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ON LIMIT CYCLES, RESONANCE AND HOMOCLINIC STRUCTURES IN ASYMMETRIC PENDULUM-TYPE EQUATION

Time-periodic perturbations of an asymmetric pendulum-type equation close to an integrable standard equation of a mathematical pendulum are considered. For an autonomous equation, the problem of limit cycles, which reduces to the study of the Poincaré–Pontryagin generating functions, is solved. A partition of the parameter plane into domains with different behavior of the phase curves is constructed. Basic phase portraits for each domain of the obtained partition are given. For a nonautonomous equation, the question of the structure of the resonance zones, to which the solution of the problem of synchronization of oscillations leads, is studied. Averaged equations of the pendulum type, describing the behavior of solutions of the original equation in individual resonance zones, are calculated and analyzed. The global behavior of solutions in cells that do not contain small neighborhoods of unperturbed separatrices is ascertained. Using the analytical Melnikov method and numerical modeling, the basic bifurcations of the nonautonomous equation associated with the appearance of nonrough homoclinic curves are studied. On the plane of the main parameters, a bifurcation diagram for the Poincaré map generated by the original equation, describing different types of homoclinic tangencies of the separatrices of the saddle fixed point, is constructed. Homoclinic zones (those domains of parameters for which homoclinic trajectories to the saddle fixed point exist) with nonsmooth bifurcation boundaries are found.

Keywords: pendulum-type equation, limit cycles, resonances, Poincaré homoclinic structures.

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Introduction

The problem of the effect of an external periodic perturbation on a self-oscillatory system is one of the classical problems in the theory of oscillations. For systems close to linear conservative ones, the main results were obtained already in the papers of A. A. Andronov, A. A. Witt, N. N. Bogolyubov, Yu. A. Mitropolsky and others. For systems close to nonlinear conservative ones, such a problem was first considered in the papers of A. D. Morozov and L. P. Shilnikov of the 70-80-ies. There are many papers devoted to various issues in the study of such problems, by now. However, many problems remain unsolved and require consideration of new examples.

In this paper we study both analytically and numerically the effect of an external periodic force in an asymmetric pendulum-type equation

$$\ddot{x} + \sin x = \varepsilon[(p_0 + p_1\dot{x} + p_2\cos nx)\dot{x} + p_3\cos p_4 t], \tag{0.1}$$

where p_0, p_1, p_2, $p_3 > 0$, $p_4 > 0$ are parameters, ε is a small positive parameter, $n \in \mathbb{N}$.

The unperturbed equation ($\varepsilon = 0$) is equivalent to the system $\dot{x} = y,\ \dot{y} = -\sin x$ that corresponds to the integrable Hamiltonian system with the Hamiltonian function $H(x, y) = y^2/2 - \cos x$. On phase cylinder $\{x(mod(2\pi)), y\}$ it has two equilibrium points: a center $(0, 0)$ and a saddle $(\pi, 0) \equiv (-\pi, 0)$.

Phase curves, which are defined by the energy integral $H(x, y) = h$, can be divided into two classes:

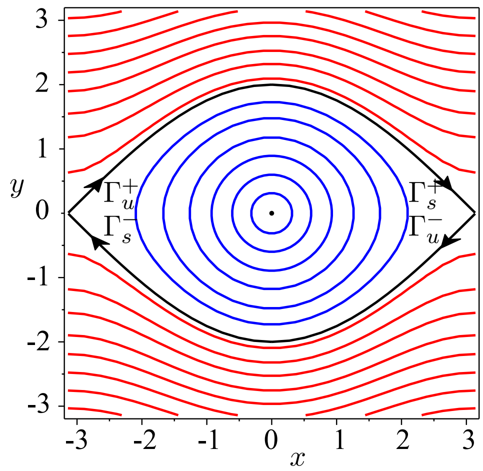

Fig. 1. Phase trajectories of the unperturbed system

- phase curves not covering phase cylinder (they correspond to the values $h \in (-1, 1)$);
- phase curves covering phase cylinder (they correspond to the values $h > 1$).

The set of phase curves (shown in blue in Fig. 1) not covering the phase cylinder forms the region G_1 of oscillatory motions of the pendulum. The set of phase curves (shown in red in Fig. 1) covering the phase cylinder forms two regions of the rotational motions of the pendulum: G_2^+ (on the upper half-cylinder) and G_2^- (on the lower half-cylinder). The oscillatory and rotational regions are separated by two separatrix loops $\Gamma^+ = \Gamma_s^+ \bigcup \Gamma_u^+$ and $\Gamma^- = \Gamma_s^- \bigcup \Gamma_u^-$ (shown in black in Fig. 1).

Solutions of the unperturbed system are known (see, for example, [1]):

$$x(k,\theta) = 2\arcsin\left(k\,\mathrm{sn}(2\mathbf{K}\theta/\pi)\right), \quad y(k,\theta) = 2k\,\mathrm{cn}(2\mathbf{K}\theta/\pi),$$
$$k^2 = (1+h)/2, \quad \omega = \pi/(2\mathbf{K}) \in (0,1)$$

for the region G_1,

$$x(k,\theta) = 2\,\mathrm{am}(\mathbf{K}\theta/\pi), \quad y(k,\theta) = \pm(2/k)\,\mathrm{dn}(\mathbf{K}\theta/\pi),$$
$$k^2 = 2/(1+h), \quad \omega = \pi/(k\mathbf{K}) \in (0,+\infty) \tag{0.2}$$

for the regions $G_2^{\pm}$. Here, ω is the frequency of motion on closed phase curves $y^2/2 - \cos x = h$, $\theta = \omega t \in [0, 2\pi]$ is the angular variable, $\mathbf{K}(k)$ is a complete elliptic integral of the first kind, k is its module.

In view of the fundamental significance of pendulum-type equations in the theory of nonlinear oscillations, many papers have been devoted to their study (see, e.g., [1–6] and references therein). Estimates of the maximal number of limit cycles for perturbed pendulum equations were obtained in [1–4]. In particular, equation (0.1) for $p_0 = p_1 = p_3 = 0$ was studied in [1–3], the main result of which is the following theorem.

Theorem 1. *There exists a sufficiently small $\varepsilon_*(n) > 0$ such that for any $\varepsilon \in (0, \varepsilon_*)$ the equation:*

$$\ddot{x} + \sin x = \varepsilon p_2 \dot{x} \cos nx, \quad n \in \mathbb{N},$$

has exactly $n-1$ rough limit cycles in the region of oscillatory motions. In the region of rotational motions, there are no limit cycles.

According to [1–3], equation (0.1) for $p_1 = p_3 = 0$, $p_0 \neq 0$:

$$\ddot{x} + \sin x = \varepsilon(p_0 + p_2 \cos nx)\dot{x}, \quad n \in \mathbb{N},$$

has exactly n limit cycles ($n - 1$ limit cycles are located in the region of oscillatory motions and one limit cycle is on the boundary of the oscillatory and rotational regions (the saddle limit cycle)) for $p_0 = p_0^* = (-1)^n(4n^2 - 1)^{-1}p_2$. Thus, we can obtain any number of self-oscillatory modes by setting the corresponding $n \in \mathbb{N}$. In [1, 3] it was also shown that the equations describing the topology of resonance zones in nonconservative systems close to the two-dimensional Hamiltonian ones belong to the class of self-oscillatory pendulum equations.

Phase synchronization problems lead to nonautonomous pendulum equations [5].

The nonautonomous equation (0.1) for $p_1 = 0$ (symmetric case), to which the problem of the interaction of two coupled pendulums was reduced, was studied in [6]. In this paper, the structure of the resonance zones was ascertained, and the conditions for the existence of Poincaré homoclinic structure were found. In addition, the problem of the passage of closed invariant curves through the main resonance under variation of perturbation frequency was studied numerically at $n = 5$.

Equation (0.1) in the case when autonomous perturbations destroy the symmetry of the unperturbed equation (that is, when $p_1 \neq 0$) has not been considered until now. The asymmetric term in the perturbation greatly complicates the study of both autonomous and nonautonomous equations.

Without loss of generality, we assume that $p_0 = -1$ in equation (0.1). Writing this equation in the form of the system and setting $n = 3$, we have

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\sin x + \varepsilon[(-1 + p_1 y + p_2 \cos 3x)y + p_3 \cos p_4 t]. \end{cases} \tag{0.3}$$

In this paper it will be shown that system (0.3) has a very rich dynamics, many of the features of which for systems close to Hamiltonian ones became known only recently.

An analysis of system (0.3) involves the solution of the following problems:

- ascertain the structures of resonance zones outside the neighborhood of unperturbed separatrices;
- determine the conditions for the existence of Poincaré homoclinic structures in a small neighborhood of unperturbed separatrices.

Solution of these problems rests upon the solution of the problem of limit cycles for the autonomous system ($p_3 = 0$). Therefore, we will start with a study of the dynamics of an autonomous system.

§ 1. Investigation of an autonomous system

Poincaré–Pontryagin generating functions

The main problem in studying system (0.3) for $p_3 = 0$ is the problem of limit cycles. Its solution results in finding real zeros of the Poincaré–Pontryagin generating functions [1]:

$$B(h(I)) \equiv B(k(h)) \equiv \frac{1}{2\pi}\int_0^{2\pi} (-1 + p_1 y + p_2 \cos 3x) y x'_\theta \, d\theta, \tag{1.1}$$

where $x = x(h(I), \theta)$, $y = y(h(I), \theta)$ are solutions of the unperturbed system on closed phase curves $y^2/2 - \cos x = h(I)$; I, θ are “action–angle” variables.

A simple real root of the equation $B(k(h)) = 0$ corresponds to a rough limit cycle of the original autonomous system [1]. Calculating the integral in (1.1), we find

$$B_1 = B_1(k(h)) = \frac{8}{105\pi}\{(1-k^2)(105+(128k^4-80k^2+3)p_2)\mathbf{K}(k) + \\ + (-105+(2k^2-1)(128k^4-128k^2+3)p_2)\mathbf{E}(k)\} \tag{1.2}$$

for the region G_1,

$$B_2 = B_2^{\pm}(k(h)) = \frac{4}{105\pi k^7}\{2(1-k^2)(-27k^4+128k^2-128)p_2\mathbf{K}(k) + \\ + (-105k^6+(2-k^2)(3k^4-128k^2+128)p_2)\mathbf{E}(k) \pm \frac{105\pi p_1}{2}(2-k^2)k^5\} \tag{1.3}$$

for the regions $G_2^{\pm}$. Here, $\mathbf{E}(k)$ is a complete elliptic integral of the second kind.

Solution of the problem of limit cycles. Oscillatory region G_1

The roots of the characteristic equation

$$\lambda^2 - \varepsilon(-1+p_2)\lambda + 1 = 0$$

determine the type of the equilibrium state $(0, 0)$ of the perturbed system. We find $\lambda_{1,2} = \frac{\varepsilon(-1+p_2)}{2} \pm \sqrt{\frac{\varepsilon^2(-1+p_2)^2}{4} - 1}$. Therefore, for sufficiently small $\varepsilon \neq 0$ and $p_2 = 1$, the equilibrium state $(0, 0)$ is a structurally unstable focus.

Let us decompose the function B_1 in a neighborhood of $k = 0$ in a power series up to terms of order k^5, using the known expansions:

$$\mathbf{K}(k) = \frac{\pi}{2}\left(1 + \frac{k^2}{4} + \frac{9k^4}{64} + O(k^5)\right), \ \mathbf{E}(k) = \frac{\pi}{2}\left(1 - \frac{k^2}{4} - \frac{3k^4}{64} + O(k^5)\right).$$

As a result, we find

$$B_1(k) = 2(-1+p_2)k^2 - \frac{1}{4}(1+35p_2)k^4 + O(k^5).$$

Since under the condition $p_2 = 1$ (the condition for the focus $(0, 0)$ to be structurally unstable) the first Lyapunov exponent $\ell_1 = 1 + 35p_2$ is nonzero, then no more than one limit cycle can be generated from such a focus [7].

From the system

$$B_1(k) = 0, \quad \frac{dB_1(k)}{dk} = 0$$

we find two bifurcation values $p_2 \approx -8.481$ and $p_2 \approx 27.273$, for each of which there is a double limit cycle.

It is not difficult to prove the following theorem on estimating the number of limit cycles of system (0.3) for $p_3 = 0$ in region G_1.

Theorem 2. *For sufficiently small ε the number of limit cycles in region G_1 of system (0.3) for $p_3 = 0$ does not exceed three.*

The proof of this theorem reduces (see [1]) to the proof of the estimate of the number of simple zeros of the function $B_1(k)$ on the interval $(0, 1)$.

Solution of the problem of limit cycles. Rotational regions $G_2^{\pm}$

From the system
$$B_2(k) = 0, \qquad \frac{dB_2(k)}{dk} = 0$$
we find the curve of double cycles in parametric form

$$p_1 = \pm\frac{4k}{5\pi}\left(\left(1024 - 1536k^2 + 620k^4 - 54k^6\right)\mathbf{KE} + \left(-128 + 256k^2 - 155k^4 + 27k^6\right)\mathbf{K}^2 + \right.$$
$$\left. + \left(-896 + 896k^2 - 161k^4\right)\mathbf{E}^2\right) / \left(\left(512 - 1024k^2 + 732k^4 - 220k^6 + 21k^8\right)\mathbf{K} + \right.$$
$$\left. + \left(-512 + 768k^2 - 380k^4 + 62k^6\right)\mathbf{E}\right),$$

$$p_2 = 21k^6\left(\left(k^2 - 2\right)\mathbf{K} + 4\mathbf{E}\right) / \left(\left(512 - 1024k^2 + 732k^4 - 220k^6 + 21k^8\right)\mathbf{K} + \right.$$
$$\left. + \left(-512 + 768k^2 - 380k^4 + 62k^6\right)\mathbf{E}\right).$$

Note that the saddle value $\sigma_s = -\varepsilon(1 + p_2)$ can vanish to zero at $p_2 = -1$, so that the double cycle can merge with the separatrix.

Estimating the number of simple zeros of the functions $B_2^{\pm}(k)$ on the interval $(0, 1)$, it is not difficult to prove the following theorem.

Theorem 3. *For sufficiently small ε the number of limit cycles in each region $G_2^{\pm}$ of system (0.3) for $p_3 = 0$ does not exceed two.*

Note that for functions $B_2^{\pm}(k)$ at $|p_1| \to 0$, we have $k \to 0$ (the limit cycle in $G_2^{\pm}$ goes to infinity).

Solution of the problem of limit cycles. Neighborhood of the separatrix loops

The magnitude of the splitting of the unperturbed separatrices under the action of the perturbation can be represented in the form $\Delta = \varepsilon\Delta_1 + O(\varepsilon^2)$. Using Melnikov formula [8], we find
$$\Delta_1 = \Delta_1(t_0) = \int_{-\infty}^{\infty} [-1 + p_1 y_s(t - t_0) + p_2 \cos 3x_s(t - t_0)] y_s^2(t - t_0)\, dt, \tag{1.4}$$
where $x_s(\tau) = 2\arcsin \operatorname{th}\tau$, $y_s(\tau) = \pm 2/\operatorname{ch}\tau$ are solutions of the unperturbed system on the separatrix. Calculating the integral in (1.4), we find
$$\Delta_1^{\pm} = 4\left[-2 \pm \pi p_1 + \frac{2}{35}p_2\right].$$

Note that $B_1(1) = -\dfrac{8}{\pi} + \dfrac{8}{35\pi}p_2$, $B_2(1) = B_2^{\pm}(1) = -\dfrac{4}{\pi} \pm 2p_1 + \dfrac{4}{35\pi}p_2$. Obviously, the expression for $\Delta_1^{\pm}$ coincides with the expression for $B_2^{\pm}(1)$ (up to a constant). From the equations $\Delta_1^{\pm} = 0$ (or $B_2^{\pm}(1) = 0$) we define the bifurcation set corresponding to the separatrix loop in the perturbed system: $2(p_2 - 35) \pm 35\pi p_1 = 0$ ("plus" ("minus") sign corresponds to a separatrix loop on the upper (lower) half-cylinder).

When $p_1 = 0$ (symmetric case), we have $B_1(1) = 2B_2(1)$ (there are two separatrix loops – on the upper and lower half-cylinders). When $p_1 \neq 0$ (asymmetric case), this equality is not satisfied. In this case, for $p_2 = 35$ ($B_1(1) = 0$), there is a separatrix loop enclosing the equilibrium state $(0, 0)$ of focus type.

From the equation $2(p_2 - 35) \pm 35\pi p_1 = 0$, under the condition $p_2 = -1$ (when the saddle value vanishes to zero) we find $p_1 = \pm\dfrac{72}{35\pi}$. As a result, we obtain the extreme point $(\pm\dfrac{72}{35\pi}, -1)$ of the double cycle curve in region $G_2^{\pm}$, respectively.

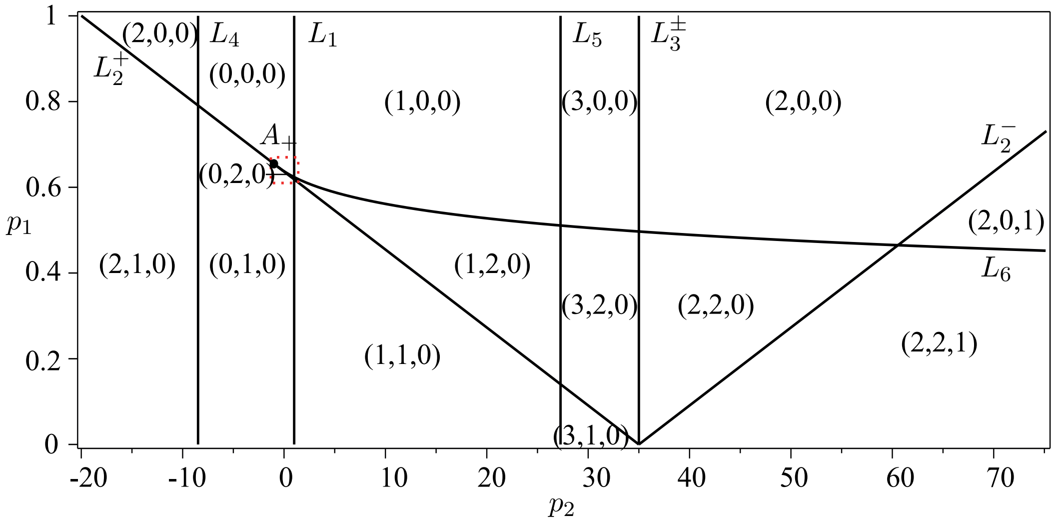

Fig. 2. Partition of the plane of parameters (p_2, p_1) into domains with different topological structures. $L_1 : p_2 = 1$ — curve of structurally unstable focus in G_1; $L_2^{\pm} : 35\pi p_1 \pm 2(p_2 - 35) = 0$ — curves of separatrix loops of a saddle in $G_2^{\pm}$; $L_3^{\pm} : p_2 = 35$ — curve of separatrix loop enclosing the equilibrium state $(0, 0)$ of focus type; $L_4 : p_2 = -8.481$ and $L_5 : p_2 = 27.273$ — double cycle curves in G_1; L_6 — double cycle curve in G_2^+; $A_+(\frac{72}{35\pi}, -1) \in L_2^+$ — extreme point of the double cycle curve L_6 in G_2^+

Global result. Bifurcation diagram on the plane of parameters (p_2, p_1)

Since the system (0.3) for $p_3 = 0$ is invariant under the change of variables $(p_1, x, y) \to (-p_1, -x, -y)$, the partition of the plane of the parameters (p_2, p_1) is symmetric to the p_2 axis. Therefore, it suffices to construct a bifurcation diagram, for example, for the upper half-plane $p_1 > 0$.

The resulting bifurcation curves divide the upper half-plane of the plane of the parameters (p_2, p_1) into 15 domains with different phase portraits topology (see Fig. 2). An enlarged fragment of this diagram is shown in Fig. 3. The introduced notation (i, j, k) means the existence of $i = \overline{0, 3}$ limit cycles in G_1, $j = \overline{0, 2}$ in G_2^+, and $k = \overline{0, 2}$ in G_2^-.

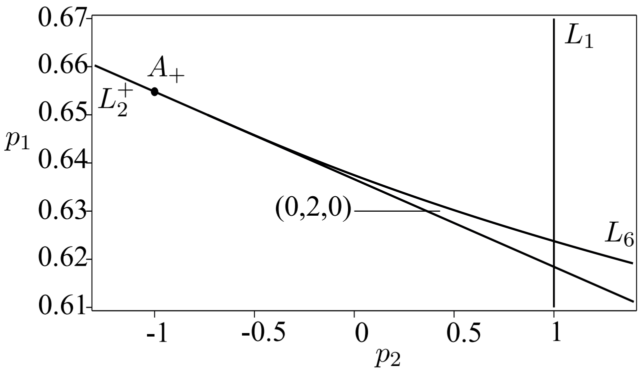

Fig. 3. Enlarged fragment of Fig. 2

Main phase portraits of the system (0.3) for $p_3 = 0$ for the values of the parameters from 15 domains are shown in Fig. 4. They are obtained using the WInSet software [9, 10]. Stable (unstable) limit cycles are shown in blue (red) color, the separatrices of the saddle are shown in black.

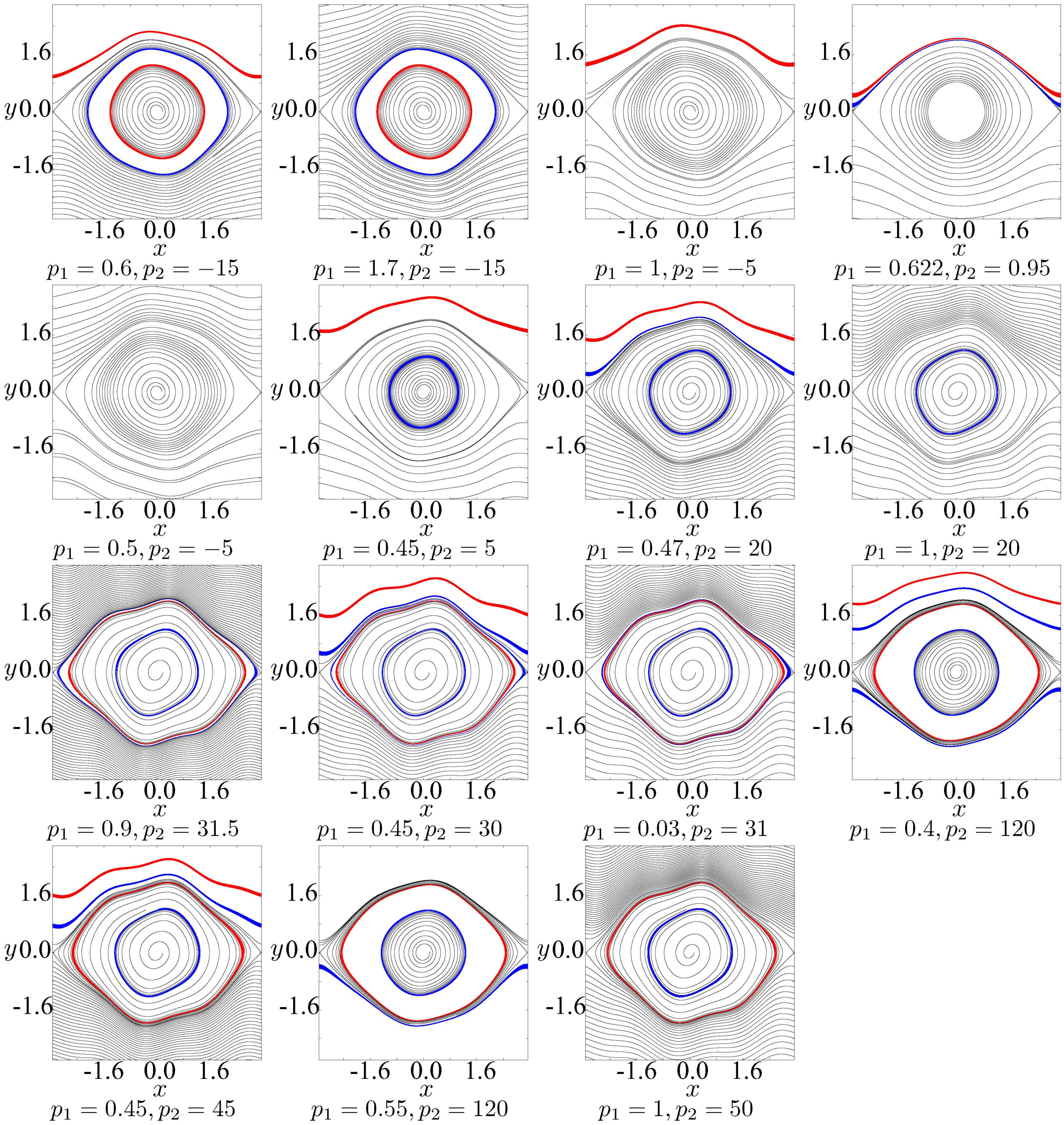

Fig. 4. Phase portraits of the system (0.3) for $p_3 = 0$ and different values of parameters p_1, p_2

§ 2. Investigation of a nonautonomous system. Resonances

First of all, in the regions separated from the unperturbed separatrices, we pass in the system (0.3) from the variables x and y to the "I action–θ angle" variables. As a result, we obtain the system

$$\begin{cases} \dot{I} = \varepsilon[(-1 + p_1 y + p_2 \cos 3x)y + p_3 \cos\varphi]x'_\theta, \\ \dot{\theta} = \omega(I) - \varepsilon[(-1 + p_1 y + p_2 \cos 3x)y + p_3 \cos\varphi]x'_I, \\ \dot{\varphi} = p_4. \end{cases} \tag{2.1}$$

Definition 1. It is said that in system (2.1) there is a resonance if the natural frequency ω and the perturbation frequency p_4 are commensurable:

$$\omega(I) = \frac{q}{p} p_4, \tag{2.2}$$

where p and q are coprime integer numbers.

The level $I = I_{pq}$, found from condition (2.2), will be called the resonance level.
It is well known [1] that in the neighborhood

$$U_\mu = \{(I, \theta) : I_{pq} - C\mu < I < I_{pq} + C\mu,\ 0 \leqslant \theta < 2\pi,\ C = \text{const} > 0\}, \quad \mu = \sqrt{\varepsilon},$$

of the individual resonance level $I = I_{pq}$ (it will be called the resonance zone) the system (2.1) reduces to an equation of the form

$$\frac{d^2 v}{d\tau^2} - bA(v, I_{pq}) = \mu\sigma(v, I_{pq})\frac{dv}{d\tau}, \tag{2.3}$$

where

$$A(v, I_{pq}) = \frac{1}{2\pi p}\int_0^{2\pi p} [(-1 + p_1 y + p_2 \cos 3x)y + p_3 \cos\varphi]x'_\theta \Big|_{\substack{x = x(I_{pq}, v + q\varphi/p)\\ y = y(I_{pq}, v + q\varphi/p)}} d\varphi,$$

$$\sigma(v, I_{pq}) = \frac{1}{2\pi p}\int_0^{2\pi p} ((-1 + 2p_1 y + p_2 \cos 3x))\Big|_{\substack{x = x(I_{pq}, v + q\varphi/p)\\ y = y(I_{pq}, v + q\varphi/p)}} d\varphi,$$

$\theta = v + q\varphi/p$, $\tau = \mu t$, $b = d\omega(I_{pq})/dI$.

It is equation (2.3) that will interest us, since it determines the topology of individual resonance zones up to terms of order μ^2. A simple stable (unstable) equilibrium state of the averaged equation (2.3) corresponds to a stable (unstable) periodic resonance solution of period $\dfrac{2\pi p}{p_4 q}$ in the initial system.

When calculating the functions $A(v, I_{pq})$ and $\sigma(v, I_{pq})$, and also the quantity b, we distinguish the following cases: 1) $(x, y) \in G_1$; 2) $(x, y) \in G_2^{\pm}$. In this connection, we represent the function $A(v, I_{pq})$ in the form $A_j(v, I_{pq}) = \widetilde{A}_j(v, I_{pq}) + B_j(I_{pq})$ and designate $\sigma = \sigma_j$, $b = b_j$, $j = 1, 2$, where the constant B_j is the Poincaré–Pontryagin generating function at the resonance value in the corresponding region.

The following definition gives a classification of the resonance levels for the averaged equation of the first approximation [1].

Definition 2. A level $I = I_{pq}$ is called splittable resonance level if the equation $A_j(v, I_{pq}) = 0$ has simple roots. The splittable resonance level $I = I_{pq}$ is called partially passable, if $B_j(I_{pq}) \neq 0$, and impassable, if $B_j(I_{pq}) = 0$. The nonsplittable resonance level $I = I_{pq}$, for which $|A_j(v, I_{pq})| > 0$, is called passable.

The first approximation of the averaged equation does not describe the behavior of the solutions of the initial system, since it is conservative, while the perturbation in the initial system is nonconservative. Therefore, it is natural to consider the averaged equation of the second approximation, the phase portraits of which for $q = 1$ are shown in Fig. 5 in the case of passable (a), partly passable (b) and impassable (c) resonances.

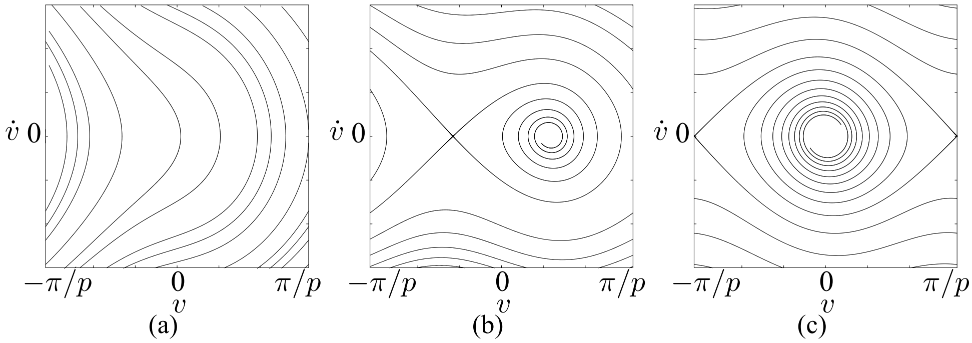

Fig. 5. Phase portraits for equation (2.3)

Averaged equation for the oscillatory region G_1

The averaged equation for the region G_1 for odd p and $q = 1$ has the form

$$\frac{d^2v}{d\tau^2} - b_1(p_3A_1\cos pv + B_1) = \mu\sigma_1\frac{dv}{d\tau},$$

where

$$A_1 = 4\frac{a^{p/2}}{1+a^p}, \quad a = \exp\left(-\pi\frac{\mathbf{K}(\sqrt{1-\rho})}{\mathbf{K}(\rho)}\right), \quad b_1 = \frac{\pi^2}{16}\frac{(1-k^2)\mathbf{K}-\mathbf{E}}{k^2(1-k^2)\mathbf{K}^3},$$

$$\sigma_1 = -1 - p_2\frac{(31-144k^2+128k^4)\mathbf{K}-(46-256k^2+256k^4)\mathbf{E}}{15\mathbf{K}},$$

B_1 is defined by formula (1.2), $k^2 = k^2(h_{pq}) = (1+h_{pq})/2$.

For even p and/or $q > 1$ we obtain the equation

$$\frac{d^2v}{d\tau^2} - b_1B_1 = \mu\sigma_1\frac{dv}{d\tau}. \tag{2.4}$$

Thus, in this case the resonance levels with odd p and $q = 1$ are split. Moreover, since the natural frequency $\omega(I)$ is a monotonic function and $\omega(I) \in (0,1)$, then it follows from the resonance relation (2.2) that $p > p_4$. Because of this, the splittable resonance levels are the resonance levels $I = I_{p1}$ for which $p > p_4$ and p is odd.

According to (2.4), for even p and/or $q > 1$, the resonance levels are passable, if $B_1(I_{pq}) \neq 0$.

Averaged equation for the rotational regions $G_2^\pm$

The averaged equation for the regions $G_2^\pm$ for $q = 1$ has the form

$$\frac{d^2v}{d\tau^2} - b_2(p_3A_2\cos pv + B_2) = \mu\sigma_2\frac{dv}{d\tau},$$

$$A_2 = A_2^\pm = \pm 2\frac{a^p}{1+a^{2p}}, \quad a = \exp\left(-\pi\frac{\mathbf{K}(\sqrt{1-\rho})}{\mathbf{K}(\rho)}\right), \quad b_2 = \frac{\pi^2}{4}\frac{\mathbf{E}}{(1-k^2)\mathbf{K}^3},$$

$$\sigma_2 = \sigma_2^\pm = -1 \pm \frac{2p_1\pi}{k\mathbf{K}} + \frac{p_2((-256+384k^2-158k^4+15k^6)\mathbf{K}+(256-256k^2+46k^4)\mathbf{E})}{15k^6\mathbf{K}},$$

B_2 is defined by formula (1.3), $k^2 = k^2(h_{pq}) = 2/(1+h_{pq})$.

For $q > 1$ we obtain the equation

$$\frac{d^2v}{d\tau^2} - b_2 B_2 = \mu\sigma_2 \frac{dv}{d\tau}. \tag{2.5}$$

Thus, in this case the resonance levels $I = I_{p1}$ (p is any number satisfying the resonance condition (2.2)) are split. According to (2.5), for $q > 1$ and $B_2(I_{pq}) \neq 0$, the resonance levels are passable.

On the global behavior of solutions outside the neighborhood of unperturbed separatrices

The resonance condition (2.2) determines a countable set of resonance levels $I = I_{pq}$. There are splittable (partially passable and impassable) and nonsplittable (passable) resonance levels. In the neighborhood of passable resonance levels the qualitative behavior of the solutions of the original system is analogous to the behavior of solutions of the autonomous system. According to the results of investigations of an autonomous perturbed system, the number of impassable resonance levels (for which $B_j(I_{pq}) = 0$, $j = 1, 2$) is finite. Namely, the number of such levels in the region G_1 is no more than three, and in each region $G_2^{\pm}$ is no more than two. It is well known [1] that in the case of a nonconservative system (which is our system under consideration), there are a finite number of partially passable resonance levels in regions that do not contain finite neighborhoods of impassable resonance levels, as well as a neighborhood of unperturbed separatrices. It follows that for a sufficiently small ε, the neighborhoods of the splittable resonance levels do not intersect. Knowledge of the qualitative behavior of solutions in the neighborhood of individual splittable resonance levels allows us to establish the global qualitative behavior of the solutions of the original system. According to [1], the separatrices of resonance saddle periodic solutions corresponding to different splittable resonance levels intersect.

Numerical analysis

Using the WInSet software [9, 10], the Poincaré map for the system (0.3) was constructed. A good correspondence of the numerical results with the theoretical study was established.

Figure 6 shows the structure of the neighborhood of the splittable resonance level $I = I_{31}$ for different values of the parameter p_4 in the region G_1. By changing the value of the parameter p_4 (for fixed values of the remaining parameters), we change the position of the resonance level relative to the level generating the limit cycle in the autonomous system. The structure of the partially passable resonance zone is shown in figures 6 (a) and (c). In addition to the separatrices of saddle periodic points of period-3, a closed invariant curve of the Poincaré map is shown here. Figure 6 (b) illustrates the case of synchronization of oscillations, when the resonance level coincides with the level in the neighborhood of which the autonomous system has a limit cycle. The dots correspond to stable periodic points of period-3, and also the unstable fixed point of the Poincaré map. Stable (unstable) separatrices are shown in red (blue).

To illustrate the global behavior of solutions, we consider the region G_2^+. Let us fix the parameter $p_2 = 120$. From the system

$$\begin{cases} B_2^+(k_1; p_1) = 0, \\ B_2^+(k_2; p_1) = 0, \\ p_4 = 1\omega(k_1) = 2\omega(k_2), \end{cases}$$

we find $p_1 \approx 0.3223$, $k_1 \approx 0.592494$, $k_2 \approx 0.9024395$ and $p_4 \approx 3.039$. Here, according to (0.2), $\omega = \pi/(k\mathbf{K})$. Under these conditions, the autonomous system has two limit cycles in region G_2^+, generated by the levels $k = k_1$ and $k = k_2$. Moreover, the level $k = k_1$ coincides with the

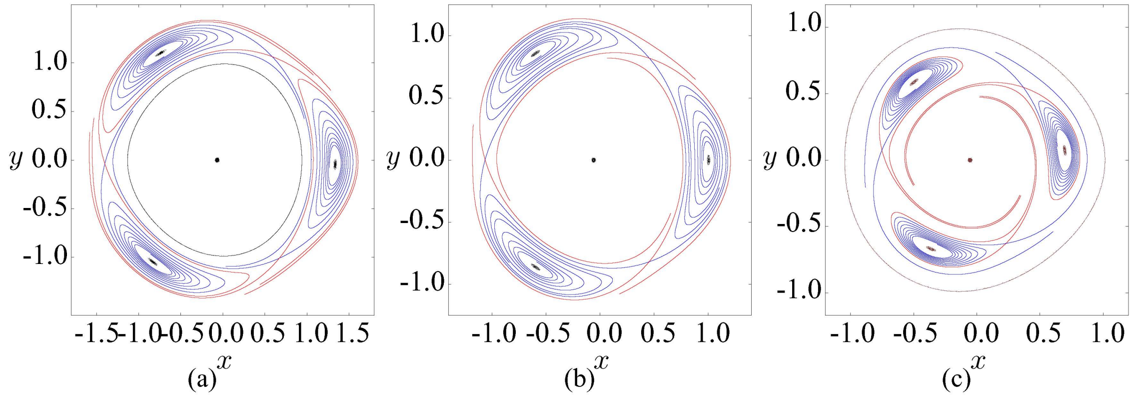

Fig. 6. The behavior of invariant curves of the Poincaré map for the system (0.3) at $\varepsilon = 0.001$, $p_1 = 0.3$, $p_2 = 5$, $p_3 = 400$ and (a) $p_4 = 2.65$; (b) $p_4 = 2.8$; (c) $p_4 = 2.9$

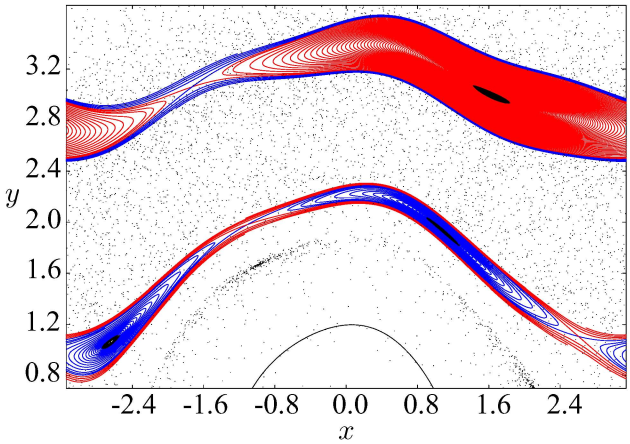

Fig. 7. The behavior of trajectories of the Poincaré map for system (0.3) in the case of two impassable resonances with $p = 1$ and $p = 2$ in region G_2^+. Here, $\varepsilon = 0.001$, $p_1 = 0.3223$, $p_2 = 120$, $p_3 = 400$, $p_4 = 3.039$

resonance level $I = I_{11}$, and the level $k = k_2$ with the resonance level $I = I_{21}$. Thus, we have two impassable resonance zones, the structure of which is shown in Fig. 7. There exist stable periodic points of period-2 (corresponding to a stable periodic solution of period $2\pi/p_4$ in the original system) and an unstable fixed point (corresponding to an unstable periodic solution of period π/p_4). All other resonance levels between these two levels in region G_2^+ will be passable. Figure 7 also shows a fragment of a closed invariant curve of the Poincaré map and an impassable resonance (with $p = 5$ and $q = 1$) in region G_1.

§ 3. Investigation of a nonautonomous system. Poincaré homoclinic structures

In this section we study the behavior of the solutions of system (0.3) in a small neighborhood of unperturbed separatrices. A key role in this study is played by the analysis of the relative position of the separatrices of the saddle fixed point of the Poincaré map.

As already noted, the unperturbed system has two separatrix loops of the saddle $(\pi, 0)$: $\Gamma^+ = \Gamma_s^+ \bigcup \Gamma_u^+$ and $\Gamma^- = \Gamma_s^- \bigcup \Gamma_u^-$. Under the action of an autonomous perturbation, the separatrices of the saddle $(\pi, 0)$ split in the general case. However, there exist values of the parameters

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$(p_2, p_1) \in L_2^+$ ($(p_2, p_1) \in L_2^-$)[1] for which the separatrix loop Γ^+ (Γ^-) is conserved in the perturbed autonomous system. In addition, for the values of the parameters $(p_2, p_1) \in L_3^\pm$ ($(p_2, p_1) \in L_3^\mp$; $L_3^\mp : p_2 = 35,\ p_1 < 0$) there exists the loop $\Gamma^\pm$ ($\Gamma^\mp$) when the separatrices Γ_s^+ and Γ_u^- (Γ_s^- and Γ_u^+) coincide[2].

It is well known that under the action of nonautonomous perturbations the separatrices of the saddle fixed point of the Poincaré map can intersect, forming the so-called Poincaré homoclinic structure. We denote by W_+^s and W_-^s (W_+^u and W_-^u) the stable (unstable) invariant curves (separatrices) of the saddle fixed point of the Poincaré map. We can distinguish two types of homoclinic structures:

1. $W_+^s \bigcap W_+^u \neq \oslash$ and/or $W_-^s \bigcap W_-^u \neq \oslash$;
2. $W_+^s \bigcap W_-^u \neq \oslash$ and/or $W_-^s \bigcap W_+^u \neq \oslash$.

The occurrence of homoclinic structures of the first type is due to the presence of a separatrix loop Γ^+ and/or Γ^-, the second type is due to the presence of a separatrix loop $\Gamma^\pm$ or $\Gamma^\mp$ in the perturbed autonomous system.

Melnikov analytical method

The problem of the existence of first type homoclinic structure is solved using the Melnikov formula [8] $\Delta(t_0) = \varepsilon\Delta_1(t_0) + O(\varepsilon^2)$, which is analogous to that used by us in determining the magnitude of the splitting of unperturbed separatrices under the action of an autonomous perturbation. Here the function $\Delta_1(t_0)$ determines (up to terms of order ε) the distance between the stable and unstable separatrices of the saddle fixed point of the Poincaré map. Before applying this formula, we transform the original system so that in the equilibrium state $(\pi, 0)$ of the saddle type of the unperturbed system the perturbation vanishes for any values of t. In system (0.3), we make the substitution $x = \xi + \varepsilon x_1(t) + O(\varepsilon^2)$, where

$$x_1(t) = -\frac{p_3}{1 + {p_4}^2}\cos{(p_4 t)},$$

replace ξ by x, as a result of which we obtain the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\sin x + \varepsilon\left[(-1 + p_1 y + p_2\cos 3x)y + \dfrac{p_3(1+\cos x)}{1 + {p_4}^2}\cos p_4 t\right]. \end{cases} \tag{3.1}$$

Applying the Melnikov formula to this system, we obtain

$$\Delta_1(t_0) = \int_{-\infty}^{\infty} \Big[(-1 + p_1 y_s(t - t_0) + p_2\cos{(3x_s(t-t_0))})y_s(t-t_0) + \\ + \frac{p_3(1 + \cos{(x_s(t-t_0))})}{1 + p_4^2}\cos p_4 t\Big]\, y_s(t - t_0)\, dt, \tag{3.2}$$

where the solution $x_s(\tau)$, $y_s(\tau)$ of the unperturbed equation on the separatrix has the form: $x_s(\tau) = 2\arcsin{(\operatorname{th}\tau)}$, $\quad y_s(\tau) = \pm\dfrac{2}{\operatorname{ch}\tau}$.

Calculating the integral in (3.2), we find

$$\Delta_1(t_0) = \Delta_1^\pm(t_0) = -8 \pm 4\pi p_1 + \frac{8}{35}p_2 \pm \frac{2\pi p_3}{\operatorname{ch}(\pi p_4/2)}\cos{(p_4 t_0)}.$$

[1] See Fig. 2.
[2] Such loops are absent in the unperturbed system.

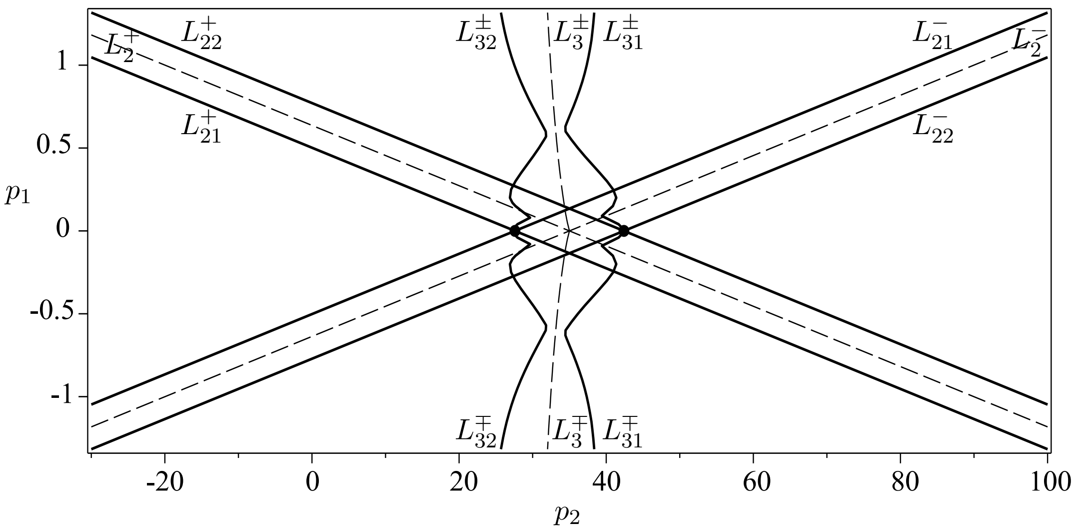

Fig. 8. Bifurcation diagram for the Poincaré map for system (3.1) at $\varepsilon = 0.001$, $p_3 = 15$, $p_4 = 3$

If $\Delta_1^{\pm}(t_0)$ is an alternating function, then there is a transversal intersection of the stable and unstable separatrices of the saddle fixed point. From the system $\Delta_1^{\pm}(t_0) = 0$, $\left(d\Delta_1^{\pm}/dt_0\right)(t_0) = 0$ we find condition

$$p_1 = \pm\frac{1}{2\pi}\left(4 - \frac{4}{35}p_2 \pm \frac{\pi p_3}{\operatorname{ch}(\pi p_4/2)}\right)$$

under which the quadratic tangency of the corresponding separatrices of the saddle fixed point takes place.

Numerical results. Bifurcation diagram

Using the WInSet and Maple 13 software, we constructed a bifurcation diagram for the Poincaré map induced by system (3.1), which describes the main cases of the relative position of the separatrices of the saddle fixed point (see Fig. 8). The bifurcation curves in this diagram are shown in bold color, the remaining lines are auxiliary. Around the auxiliary lines (on which the structurally unstable structures of the autonomous equation take place), certain regions arise which we call homoclinic zones.

The behavior of the separatrices of the saddle fixed point of the Poincaré map for system (3.1) on different curves of this bifurcation diagram are shown in Fig. 9. Homoclinic structures on curves $L_3^{\mp}$, $L_{31}^{\mp}$, $L_{32}^{\mp}$ are not shown here, since they can be obtained by rotating angle π of the corresponding structure on curves $L_3^{\pm}$, $L_{31}^{\pm}$, $L_{32}^{\pm}$. This is due to the inherited symmetry of the partition of the parameters plane for the autonomous case.

Of particular interest is the homoclinic zones with piecewise smooth boundaries. For the first time the structure of such zones and its boundaries for a two-parameter family of mappings with the figure-eight of a dissipative saddle was described in [11]. Then in [12] similar results were obtained for an asymmetric Duffing–Van-der-Pol equation, close to an integrable one with a homoclinic figure-eight. In this paper, the structure of such homoclinic zones was studied in detail using numerical analysis. Such features of the dynamics of systems close to Hamiltonian ones have not been known before. Comparing the results obtained here with the results of papers [11, 12], we conclude that for an asymmetric pendulum-type equation, the structure of homoclinic zones with piecewise smooth boundaries is similar.

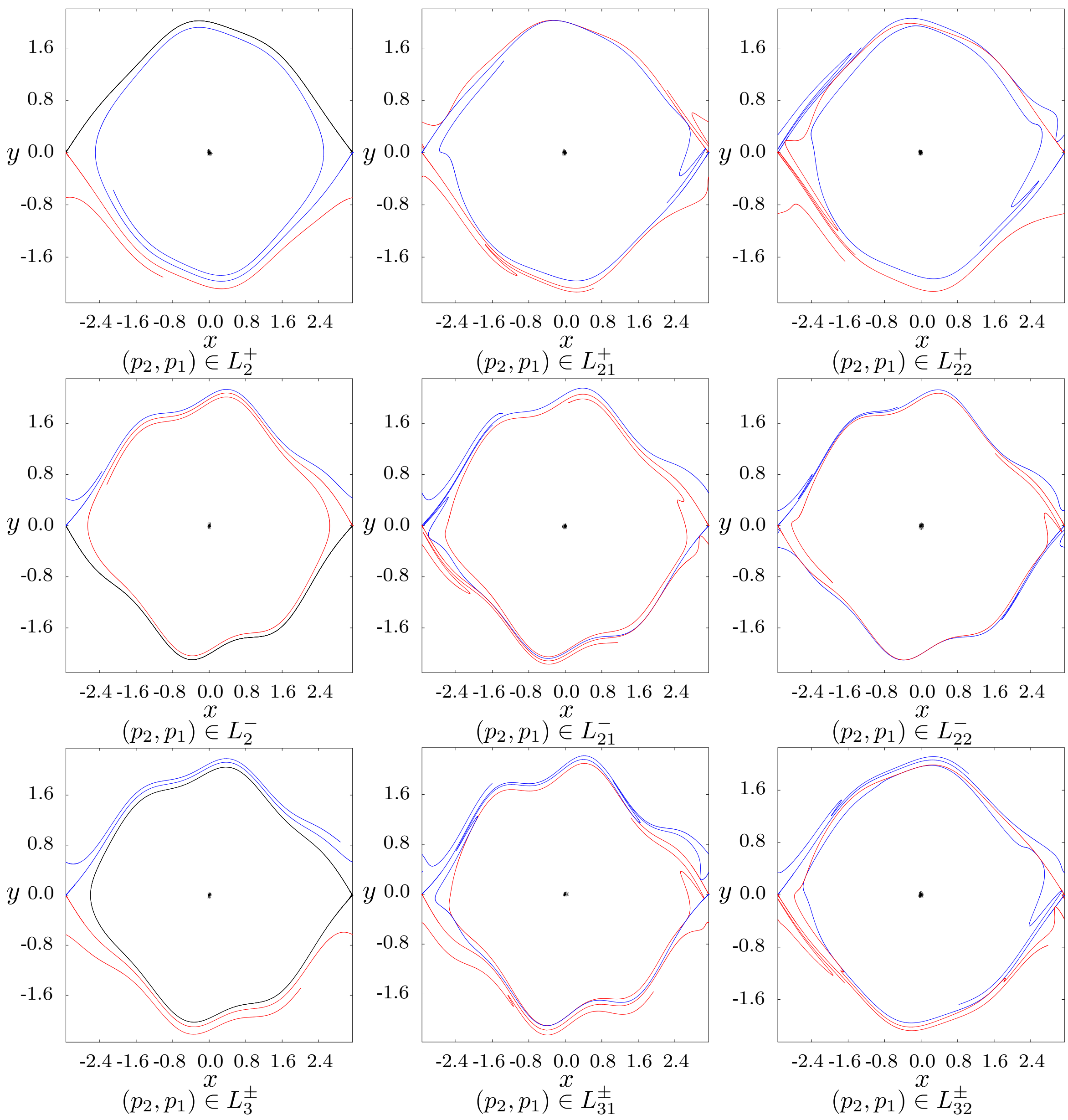

Fig. 9. Behavior of the separatrices of the saddle fixed point of the Poincaré map for system (3.1) on different curves of the bifurcation diagram in Fig. 8

Namely, the boundaries of such zones are formed by an infinite number of bifurcation curves with different quadratic homoclinic tangencies, which accumulate to the bold points on the bifurcation diagram. The points of the transversal intersection of these curves, that is, the points at which the smoothness of the boundaries of homoclinic zones under study is violated, correspond to double quadratic tangencies. After crossing, these curves continue into the homoclinic zone and terminate at points of cubic homoclinic tangencies (for details, see [11, 12]).

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REFERENCES

1. Morozov A.D. *Quasi-conservative systems: cycles, resonances and chaos*, World Scientific, 1998. https://doi.org/10.1142/3238
2. Morozov A.D. Limit cycles and chaos in equations of the pendulum type, *Journal of Applied Mathematics and Mechanics*, 1989, vol. 53, issue 5, pp. 565–572. https://doi.org/10.1016/0021-8928(89)90101-9
3. Morozov A.D. On resonances, pendulum equations, limit cycles and chaos, *Nonlinear Waves 3. Research Reports in Physics*, Eds.: A.V. Gaponov-Grekhov, M.I. Rabinovich, J. Engelbrecht. Berlin–Heidelberg: Springer, 1990, pp. 276–282. https://doi.org/10.1007/978-3-642-75308-4_25
4. Gasull A., Geyer A., Manosas F. On the number of limit cycles for perturbed pendulum equations, *Journal of Differential Equations*, 2016, vol. 261, issue 3, pp. 2141–2167. https://doi.org/10.1016/j.jde.2016.04.025
5. Shakhgil'dyan V.V., Belyustina L.N. (Eds.) *Sistemy fazovoi sinkhronizatsii* (Systems of phase synchronization), Moscow: Radio i Svyaz', 1982.
6. Korolev S.A., Morozov A.D. On periodic perturbations of self-oscillating pendulum equations, *Russian Journal of Nonlinear Dynamics*, 2010, vol. 6, no. 1, pp. 79–89 (in Russian). https://doi.org/10.20537/nd1001006
7. Bautin N.N., Leontovich E.A. *Metody i priemy kachestvennogo issledovaniya dinamicheskikh sistem na ploskosti* (Methods and techniques of qualitative investigation of dynamical systems on a plane), Moscow: Nauka, 1976.
8. Mel'nikov V.K. On the stability of a center for time-periodic perturbations, *Trudy Moskovskogo Matematicheskogo Obshchestva*, 1963, vol. 12, pp. 3–52 (in Russian). http://mi.mathnet.ru/eng/mmo137
9. Morozov A.D., Dragunov T.N., Boykova S.A., Malysheva O.V. *Invariant sets for windows*, World Scientific, 1999. https://doi.org/10.1142/4220
10. Morozov A.D., Dragunov T.N. *Vizualizatsiya i analiz invariantnykh mnozhestv dinamicheskikh sistem* (Visualization and analysis of invariant sets of dynamical systems), Moscow–Izhevsk: Institute of Computer Science, 2003.
11. Gonchenko S.V., Simo C., Vieiro A. Richness of dynamics and global bifurcations in systems with a homoclinic figure-eight, *Nonlinearity*, 2013, vol. 26, no. 3, pp. 621–678. https://doi.org/10.1088/0951-7715/26/3/621
12. Kostromina O.S. On the investigation of the bifurcation and chaotic phenomena in the system with a homoclinic "figure-eight", *Russian Journal of Nonlinear Dynamics*, 2016, vol. 12, no. 1, pp. 31–52 (in Russian). https://doi.org/10.20537/nd1601003

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О. С. Костромина

О предельных циклах, резонансных и гомоклинических структурах в асимметричном уравнении маятникового типа

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Рассматриваются периодические по времени возмущения асимметричного уравнения маятникового типа, близкого к интегрируемому стандартному уравнению математического маятника. Для автономного уравнения решается проблема предельных циклов, которая сводится к исследованию порождающих функций Пуанкаре–Понтрягина. Строится разбиение плоскости параметров на области с разным поведением фазовых кривых. Даются основные фазовые портреты для каждой области полученного разбиения. Для неавтономного уравнения изучается вопрос о структуре резонансных зон, к которому приводит решение задачи о синхронизации колебаний. Вычисляются усредненные уравнения маятникового типа, описывающие поведение решений исходного уравнения в индивидуальных резонансных зонах, и проводится их анализ. Устанавливается глобальное поведение решений в ячейках, не содержащих малых окрестностей невозмущенных сепаратрис. С помощью аналитического метода Мельникова и численного моделирования изучаются основные бифуркации неавтономного уравнения, связанные с возникновением негрубых гомоклинических кривых. На плоскости основных параметров строится бифуркационная диаграмма для отображения Пуанкаре, порожденного исходным уравнением, описывающая различные типы гомоклинических касаний сепаратрис седловой неподвижной точки. Обнаруживаются гомоклинические зоны (те области параметров, для которых существуют гомоклинические траектории к седловой неподвижной точки) с негладкими бифуркационными границами.

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СПИСОК ЛИТЕРАТУРЫ

1. Morozov A.D. Quasi-conservative systems: cycles, resonances and chaos. World Scientific, 1998. https://doi.org/10.1142/3238
2. Morozov A.D. Limit cycles and chaos in equations of the pendulum type // Journal of Applied Mathematics and Mechanics. 1989. Vol. 53. Issue 5. P. 565–572. https://doi.org/10.1016/0021-8928(89)90101-9
3. Morozov A.D. On resonances, pendulum equations, limit cycles and chaos // Nonlinear Waves 3. Research Reports in Physics / A.V. Gaponov-Grekhov, M.I. Rabinovich, J. Engelbrecht. Berlin–Heidelberg: Springer, 1990. P. 276–282. https://doi.org/10.1007/978-3-642-75308-4_25
4. Gasull A., Geyer A., Manosas F. On the number of limit cycles for perturbed pendulum equations // Journal of Differential Equations. 2016. Vol. 261. Issue 3. P. 2141–2167. https://doi.org/10.1016/j.jde.2016.04.025
5. Шахгильдян В.В., Белюстина Л.Н. (Ред.) Системы фазовой синхронизации. М.: Радио и связь, 1982.
6. Королев С.А., Морозов А.Д. О периодических возмущениях автоколебательных маятниковых уравнений // Нелинейная динамика. 2010. Т. 6. № 1. С. 79–89. https://doi.org/10.20537/nd1001006

7. Баутин Н.Н., Леонтович Е.А. Методы и приемы качественного исследования динамических систем на плоскости. М.: Наука, 1976.
8. Мельников В.К. Об устойчивости центра при периодических по времени возмущениях // Труды Московского математического общества. 1963. Т. 12. С. 3–52.
http://mi.mathnet.ru/mmo137
9. Morozov A.D., Dragunov T.N., Boykova S.A., Malysheva O.V. Invariant sets for windows. World Scientific, 1999. https://doi.org/10.1142/4220
10. Морозов А.Д., Драгунов Т.Н. Визуализация и анализ инвариантных множеств динамических систем. М.–Ижевск: Институт компьютерных исследований, 2003.
11. Gonchenko S.V., Simo C., Vieiro A. Richness of dynamics and global bifurcations in systems with a homoclinic figure-eight // Nonlinearity. 2013. Vol. 26. No. 3. P. 621–678.
https://doi.org/10.1088/0951-7715/26/3/621
12. Костромина О.С. К исследованию бифуркационных и хаотических явлений в системе с гомоклинической «восьмеркой» // Нелинейная динамика. 2016. Т. 12. № 1. С. 31–52.
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