\infty)}$-injective.*

P r o o f. By Lemma 1, Lemma 6 and Theorem 6, we have the equivalence of (1), (2), and (3).

(1) ⇔ (4). By Corollary 15.

Corollary 18 (see **[6, Theorem 16.16, p. 98]**). *Let τ be any hereditary torsion theory and let E be τ-injective module. Then the following conditions are equivalent.*

(1) *E is countably $\sum$-τ-injective.*

(2) *Every ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ of E-annihilators in R, where $I_\infty = \bigcup_{j=1}^{\infty} I_j$ is τ-dense left ideal in R, terminates.*

(3) *The following conditions hold.*

(a) *Every ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ of E-annihilators in R, where I_j is τ-dense left ideals of R $\forall j \in \mathbb{N}$, terminates.*

(b) *For every ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ of E-annihilators in R, where $I_\infty = \bigcup_{j=1}^{\infty} I_j$ is τ-dense left ideal in R, there exists a positive integer n such that I_n is τ-dense in R.*

(4) *E is $\sum$-τ-injective.*

P r o o f. By taking a P-filter $\rho = \rho_\tau$ and applying Corollary 17.

Corollary 19 (**see [11, Proposition 3, p. 184]**). *The following conditions on an injective module E are equivalent.*

(1) *E is countably $\sum$-injective.*
(2) *R satisfies the ACC on the E-annihilators left ideals.*
(3) *E is $\sum$-injective.*

P r o o f. By taking $\rho = \Re$ and applying Corollary 17.

Corollary 20. *Let $\mathcal{L}$ satisfy conditions (α), (μ), and (δ), and let $\{E_i \mid 1 \leq i \leq n\}$ be a family of modules. If E_i is $\sum$-$\mathcal{L}$-injective $\forall i = 1, 2, \ldots, n$, then $\bigoplus_{i=1}^n E_i$ is $\sum$-$\mathcal{L}$-injective.*

P r o o f. By Corollary 5 and Corollary 15.

Corollary 21. *Let ρ_1 and ρ_2 be any two P-filters and let $\{E_i \mid 1 \leq i \leq n\}$ be a family of modules. If E_i is $\sum$-$\mathcal{L}_{(\rho_1,\rho_2)}$-injective $\forall i = 1, 2, \ldots, n$, then $\bigoplus_{i=1}^n E_i$ is $\sum$-$\mathcal{L}_{(\rho_1,\rho_2)}$-injective.*

P r o o f. By Lemma 1 and Corollary 20.

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Об $\mathcal{L}$-инъективных модулях

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Ключевые слова: инъективный модуль, обобщенный критерий Фукса, наследственная теория кручения, t-плотный, прерадикал, естественный класс.

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Пусть $\mathcal{M} = \{(M, N, f, Q) \mid M, N, Q \in R\text{-Mod}, N \leq M, f \in \mathrm{Hom}_R(N, Q)\}$ и пусть $\mathcal{L}$ — непустой подкласс $\mathcal{M}$. Jirásko ввел понятие $\mathcal{L}$-инъективного модуля как обобщение инъективного модуля: модуль Q называется $\mathcal{L}$-инъективным, если для каждого $(B, A, f, Q) \in \mathcal{L}$ существует гомоморфизм $g\colon B \to Q$ такой, что $g(a) = f(a)$ для всех $a \in A$. Целью данной работы является изучение $\mathcal{L}$-инъективных модулей и некоторых связанных с ними понятий. Даны некоторые характеристики $\mathcal{L}$-инъективных модулей. Приводится версия критерия Бэра для $\mathcal{L}$-инъективности. В качестве обобщений M-инъективных модулей вводятся понятия $\mathcal{L}$-M-инъективного модуля и s-$\mathcal{L}$-M-инъективного модуля и даются некоторые результаты о них. Дана наша версия обобщенного критерия Фукса. Получены условия, при которых класс $\mathcal{L}$-инъективных модулей замкнут относительно прямых сумм. Наконец, мы вводим и изучаем понятие $\sum$-$\mathcal{L}$-инъективности как обобщение $\sum$-инъективности и $\sum$-τ-инъективности.

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