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ON NONLOCAL PERTURBATION OF THE PROBLEM ON EIGENVALUES OF DIFFERENTIATION OPERATOR ON A SEGMENT

This work is devoted to the construction of a characteristic polynomial of the spectral problem of a first-order differential equation on an interval with a spectral parameter in a boundary value condition with integral perturbation which is an entire analytic function of the spectral parameter. Based on the characteristic polynomial formula, conclusions about the asymptotics of the spectrum of the perturbed spectral problem are established.

Keywords: differentiation operator, boundary value conditions, integral perturbation, function of bounded variation, characteristic polynomial, entire functions, zeros, eigenvalues, asymptotics.

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Introduction

The works [4,9,22,23] are devoted to the study of zeros of entire functions having an integral representation. Sometimes entire functions coincide with quasi-polynomials, zeros of which were investigated in [18,26]. Connection between the zeros of quasi-polynomials and spectral problems is reflected in [1, 5, 6, 14, 15, 20]. Eigenvalue problems for some classes of differential operators on an interval are reduced to a similar problem. In particular, a spectral problem for a first-order equation on an interval with a spectral parameter in a boundary value condition with integral perturbation leads to the studied problem. Asymptotic properties of entire functions with a given distribution law of roots are deeply studied in the Doctoral Thesis of V. B. Sherstyukov, on the basis of which the paper [25] was published. Questions of location of zeros of an entire function: on one ray, on a straight line, on several rays, in an angle or arbitrarily on the complex plane were studied in [2, 3, 6, 9, 20, 22, 23].

Meromorphic functions of completely regular growth in the upper half-plane with respect to the growth function are studied in one of the last works of K. G. Malyutin and M. V. Kabanko [17]. This paper is devoted to the construction of a characteristic polynomial of the spectral problem for the differentiation operator on an interval with linear occurrence of a spectral parameter in the perturbed boundary value condition, which is an entire holomorphic function. On the basis of the characteristic polynomial formula, conclusions about the asymptotic behavior of the spectrum of the perturbed spectral problem are established. The considered problem belongs to the nonlocal type of spectral problems. Such problems have been studied many times before. Among the recent publications, we note [11, 12, 16, 21]. The main fundamental feature of such tasks is their non-self-adjointness. This causes the main difficulties in considering them.

§ 1. Problem statement and main result

In the space $W_2^1(-1,1)$, we consider the eigenvalue problem for the differentiation operator:

$$L_1 y = y'(t) = \lambda y(t), \tag{1.1}$$

with the “perturbed” boundary value condition:

$$y(-1) - y(1) = \lambda \int_{-1}^{1} y(t)\,\Phi(t)\,dt, \tag{1.2}$$

where $\Phi(t)$ is a function with bounded variation and $\Phi(-1) - \Phi(1) = 1,$ λ is a complex number, spectral parameter.

It is required to find complex values of λ, at which the operator equation (1.1) has a nonzero solution.

In the case when $\lambda = 0$ we have $y(t) = C$, i. e., $\lambda_0 = 0$ is an eigenvalue of the operator L_1.

In the case when $\lambda \neq 0$, general solution of the equation (1.1) can be represented by the formula $y(t) = Ce^{\lambda t}, \forall C > 0$, and satisfying it by the boundary value problem (1.2), we get the characteristic determinant of the spectral problem (1.1), (1.2):

$$\Delta_1(\lambda) = e^{-\lambda} - e^{\lambda} - \lambda \int_{-1}^{1} e^{\lambda t}\Phi(t)dt,$$

which is an entire analytical function of the variable $\lambda = x + iy$, $\operatorname{Re}\lambda = x,$ $\operatorname{Im}\lambda = y$.

In the case when $\Phi(t) = 0$, the spectral problem (1.1), (1.2) is reduced to

$$L_0 y = y'(t) = \lambda y(t), \quad -1 < t < 1, \quad y(-1) = y(1), \tag{1.3}$$

moreover, the characterictic determinant of the “unperturbed” spectral problem (1.3) is the entire function: $\Delta_0(\lambda) = \dfrac{e^{-\lambda} - e^{\lambda}}{\lambda}$.

Numbers $\lambda_n^0 = in\pi$, $n = \pm 1, \pm 2, \pm 3, \pm 4, \ldots,$ are eigenvalues, moreover, $\forall C > 0$ $y_{n_0}^0 = Ce^{in\pi t}$ are eigen functions of the “unperturbed” operator L_0, which form a complete orthonormal system in $L_2(-1, 1)$.

In the case $\Phi(t)$ is a function of bounded variation and $\Phi(-1) = \Phi(1) = 1,$ equating the characteristic determinant $\Delta_1(\lambda)$ of the “perturbed” spectral problem (1.1), (1.2) to zero, we investigate distribution of zeros of the entire analytic function, which adequately determines eigenvalues of the operator L_1. Due to the well-known theorem [19], which says that any function with bounded variation has almost everywhere a finite derivative, we take integral from the expression $\int_{-1}^{1} e^{\lambda t}\Phi(t)\,dt$, by using the method of integration by parts.

Then the function $\Delta_1(\lambda)$ takes the following form:

$$\Delta_1(\lambda) = \frac{2\left(e^{\lambda} - e^{-\lambda}\right)}{\lambda} - \frac{1}{\lambda}\int_{-1}^{1-\delta} e^{\lambda t} d\Phi(t) - \frac{1}{\lambda}\int_{1-\delta}^{1} e^{\lambda t} d\Phi(t).$$

Further, by using the known Rouche theorem [24], and due to this theorem, we introduce the function $f(\lambda) = \frac{2(e^{\lambda} - e^{-\lambda})}{\lambda} = \Delta_0(\lambda)$, and the difference $g(\lambda) = \Delta_1(\lambda) - f(\lambda)$. Let us show that the function $\Delta_1(\lambda)$ has no zeros out of the strip $|\operatorname{Re}\lambda| < k$ at some k. To do this we estimate the function $\Delta_1(\lambda)$ from below:

$$|\Delta_1(\lambda)| \geq \frac{2}{|\lambda|}e^{|\lambda|} - \frac{2}{|\lambda|}e^{-|\lambda|} - \frac{1}{|\lambda|}\int_{-1}^{1-\delta} e^{|\lambda| t} d|\Phi(t)| - \frac{1}{|\lambda|}\int_{1-\delta}^{1} e^{|\lambda| t} d|\Phi(t)|. \tag{1.4}$$

Therefore, we estimate the function $f(\lambda)$ from below, while the remaining terms of the function $g(\lambda)$ are estimated from above:

$$\begin{aligned} &|f(\lambda)| \geq \frac{2}{|\lambda|}e^{x} - \frac{2}{|\lambda|}e^{-x}, \\ &|g(\lambda)| \leq \left(\int_{-1}^{1-\delta} e^{xt} d|\Phi(t)| + \int_{1-\delta}^{1} e^{xt} d|\Phi(t)|\right). \end{aligned} \tag{1.5}$$

We consider separately the estimation of each integral from (1.5). For this we use boundedness of variation of the function $\Phi(t)$. Then the first integral from (1.5) is estimated as follows:

$$\int_{-1}^{1-\delta} e^{xt} d\,|\Phi(t)| \le e^{x(1-\delta)} \int_{-1}^{1-\delta} |d\Phi(t)| \le e^{x(1-\delta)} \int_{-1}^{1} |d\Phi(t)| = e^{x(1-\delta)} H,$$

where $H = \int_{-1}^{1} |d\Phi(t)|$ is a constant.

Now we consider estimation of the second integral:

$$\int_{1-\delta}^{1} e^{xt} |d\Phi(t)| \le e^{x} \int_{1-\delta}^{1} |d\Phi(t)| \le e^{x} \varepsilon(\delta), \tag{1.6}$$

moreover $\varepsilon(\delta) \to 0$ as $\delta \to 0$. Thus, due to (1.4)–(1.6), we come to the estimation:

$$\Delta_1(\lambda) \ge \frac{2}{|\lambda|}\left(e^{x} - e^{-x}\right) - \frac{1}{|\lambda|} e^{x(1-\delta)} H - \frac{1}{|\lambda|} e^{x} \varepsilon(\delta), |\lambda||\Delta_1(\lambda)| = |\Delta(\lambda)|.$$

$|\operatorname{Re}\Delta(\lambda)| \ge |e^{x} - e^{-x}O(1)| > e^{x}$, for $\operatorname{Re}\lambda = x \ge k$ that is $\Delta_1(\lambda)$ has no zeroes for these values of x. Similar reasoning is for negative x. We formulate the obtained result in the form of the following theorem.

Theorem 1. *If $\Phi(t)$ is a function of bounded variation and $\Phi(-1) = \Phi(1) = 1$, then all eigenvalues of the "perturbed" operator of differentiation L_1 belong to the strip $|\operatorname{Re}\lambda| = |x| < k$ for some k, where $\lambda = x + iy$.*

Calculation of zeros of the function $f(\lambda)$ implies $\lambda_n^0 = in\pi$, $n = \pm 1, \pm 2, \ldots$, which coincide with zeros of the function $\Delta_0(\lambda)$, otherwise they coincide with eigenvalues of the operator L_0, i. e., "unperturbed" spectral problem (1.3). We consider a square T with the side 2ε centered at the point λ_n^0 on the complex plane λ. We choose a minimal $\varepsilon > 0$ such that conditions of the Rouche theorem [24] hold for the functions $f(\lambda), g(\lambda)$ on sides of the square T. Let us show that conditions of the Rouche theorem $|g(\lambda)| < |f(\lambda)|$ hold on sides of the square T, for this we compare majorant of the function $g(\lambda)$ with minorant of the function $f(\lambda)$:

$$\max_T |g(\lambda)| < \min_T |f(\lambda)|.$$

We estimate the function $f(\lambda)$ from below:

$$|f(\lambda)| = \frac{2}{|\lambda|}\left(e^{\lambda} - e^{-\lambda}\right) \ge \min_T \frac{2}{|\lambda|}\left|e^{\lambda} - e^{-\lambda}\right| = \frac{2}{\lambda_0^*}\left|e^{\lambda^*} - e^{-\lambda^*}\right|,$$

where $\lambda^* \in T$. The last equality follows from the fact that $|f(\lambda)|$ is a continuous function, and the square T is compact.

We estimate the function $g(\lambda)$ on sides of the square T. Imaginary axis divides the square T into two equal parts. We estimate the function $g(\lambda)$ on the right half of the square

$$|g(\lambda)| \le \frac{1}{|\lambda|} \int_{-1}^{1-\delta} e^{xt} d\,|\Phi(t)| + \frac{1}{|\lambda|} \int_{1-\delta}^{1} e^{xt} d\,|\Phi(t)|.$$

We consider estimation of each term separately.

Let us estimate the first term. We choose $\frac{2}{n} \ge \delta \ge \frac{1}{n}$, then, due to the fact that $x \ge 0$, $-1 \le t \le 1 - \delta$, we obtain the inequality:

$$\frac{1}{|\lambda|} \int_{-1}^{1-\delta} e^{xt} d\,|\Phi(t)| \le \frac{1}{|\lambda|} e^{x(1-\delta)} \int_{-1}^{1-\delta} d\,|\Phi(t)| \le \frac{1}{|\lambda|} e^{x(1-\frac{1}{n})} \int_{-1}^{1-\frac{1}{n}} d\,|\Phi(t)| = \frac{1}{|\lambda|} C_1 e^{x(1-\frac{1}{n})},$$

where $C_1 = \int_{-1}^{1} d\,|\Phi(t)|$ is a constant.

We estimate the second term. Since $\max(xt) = x$, we have:

$$\frac{1}{|\lambda|}\int_{1-\delta}^{1} e^{xt} d\,|\Phi(t)| \le \frac{1}{|\lambda|}e^{x}\int_{1-\delta}^{1} d\,|\Phi(t)| \ \le \frac{1}{|\lambda|}e^{x}\int_{1-\frac{2}{n}}^{1} d\,|\Phi(t)| \ \le \frac{1}{|\lambda|}e^{x}\mu\left(\frac{1}{n}\right),$$

as $n \gg 1$. It is obvious that $\mu\left(\frac{1}{n}\right) \to 0$ as $n \to \infty$.

Now we estimate the function $g(\lambda)$ from above by the left half of the square:

$$|g(\lambda)| \le \frac{1}{|\lambda|}\int_{-1}^{-1+\delta} e^{xt} d\,|\Phi(t)| + \frac{1}{|\lambda|}\int_{-1+\delta}^{1} e^{xt} d\,|\Phi(t)|\,.$$

Estimation of the first term has the following form:

$$\frac{1}{|\lambda|}\int_{-1}^{-1+\delta} e^{xt} d\,|\Phi(t)| \le \frac{1}{|\lambda|}e^{-x}\int_{-1}^{-1+\delta} d\,|\Phi(t)| \le \frac{1}{|\lambda|}e^{-x}\int_{-1}^{-1+\frac{2}{n}} d\,|\Phi(t)| \le \frac{1}{|\lambda|}e^{-x}\mu\left(\frac{1}{n}\right),$$

as $n \gg 1$.

Here, as $n \to \infty$, the quantity $\mu\left(\frac{1}{n}\right) \to 0$. The second term is estimated as follows:

$$\frac{1}{|\lambda|}\int_{-1+\delta}^{1} e^{xt} d\,|\Phi(t)| \le \frac{1}{|\lambda|}e^{x(-1+\delta)}\int_{-1+\delta}^{1} d\,|\Phi(t)| \le \frac{1}{|\lambda|}e^{x(-1+\frac{1}{n})}\int_{-1+\frac{1}{n}}^{1} d\,|\Phi(t)| \le \frac{1}{|\lambda|}e^{x(-1+\frac{1}{n})}C_1,$$

where $C_1 = \int_{-1}^{1} d\,|\Phi(t)|$ is a constant.

Taking into account these inequalities, we choose $\varepsilon > 0$ such that the following inequality holds on sides of the square T:

$$\frac{2}{|\lambda_0^*|}\left|e^{\lambda^*} - e^{-\lambda^*}\right| > \frac{1}{|\lambda|}e^{|x|}C\left(\mu\left(\frac{1}{n}\right) + e^{-\frac{1}{n}|x|}\right).$$

To the left side of the inequality, we apply the Lagrange theorem on finite increments [13], then the following inequality is sufficient:

$$\frac{2}{|\lambda_0^*|}H\varepsilon > \frac{1}{|\lambda|}e^{|x|}C\left(\mu\left(\frac{1}{n}\right) + e^{-\frac{1}{n}|x|}\right),$$

where H is a constant, which bounds the derivative $\left(e^{\lambda} - e^{-\lambda}\right)$ from below on sides of the square T; e^{x} is bounded, since from $-\varepsilon < \operatorname{Re}\lambda < \varepsilon$ it follows that $e^{-\varepsilon} < e^{x} < e^{\varepsilon}$.

From boundedness of the differences $\lambda - \lambda_n^0$, $\lambda^* - \lambda_n^0$, it follows that the relation $|\frac{\lambda}{\lambda^*}|$ is bounded as $n \to \infty$.

Since $e^{-\frac{1}{n}x} > \mu\left(\frac{1}{n}\right)$, consequently, we obtain that $\varepsilon \approx \underline{\underline{o}}(1)$. Therefore, we prove the next theorem.

Theorem 2. *Let $\Phi(t)$ be a function of bounded variation and $\Phi(-1) = \Phi(1) = 1$. Then zeros of the function $\Delta_1(\lambda)$, that is, eigenvalues of the operator L_1 — "perturbed" spectral problem* (1.1), (1.2) *form a countable set and have the asymptotic $\lambda_n^0 = in\pi + \underline{\underline{o}}(1)$ as $n \to \infty$.*

Remark 1. One of the features of the considered problem is that conjugate to (1.1), (1.2) is the spectral problem for the loaded differential equation:

$$L_1^*\vartheta = \vartheta'(t) + \Phi(t)\,\overline{\lambda}\vartheta(-1) = \overline{\lambda}\vartheta(t)\,, \quad \vartheta(-1) = \vartheta(1)\,.$$

In [10], eigenvalues of a loaded differential operator of differentiation with general boundary value conditions on an interval have been found, and in [7, 8], questions of stability of basis properties of root vectors of a loaded multiple differentiation operator have been investigated in the space $L_2(0, 1)$.

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Н. С. Иманбаев

О нелокальном возмущении задачи на собственные значения оператора дифференцирования на отрезке

Ключевые слова: оператор дифференцирования, краевые условия, интегральное возмущение, функция ограниченной вариации, характеристический многочлен, целые аналитические функции, нули целой функции, собственные значений, асимптотика.

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Построен характеристический многочлен спектральной задачи дифференциального уравнения первого порядка на отрезке со спектральным параметром в краевом условии с интегральным возмущением, которое является целой аналитической функцией от спектрального параметра. На основе формулы характеристического многочлена доказаны выводы об асимптотике спектра возмущенной спектральной задачи.

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