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HOLOMORPHIC CONTINUATION INTO A MATRIX BALL OF FUNCTIONS DEFINED ON A PIECE OF ITS SKELETON

The question of the possibility of holomorphic continuation into some domain of functions defined on the entire boundary of this domain has been well studied. The problem of describing functions defined on a part of the boundary that can be extended holomorphically into a fixed domain is attracting more interest. In this article, we reformulate the problem under consideration: *Under what conditions can we extend holomorphically to a matrix ball the functions given on a part of its skeleton?* We describe the domains into which the integral of the Bochner–Hua Luogeng type for a matrix ball can be extended holomorphically. As the main result, we present the criterion of holomorphic continuation into a matrix ball of functions defined on a part of the skeleton of this matrix ball. The proofs of several results are briefly presented. Some recent advances are highlighted. The results obtained in this article generalize the results of L. A. Aizenberg, A. M. Kytmanov and G. Khudayberganov.

Keywords: matrix ball, Shilov's boundary, Bochner–Hua Luogeng integral, Hardy space, holomorphic continuation, orthonormal system.

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Various solutions to this problem can be found in the book [1] for one dimensional and multidimensional cases. A holomorphic function does not have isolated singular points in case $n \geq 2$, the singularities of such functions must go to the boundary of the domain or extend to infinity. The question of the possibility of holomorphic continuation to the domain of functions defined on the entire boundary of this domain is well studied (see, for example, [1–9]). The theory of functions of several complex variables or multidimensional complex analysis, by now has a fairly rigorously constructed theory (see, for example, [10]). At the same time, many questions of classical (one-dimensional) complex analysis still do not have unambiguous multidimensional analogs. This is due to the complex structure of the multidimensional complex space, the ambiguity (overdetermination) of the Cauchy–Riemann conditions, the absence of a universal Cauchy integral formula, etc. In the works of E. Cartan [11], Hua Luogeng [12], the matrix approach of the presentation of the theory of multivariate complex analysis is widely used. Here some investigations of a matrix ball associated with classical domains and related questions of function theory and geometry are presented. The importance of studying matrix balls is that they are not reducible, i. e., these domains are, in a sense, model regions of multidimensional space. In this paper, we present the latest results in multivariate complex analysis related to matrix balls. It is known that in multivariable complex analysis, integral representations are a powerful constructive apparatus and have important applications. Matrix calculus is currently widely used in various domains of mathematics, mechanics, theoretical physics, electronics, telemechanics, etc. At present, the investigation and study of the boundary properties of the integral formulas of Bochner–Hua Luogeng, Bergman, Cauchy–Szegő, Poisson for matrix domains, as well as finding the Car-leman formula for a matrix ball is an urgent problem. The goal of our work is to obtain a criterion for holomorphic extendibility into a matrix ball for functions defined on a part of the Shilov boundary (skeleton) of a matrix ball, which is close in spirit to the criterion of L. A. Aizenberg, A. M. Kytmanov [3], and G. Khudayberganov [4].

Introduction and preliminaries

Consider the space of complex m^2 variables, denoted by $\mathbb{C}^{m^2}$. In some cases it is convenient to represent the point Z of this space in the form of square $[m \times m]$ matrices, i. e., in the form $Z = (z_{ij})_{i,j=1}^{m}$. With this representation of points, the space $\mathbb{C}^{m^2}$ will be denoted by $\mathbb{C}[m\times m]$. Direct product $\underbrace{\mathbb{C}[m \times m] \times \cdots \times \mathbb{C}[m \times m]}_{n}$ of n instances of $[m\times m]$ matrices we denote by $\mathbb{C}^n[m\times m]$.

Let $Z = (Z_1, \ldots, Z_n)$ be a vector composed of square matrices Z_j of order m, considered over the field of complex numbers $\mathbb{C}$. Let us write the elements of the vector $Z = (Z_1, \ldots, Z_n)$ as points z of the space $\mathbb{C}^{nm^2}$:

$$z = \left\{ z_{11}^{(1)}, \ldots, z_{1m}^{(1)}, \ldots, z_{m1}^{(1)}, \ldots, z_{mm}^{(1)}, \ldots, z_{11}^{(n)}, \ldots, z_{1m}^{(n)}, \ldots, z_{m1}^{(n)}, \ldots, z_{mm}^{(n)} \right\} \in \mathbb{C}^{nm^2}. \quad (0.1)$$

Hence, we can assume that Z is an element of the space $\mathbb{C}^n\left[m \times m\right]$, i. e., we arrive to the isomorphism $\mathbb{C}^n\left[m \times m\right] \cong \mathbb{C}^{nm^2}$.

Let's define the matrix "scalar" product:

$$\langle Z, W \rangle = Z_1 W_1^* + \cdots + Z_n W_n^*,$$

where W_j^* is the conjugate and transposed matrix for the W_j matrix.

It is known (see [6]) that the matrix balls $\mathbb{B}_{m,n}^{(1)}$, $\mathbb{B}_{m,n}^{(2)}$ and $\mathbb{B}_{m,n}^{(3)}$ of the first, second and third types have the forms, respectively:

$$\mathbb{B}_{m,n}^{(1)} = \left\{(Z_1, \ldots, Z_n) = Z \in \mathbb{C}^n\left[m \times m\right] \colon I - \langle Z, Z \rangle > 0\right\},$$
$$\mathbb{B}_{m,n}^{(2)} = \{Z \in \mathbb{C}^n\left[m \times m\right] \colon I - \langle Z, Z \rangle > 0, \quad \forall Z'_\nu = Z_\nu,\ \nu = 1, \ldots, n\},$$

and

$$\mathbb{B}_{m,n}^{(3)} = \Big\{(Z \in \mathbb{C}^n\left[m \times m\right] \colon I + \langle Z, Z \rangle > 0,\ \forall\ Z_\nu' = -Z_\nu, \quad \nu = 1, \ldots, n\Big\}.$$

The skeletons (Shilov's boundary) of matrix balls $\mathbb{B}_{m,n}^{(k)}$, denote by $\mathbb{X}_{m,n}^{(k)}$, $\ k = 1, 2, 3$, i. e.,

$$\mathbb{X}_{m,n}^{(1)} = \left\{Z \in \mathbb{C}^n\left[m \times m\right] \colon \langle Z, Z \rangle = I\right\},$$
$$\mathbb{X}_{m,n}^{(2)} = \{Z \in \mathbb{C}^n\left[m \times m\right] \colon \langle Z, Z \rangle = I, \quad Z'_v = Z_\nu,\ \nu = 1, 2, \ldots, n\},$$
$$\mathbb{X}_{m,n}^{(3)} = \Big\{Z \in \mathbb{C}^n\left[m \times m\right] \colon I + \langle Z, Z \rangle = 0,\ Z_\nu' = -Z_\nu, \quad \nu = 1, 2, \ldots, n\Big\}.$$

Note that $\mathbb{B}_{1,1}^{(1)}$, $\mathbb{B}_{1,1}^{(2)}$ and $\mathbb{B}_{2,1}^{(3)}$ are unit discs, and $\mathbb{X}_{1,1}^{(1)}$, $\mathbb{X}_{1,1}^{(2)}$, and $\mathbb{X}_{2,1}^{(3)}$ are unit circles in the complex plane $\mathbb{C}$.

If $n = 1$, $m > 1$, then $\mathbb{B}_{m,1}^{(k)}$, $\ k = 1, 2, 3$, are classical domains of the first, second and third types (according to the classification of E. Cartan (see [11])), and the skeletons $\mathbb{X}_{m,1}^{(1)}$, $\mathbb{X}_{m,1}^{(2)}$, and $\mathbb{X}_{m,1}^{(3)}$ are unitary, symmetric unitary and skew-symmetric unitary matrices, respectively.

The first type of matrix balls was considered by A. G. Sergeev, G. Khudayberganov, S. Kosbergenov (see [6, 13–16]). In [6] the volumes of a matrix ball of the first type and its skeleton are calculated:

$$V\left(\mathbb{B}_{m,n}^{(1)}\right) = \pi^{nm^2} \frac{1!2!\ldots(mn-1)!}{m!\,(m+1)!\ldots(m+mn-1)!}, \qquad V\left(\mathbb{X}_{m,n}^{(1)}\right) = \frac{(2\pi)^{nm^2 - \frac{m(m-1)}{2}}}{(mn-m)!\ldots(mn-1)!}.$$

Integral formulas for a matrix ball of the second type were obtained in [17, 18], and the third type was studied and integral formulas were found in [19,20]. In [21] the volumes of a matrix ball of the second and third types are calculated. The full volumes of these domains are necessary to find the kernels of the integral formulas for these domains (the Bergman, Cauchy-Szegő, Poisson kernels, etc. (see eg [17, 22])).

Note that the matrix balls $\mathbb{B}^{(1)}_{m,n}$, $\mathbb{B}^{(2)}_{m,n}$, $\mathbb{B}^{(3)}_{m,n}$ are complete circular convex bounded domains. In addition, the domains $\mathbb{B}^{(1)}_{m,n}$, $\mathbb{B}^{(2)}_{m,n}$, $\mathbb{B}^{(3)}_{m,n}$ and their skeletons $\mathbb{X}^{(1)}_{m,n}$, $\mathbb{X}^{(2)}_{m,n}$, $\mathbb{X}^{(3)}_{m,n}$ are invariant under unitary transformations (see [6, 18]).

§ 1. Holomorphic continuation from a part of the skeleton of the matrix ball $\mathbb{B}^{(1)}_{m,n}$

For convenience, denote $\mathbb{B}^{(1)}_{m,n}$ by $\mathbb{B}_{m,n}$, and $\mathbb{X}^{(1)}_{m,n}$ by $\mathbb{X}_{m,n}$. Let $L^2(\mathbb{X}_{m,n}, d\mu)$ be the space of functions f, that are square-integrable in the normalized Lebesgue measure $d\mu$ on skeleton $\mathbb{X}_{m,n}$, which is the Haar measure and therefore, it is invariant under rotations, and the Hardy class $H^2(\mathbb{B}_{m,n})$ consists of all functions f, that are holomorphic in the domain $\mathbb{B}_{m,n}$ for that

$$\|f\|_{H^2} = \sup_{0<r<1}\left(\int\limits_{\mathbb{X}_{m,n}} |f(rZ)|^2\, d\mu\right)^{1/2} < \infty.$$

Since $\mathbb{B}_{m,n}$ is a bounded complete circular domain, the functions f of class $H^2(\mathbb{B}_{m,n})$ have the following properties (see [1, chapter 6], [16, 24]):

1^0. For almost all (as μ) $Z \in \mathbb{X}_{m,n}$ slice functions $f_Z(\lambda) = f(\lambda Z)$ belong to the space H^2 in the unit circle $\Delta = \{\lambda \in \mathbb{C}^1 : |\lambda| < 1\}$;

2^0. The function f has radial boundary values

$$\lim_{r\to 1-0} f(rZ) = f^*(Z), \quad Z \in \mathbb{X}_{m,n},$$

moreover, these boundary values f^* belong to the class $L^2(\mathbb{X}_{m,n}, d\mu)$;

3^0. The following formula is valid:

$$\lim_{r\to 1-0} \int\limits_{\mathbb{X}_{m,n}} |f(rZ)|\, d\mu = \int\limits_{\mathbb{X}_{m,n}} |f^*(Z)|\, d\mu;$$

4^0. If the slice function $f_Z(\lambda)$ of some holomorphic in $\mathbb{B}_{m,n}$ functions f belong to the Hardy class H^2 in the unit disk for almost all $Z \in \mathbb{X}_{m,n}$ and the radial boundary values of f^* are in $L^2(\mathbb{X}_{m,n}, d\mu)$, then $f \in H^2(\mathbb{B}_{m,n})$;

5^0. The function $f \in H^2(\mathbb{B}_{m,n})$ can be restored to $\mathbb{B}_{m,n}$ by its radial boundary values f^* using the Bochner–Hua Luogeng formula[1]

$$f(Z) = \int\limits_{\mathbb{X}_{m,n}} \det{}^{-mn}\left(I^{(m)} - \langle Z, U\rangle\right) f(U)\, d\mu. \tag{1.1}$$

6^0. If the set $V \subset \mathbb{X}_{m,n}$ has positive measure ($\mu(V) > 0$), then V is the set of uniqueness for the Hardy class $H^2(\mathbb{B}_{m,n})$;

7^0. The Hardy class $H^2(\mathbb{B}_{m,n})$ is invariant under automorphisms of the ball $\mathbb{B}_{m,n}$.

Let us write the elements of the vector $Z = (Z_1, \ldots, Z_n) \in \mathbb{B}_{m,n}$ in the form (0.1) and by $z^{[\alpha]}$ we denote a vector with components

$$\sqrt{\frac{|\alpha|!}{\alpha_1!\alpha_2!\ldots\alpha_{nm^2}!}}\left(z^{(1)}_{11}\right)^{\alpha_1}\cdot\ldots\cdot\left(z^{(1)}_{1m}\right)^{\alpha_m}\cdot\ldots\cdot\left(z^{(n)}_{mm}\right)^{\alpha_{nm^2}}, \quad |\alpha| = \sum_{i=1}^{nm^2}\alpha_i, \quad \alpha_i \geq 0. \tag{1.2}$$

[1] Futher, we will denote the radial boundary functions f^* of the Hardy class also by f.

The dimension of the subspace generated by the vector $z^{[\alpha]}$ is equal to the dimension of the direct sum of subspaces with dimensions (see [12,25,26])

$$q(\alpha_1,\alpha_2,\ldots,\alpha_m)=N(\alpha_1,\alpha_2,\ldots,\alpha_m)\cdot N(\alpha_1,\alpha_2,\ldots,\alpha_m,0,\ldots,0)$$

and it is equal to

$$\sum_{\substack{\alpha_1+\alpha_2+\ldots+\alpha_m=|\alpha|\\ \alpha_1\geq\alpha_2\geq\ldots\geq\alpha_m\geq 0}} N(\alpha_1,\alpha_2,\ldots,\alpha_m)\cdot N(\alpha_1,\alpha_2,\ldots,\alpha_m,0,\ldots,0)=\frac{(nm^2+|\alpha|-1)!}{\alpha!(nm^2-1)!}$$

where

$$N(\alpha_1,\alpha_2,\ldots,\alpha_m)=\frac{D(\alpha_1+m-1,\alpha_2+m-2,\ldots,\alpha_{m-1}+1,\alpha_m)}{D(m-1,m-2,\ldots,1,0)},$$
$$D(\alpha_1,\alpha_2,\ldots,\alpha_m)=\prod_{1\leq i<j\leq m}(\alpha_i-\alpha_j),\quad m\geq 2.$$

Obviously (1.2) contains all monomials of degree α, that is, any polynomial in

$$z_{11}^{(1)},\ldots,z_{1m}^{(1)};\ldots;z_{m1}^{(1)},\ldots,z_{mm}^{(1)};\ldots;z_{11}^{(n)},\ldots,z_{1m}^{(n)};\ldots;z_{m1}^{(n)},\ldots,z_{mm}^{(n)}$$

is a linear combination of expressions like (1.2), if α takes the values 0,1,2,. . . .

Let us denote by

$$\varphi^{(p)}_{\alpha_1,\alpha_2,\ldots,\alpha_m}(Z),\quad p=1,2,\ldots,q(\alpha_1,\alpha_2,\ldots,\alpha_m),$$

components of the vector $z^{[\alpha]}$.

In [6] it was proved that the system of functions

$$(\rho_\alpha)^{-\frac{1}{2}}\varphi^{(p)}_\alpha(Z),\;\; p=1,2,\ldots,q(\alpha_1,\alpha_2,\ldots,\alpha_m),\quad \alpha=0,1,2,\ldots,$$

is an orthonormal system in the domain $\mathbb{B}_{m,n}$, where

$$\rho_\alpha=\int\limits_{\mathbb{B}_{m,n}}\left|\varphi^{(p)}_\alpha(Z)\right|^2 d\nu,\qquad d\nu=\prod_{k=1}^{n}\prod_{1\leq i\leq j\leq m}dx^{(k)}_{i,j}dy^{(k)}_{i,j}$$

and the system of functions

$$(\delta_\alpha)^{-\frac{1}{2}}\varphi^{(p)}_\alpha(U),\quad p=1,2,\ldots,q(\alpha_1,\alpha_2,\ldots,\alpha_m),\quad \alpha=0,1,2,\ldots,\tag{1.3}$$

forms a complete orthonormal system on $\mathbb{X}_{m,n}$, where $\varphi^{(p)}_\alpha(U)$, $p=1,2,\ldots,q(\alpha_1,\alpha_2,\ldots,\alpha_m)$, $\alpha=0,1,2,\ldots$ are the components of the vector $u^{[\alpha]}$ ($u=(u_{11}^{(1)},\ldots,u_{1m}^{(1)},\ldots,u_{m1}^{(n)},\ldots,u_{mm}^{(n)})$) and

$$\delta_\alpha=\int\limits_{\mathbb{X}_{m,n}}\left|\varphi^{(p)}_\alpha(U)\right|^2 d\mu.$$

Theorem 1 (see [6]). *Let $f\,(U)$ be an integrable function in $\mathbb{X}_{m,n}$ and let*

$$a^p_\alpha=\frac{V(\mathbb{X}_{m,n})}{\sqrt{\delta_\alpha}}\cdot\int\limits_{\mathbb{X}_{m,n}}f(U)\overline{\varphi^{(p)}_\alpha(U)}\,d\mu\tag{1.4}$$

be the Fourier coefficients of this function with respect to the orthonormal system (1.3)*. Then the integral* (1.1) *represents in $\mathbb{B}_{m,n}$ a holomorphic function that expands in this domain in a series*

$$\sum_{\alpha\geq 0}\sum_{p=1}^{q(\alpha)}a^p_\alpha\frac{\varphi^{(p)}_\alpha(Z)}{\sqrt{\delta_\alpha}}.\tag{1.5}$$

In other words, due to (1.1) any function $f \in H^2(\mathbb{B}_{m,n})$ can be expanded into the series (1.5) with coefficients (1.4), while the series (1.5) converges uniformly on every compact set from $\mathbb{B}_{m,n}$.

Consider the Bochner–Hua Luogeng type integral

$$F(Z) = \int\limits_{X_{m,n}} \frac{f(U)}{\det^{mn}\left(I^{(m)} - \langle Z, U\rangle\right)} d\mu, \tag{1.6}$$

where $Z \in \mathbb{C}^n[m \times m]$ for a given integrable function $f(U)$ (we will try to find out for which Z the integral (1.6) exists). Consider the domain $\mathbb{B}^-_{m,n}$, defined by

$$\mathbb{B}^-_{m,n} = \{Z \in \mathbb{C}^n[m \times m] \colon I - \langle Z, Z\rangle < 0\}.$$

The following statement is true.

Proposition 1. *For points Z such that* $\det(\langle Z, Z\rangle) \neq 0$*, the mapping*

$$W_k = (\langle Z, Z\rangle)^{-1} Z_k \quad (k = 1, \ldots, n)$$

maps the domain $\mathbb{B}_{m,n}$ to the domain $\mathbb{B}^-_{m,n}$, while the points $U \in \mathbb{X}_{m,n}$ go onto themselves.

P r o o f. Let $Z \in \mathbb{B}_{m,n}$. The matrix $\langle Z, Z\rangle$ is Hermitian and $(\langle Z, Z\rangle)^{-1} = (\langle Z, Z\rangle^*)^{-1} = \left(\langle Z, Z\rangle^{-1}\right)^*$. Then

$$\begin{gathered}\langle W, W\rangle = W_1W_1^* + \ldots + W_nW_n^* = \\ = (\langle Z, Z\rangle)^{-1} Z_1 Z_1^* \left((\langle Z, Z\rangle)^{-1}\right)^* + \ldots + (\langle Z, Z\rangle)^{-1} Z_n Z_n^* \left((\langle Z, Z\rangle)^{-1}\right)^* = \\ = (\langle Z, Z\rangle)^{-1} \left(Z_1Z_1^* + \ldots + Z_nZ_n^*\right) \left((\langle Z, Z\rangle)^{-1}\right)^* = \\ = (\langle Z, Z\rangle)^{-1} \langle Z, Z\rangle \left((\langle Z, Z\rangle)^{-1}\right)^* = \left((\langle Z, Z\rangle)^{-1}\right)^* = (\langle Z, Z\rangle^*)^{-1} = (\langle Z, Z\rangle)^{-1}.\end{gathered}$$

This implies

$$\langle W, W\rangle - I = (\langle Z, Z\rangle)^{-1} - I = (\langle Z, Z\rangle)^{-1} - (\langle Z, Z\rangle)^{-1}\langle Z, Z\rangle = (\langle Z, Z\rangle)^{-1}(I - \langle Z, Z\rangle).$$

Note that all eigenvalues of the matrix $\langle Z, Z\rangle$ are less than one, since $I - \langle Z, Z\rangle > 0$. Whence $\det\left((\langle Z, Z\rangle)^{-1}\right) > 1$ and then the matrix $\langle W, W\rangle - I$ is definite positive. It means $W = (W_1, W_2, \ldots, W_n) \in \mathbb{B}^-_{m,n}$. Taking this into account, we obtain the first statement of the proposition.

Now let $U \in \mathbb{X}_{m,n}$, i. e., $\langle U, U\rangle = I$. Then

$$W_k = (\langle U, U\rangle)^{-1} U_k = U_k \Rightarrow W \in \mathbb{X}_{m,n}. \qquad \square$$

Now, by $F^+(Z)$ we denote the value of the integral (1.6) for $Z \in \mathbb{B}_{m,n}$, and $F^-(Z)$ is the value of the integral (1.6) for $Z \in \mathbb{B}^-_{m,n}$.

The next statement is true.

Theorem 2. *The integral* (1.6) *makes sense in each of the following domains:*

$$\mathbb{B}_{m.n} = \{Z \in \mathbb{C}^n[m \times m] \colon I - \langle Z, Z\rangle > 0\},$$
$$\mathbb{B}^-_{m,n} = \{Z \in \mathbb{C}^n[m \times m] \colon I - \langle Z, Z\rangle < 0\}.$$

P r o o f. Now we will study the integrals of the Bochner–Hua Luogeng type more precisely, i. e., for a detailed study of $F^{\pm}(Z)$ we need to expand the definition of the functions $\varphi^{(p)}_{\alpha_1,\alpha_2,\ldots,\alpha_m}(U)$ in case $p=1,2,\ldots,q(\alpha_1,\alpha_2,\ldots,\alpha_m)$, $\alpha=0,1,2,\ldots$ have any signs.

If $f\left(U\right)$ has a Fourier series of the form

$$f\left(U\right)=\sum_{\alpha_1\geq\ldots\geq\alpha_m\geq 0}\sum_{p=1}^{q(\alpha)}a^{p}_{\alpha}\frac{\varphi^{(p)}_{\alpha}(U)}{\sqrt{\delta_\alpha}},$$

then by the theorem 1 for all $Z\in\mathbb{B}_{m,n}$ we have

$$F^{+}\left(Z\right)=\sum_{\alpha_1\geq\ldots\geq\alpha_m\geq 0}\sum_{p=1}^{q(\alpha)}a^{p}_{\alpha}\frac{\varphi^{(p)}_{\alpha}(Z)}{\sqrt{\delta_\alpha}}. \tag{1.7}$$

Moreover, if $f(U)\in L^2\left(\mathbb{X}_{m,n},d\mu\right)$, then $F^{+}(Z)\in H^2\left(\mathbb{B}_{m,n}\right)$. Using the equality [12, p. 114] we obtain

$$\varphi^{(p)}_{\alpha_1,\ldots,\alpha_m}(U)=\varphi^{(p)}_{\alpha_1-\alpha_m,\alpha_2-\alpha_m,\ldots,\alpha_{m-1}-\alpha_m,0}(U)(\det U)^{\alpha_m}, \tag{1.8}$$

for any $\alpha_1\geq\ldots\geq\alpha_m$. The system of functions $\left\{\varphi^{(p)}_{\alpha_1,\ldots,\alpha_m}\right\}$ is complete and orthogonal in $L^2\left(\mathbb{X}_{m,n},d\mu\right)$ (see [6]).

Let $Z\in\mathbb{B}^{-}_{m,n}$ and $f\left(U\right)\in L^2\left(\mathbb{X}_{m,n},d\mu\right)$. Then we have

$$F^{-}(Z)=\int\limits_{X_{m,n}}\frac{f\left(U\right)}{\det^{mn}\left(I^{(m)}-\left\langle\left(\langle Z,Z\rangle\right)^{-1}Z,U\right\rangle\right)}\,d\mu.$$

In [6] the Cauchy–Szegő kernel has the following form

$$C(\widetilde{Z},U)=\frac{1}{V(\mathbb{X}_{m,n})}\det{}^{-mn}(I^{(m)}-\left\langle\widetilde{Z},U\right\rangle=\sum_{\alpha\geq 0}\sum_{i,j=1}^{q(\alpha)}\varphi^{(\alpha)}_{i,j}(\widetilde{Z})\overline{\varphi^{(\alpha)}_{i,j}(U)}.$$

Using (1.8) and noting that $\widetilde{Z}=(\langle Z,Z\rangle)^{-1}Z$, we have

$$\frac{1}{\det^{mn}\left(I^{(m)}-\left\langle\left(\langle Z,Z\rangle\right)^{-1}Z,U\right\rangle\right)}=V\left(\mathbb{X}_{m,n}\right)\sum_{\alpha\geq 0}\sum_{i,j=1}^{q(\alpha)}\varphi^{(\alpha)}_{i,j}\left(\left(\langle Z,Z\rangle\right)^{-1}Z\right)\overline{\varphi^{(\alpha)}_{i,j}(U)}.$$

Multiplying the last expression by $f\left(U\right)$ and integrating term by term in measure $d\mu$, we obtain

$$\begin{aligned}F^{-}(Z)&=\int\limits_{\mathbb{X}_{m,n}}f\left(U\right)V\left(\mathbb{X}_{m,n}\right)\sum_{\alpha\geq 0}\sum_{i,j=1}^{q(\alpha)}\varphi^{(\alpha)}_{i,j}\left(\left(\langle Z,Z\rangle\right)^{-1}Z\right)\overline{\varphi^{(\alpha)}_{i,j}(U)}d\mu=\\&=\sum_{\alpha\geq 0}\sum_{i,j=1}^{q(\alpha)}\varphi^{(\alpha)}_{i,j}\left(\left(\langle Z,Z\rangle\right)^{-1}Z\right)\left[V\left(\mathbb{X}_{m,n}\right)\int\limits_{\mathbb{X}_{m,n}}f\left(U\right)\overline{\varphi^{(\alpha)}_{i,j}(U)}d\mu\right]=\\&=(-1)^{mn}\sum_{\alpha_1\geq\ldots\geq\alpha_m\geq 0}\sum_{p=1}^{q(\alpha)}a^{(p)}_{-\alpha_m-m,\ldots,-\alpha_1-m}\frac{\varphi^{(p)}_{-\alpha_m-m,\ldots,-\alpha_1-m}\left(\left(\langle Z,Z\rangle\right)^{-1}Z\right)}{\sqrt{\delta_\alpha}}.\end{aligned} \tag{1.9}$$

This means that $F^- \in H^2\left(\mathbb{B}^-_{m,n}\right)$. Therefore, in the domain $\mathbb{B}^-_{m,n}$ integral (1.6) represents a holomorphic function of $\left(\langle Z, Z\rangle\right)^{-1}Z$, which has a decomposition

$$F^-(Z) = (-1)^{mn} \sum_{\alpha_1\geq\ldots\geq\alpha_m\geq 0} \sum_{p=1}^{q(\alpha)} a^{(p)}_{-\alpha_m-m,\ldots,-\alpha_1-m} \frac{\varphi^{(p)}_{-\alpha_m-m,\ldots,-\alpha_1-m}\left(\left(\langle Z, Z\rangle\right)^{-1}Z\right)}{\sqrt{\delta_\alpha}}. \qquad \square$$

Remark 1. In other cases, there is always $U \in \mathbb{X}_{m,n}$, such that

$$\det(I - \langle Z, U\rangle) = 0,$$

i. e., the integral (1.6) is undefined.

Let us denote by $\{\varphi\}$ the system of functions defined by the equality (1.8), but not included in the expansions (1.7) and (1.9).

The following multidimensional matrix analogue of the Sokhotskii formula takes place.

Theorem 3. *If $f \in L^2(\mathbb{X}_{m,n})$ and f is orthogonal to the system of functions $\{\varphi\}$, then $F^+ \in H^2(\mathbb{B}_{m,n})$, and $F^- \in H^2\left(\mathbb{B}^-_{m,n}\right)$, moreover, almost everywhere on $\mathbb{X}_{m,n}$ there exist radial limits of the functions $F^\pm$ and the following equality holds almost everywhere:*

$$F^+(Z)\big|_{\mathbb{X}_{m,n}} + (-1)^{mn} F^-(Z)\big|_{\mathbb{X}_{m,n}} = f(Z)|_{\mathbb{X}_{m,n}}, \quad Z \in \mathbb{X}_{m,n}.$$

P r o o f. Let $0 < r < 1$. Then, for all $Z \in \mathbb{B}_{m,n}$ just

$$F^+(rZ) = \sum_{\alpha_1\geq\ldots\geq\alpha_m\geq 0} \sum_{p=1}^{q(\alpha)} a^p_\alpha \frac{\varphi^{(p)}_\alpha(Z)}{\sqrt{\delta_\alpha}} r^{|\alpha|}, \tag{1.10}$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_m$, and the series (1.10) converges uniformly.

Since $f \in L^2(\mathbb{X}_{m,n})$, then $F^+ \in H^2(\mathbb{B}_{m,n})$. Then the function $F^+(Z)$ has a radial limit almost everywhere on $\mathbb{X}_{m,n}$ and $F^+(Z)|_{\mathbb{X}_{m,n}} \in L^2(\mathbb{X}_{m,n})$.

Now for $Z \in \mathbb{B}^-_{m,n}$ consider the decomposition

$$F^-(\frac{1}{r}Z) = (-1)^{mn} \sum_{\alpha_1\geq\ldots\geq\alpha_m\geq 0} \sum_{p=1}^{q(\alpha)} a^{(p)}_{-\alpha_m-m,\ldots,-\alpha_1-m} \frac{\varphi^{(p)}_{-\alpha_m-m,\ldots,-\alpha_1-m}\left(\left(\langle Z, Z\rangle\right)^{-1}Z\right)}{\sqrt{\delta_\alpha}} r^{|\alpha|+m^2}.$$

This series converges uniformly for all $Z \in \mathbb{B}^-_{m,n}$ and by virtue of $f \in L^2(\mathbb{X}_{m,n})$ we get that $F^-(Z) \in H^2\left(\mathbb{B}^-_{m,n}\right)$. Then the function $F^-(Z)$ has a radial limit almost everywhere on $\mathbb{X}_{m,n}$ and $F^-(Z)|_{\mathbb{X}_{m,n}} \in L^2(\mathbb{X}_{m,n})$. Hence,

$$F^+(Z)\big|_{\mathbb{X}_{m,n}} + (-1)^{mn} F^-(Z)\big|_{\mathbb{X}_{m,n}} = \sum_{\alpha_1\geq\ldots\geq\alpha_m\geq 0} \sum_{p=1}^{q(\alpha)} a^p_\alpha \frac{\varphi^{(p)}_\alpha(U)}{\sqrt{\delta_\alpha}} = f(U),$$

equality holds almost everywhere, and this proves the statement of the theorem. $\square$

Theorem 4. *If $f \in C^\alpha(\mathbb{X}_{m,n}), \alpha > \frac{nm^2-2}{2} \geq 0$[2] and f is orthogonal to the system of functions $\{\varphi\}$, then the functions $F^+(Z)$ and $F^-(Z)$ extend continuously to $\mathbb{X}_{m,n}$ to a function from the class $C^{\alpha-\frac{nm^2}{2}+1-\varepsilon}(\mathbb{X}_{m,n})$, where ε is any positive number and*

$$F^+(Z)\big|_{\mathbb{X}_{m,n}} + (-1)^{mn} F^-(Z)\big|_{\mathbb{X}_{m,n}} = f(Z)|_{\mathbb{X}_{m,n}}.$$

[2]Jöricke's theorem [27] the estimate is not improved in the Hölder norm scale.

P r o o f. Since $f \in C^{\alpha}(\mathbb{X}_{m,n})$, it can be represented as

$$f(U) = \sum_{\varphi^{(p)}_{\alpha_1,\ldots,\alpha_m} \notin \{\varphi\}} \sum_{p=1}^{q(\alpha)} a^{(p)}_{\alpha} \frac{\varphi^{(p)}_{\alpha_1,\ldots,\alpha_m}(U)}{\sqrt{\delta_\alpha}}.$$

For $Z \in \mathbb{B}_{m,n}$ the decomposition

$$F^{+}(Z) = \sum_{\alpha_1,\ldots,\alpha_m \geq 0} \sum_{p=1}^{q(\alpha)} a^{(p)}_{\alpha} \frac{\varphi^{(p)}_{\alpha_1,\ldots,\alpha_m}(Z)}{\sqrt{\delta_\alpha}}.$$

For $Z \in \mathbb{B}^{-}_{m,n}$ the decomposition

$$F^{-}(Z) = (-1)^{mn} \sum_{\alpha_1 \geq \ldots \geq \alpha_m \geq 0} \sum_{p=1}^{q(\alpha)} a^{(p)}_{-\alpha_m - m,\ldots,-\alpha_1 - m} \frac{\varphi^{(p)}_{-\alpha_m - m,\ldots,-\alpha_1 - m}\left((\langle Z, Z\rangle)^{-1} Z\right)}{\sqrt{\delta_\alpha}}.$$

By Jöricke's theorem [27], as well as the theorem from [28] the functions $F^{+}(Z)$ and $F^{-}(Z)$ extend continuously to $\mathbb{X}_{m,n}$, to a function from the class $C^{\alpha - \frac{nm^2}{2} + 1 - \varepsilon}(\mathbb{X}_{m,n})$, $\varepsilon > 0$. Hence,

$$F^{+}(Z)\big|_{\mathbb{X}_{m,n}} + (-1)^{mn} F^{-}(Z)\big|_{\mathbb{X}_{m,n}} = \sum_{\alpha_1,\ldots,\alpha_m \geq 0} \sum_{p=1}^{q(\alpha)} a^{(p)}_{\alpha} \frac{\varphi^{(p)}_{\alpha_1,\ldots,\alpha_m}(Z)}{\sqrt{\delta_\alpha}} +$$

$$+ (-1)^{mn} \sum_{\alpha_1 \geq \ldots \geq \alpha_m \geq 0} \sum_{p=1}^{q(\alpha)} a^{(p)}_{-\alpha_m - m,\ldots,-\alpha_1 - m} \frac{\varphi^{(p)}_{-\alpha_m - m,\ldots,-\alpha_1 - m}\left((\langle Z, Z\rangle)^{-1} Z\right)}{\sqrt{\delta_\alpha}} =$$

$$= \sum_{\alpha_1,\ldots,\alpha_m \geq 0} \sum_{p=1}^{q(\alpha)} a^{(p)}_{\alpha} \frac{\varphi^{(p)}_{\alpha_1,\ldots,\alpha_m}(Z)}{\sqrt{\delta_\alpha}} + (-1)^{mn} \sum_{\alpha_1 \geq \ldots \geq \alpha_m \geq 0} \sum_{p=1}^{q(\alpha)} a^{(p)}_{-\alpha_m - m,\ldots,-\alpha_1 - m} \cdot$$

$$\cdot \frac{\varphi^{(p)}_{-\alpha_m - m,\ldots,-\alpha_1 - m}(Z^{*})}{\sqrt{\delta_\alpha}} = f(Z).$$

□

Theorem 5. *In order for the function $f \in L^2(\mathbb{X}_{m,n})$ to continue to a function from the class $H^2(\mathbb{B}_{m,n})$, it is necessary and sufficient that f be orthogonal to the system of functions $\{\varphi\}$ and $F^{-}(Z) = 0$ in $\mathbb{B}^{-}_{m,n}$.*

P r o o f. *Necessity.* Let f extend to a function from the class $H^2(\mathbb{B}_{m,n})$. Consider the integral

$$\int_{\mathbb{X}_{m,n}} \frac{f(U)}{\det^{mn}\left(I^{(m)} - \langle Z, U\rangle\right)} d\mu = \begin{cases} F^{+}(Z), & Z \in \mathbb{B}_{m,n}, \\ F^{-}(Z), & Z \in \mathbb{B}^{-}_{m,n}. \end{cases}$$

Then

$$F^{+}(Z) = \sum_{\alpha_1,\ldots,\alpha_m \geq 0} \sum_{p=1}^{q(\alpha)} a^{(p)}_{\alpha} \frac{\varphi^{(p)}_{\alpha_1,\ldots,\alpha_m}(Z)}{\sqrt{\delta_\alpha}},$$

where the coefficients are determined by the formula

$$a^{(p)}_{\alpha} = \frac{V(\mathbb{X}_{m,n})}{\sqrt{\delta_\alpha}} \cdot \int_{\mathbb{X}_{m,n}} f(U)\overline{\varphi^{(p)}_{\alpha}(U)}\, d\mu.$$

The radial boundary values of F^+ on $\mathbb{X}_{m,n}$ coincide with f, which implies that f is orthogonal to functions from the system functions $\{\varphi\}$. Now consider the integral

$$\int\limits_{\mathbb{X}_{m,n}} \frac{f(U)}{\det^{mn}\left(I^{(m)} - \langle Z, U\rangle\right)} d\mu, \quad Z \in \mathbb{B}_{m,n},$$

and there we make the replacement $Z \to (\langle Z, Z\rangle)^{-1} Z$. Then, by Proposition 1, $\mathbb{B}_{m,n}$ goes to $\mathbb{B}^-_{m,n}$. Since the integral under consideration is invariant under unitary transformations and by the equality (1.9), due to the proof of Theorem 2, we have

$$F^-(Z) = \int\limits_{\mathbb{X}_{m,n}} \frac{f(U)}{\det^{mn}\left(I^{(m)} - \langle Z, U\rangle\right)} d\mu =$$

$$= (-1)^{mn} \sum_{\alpha_1 \geq \ldots \geq \alpha_m \geq 0} \sum_{p=1}^{q(\alpha)} a^{(p)}_{-\alpha_m - m, \ldots, -\alpha_1 - m} \frac{\varphi^{(p)}_{-\alpha_m - m, \ldots, -\alpha_1 - m}\left((\langle Z, Z\rangle)^{-1} Z\right)}{\sqrt{\delta_\alpha}}.$$

Therefore, $F^-(Z) = 0$ in $\mathbb{B}^-_{m,n}$.

Sufficiency. Let f be orthogonal to the system $\{\varphi\}$ and $F^-(Z) = 0$ in $\mathbb{B}^-_{m,n}$. Then, by the Theorem 3, we have the following equality almost everywhere

$$F^+(Z)\big|_{\mathbb{X}_{m,n}} + (-1)^{mn} F^-(Z)\big|_{\mathbb{X}_{m,n}} = f(Z)|_{\mathbb{X}_{m,n}}.$$

From that we get that $F^+(Z)|_{\mathbb{X}_{m,n}} = f(Z)|_{\mathbb{X}_{m,n}}$. Therefore, the function $F^+(Z) \in H^2(\mathbb{B}_{m,n})$ is holomorphic continuation of the function f. ☐

Now let us reformulate the considered problem: *Under what conditions can we extend holomorphically to the matrix ball the functions given on a part of its skeleton?*

Theorem 6. *Let $V \subset \mathbb{X}_{m,n}$ be an open set and the function $f \in L^2(V)$ defined on V is orthogonal to the system $\{\varphi\}$. Then, for the continuation of f holomorphically into $\mathbb{B}_{m,n}$, it is necessary and sufficient that $F^-(Z)$ continues holomorphically into $\mathbb{B}_{m,n}$.*

P r o o f. *Necessity.* Let f extend holomorphically into $\mathbb{B}_{m,n}$, i. e., there exists $F \in \mathcal{O}(\mathbb{B}_{m,n})$ such that

$$F^+(Z)\big|_V = f,$$

where $\mathcal{O}(\mathbb{B}_{m,n})$ is the space of holomorphic functions in $\mathbb{B}_{m,n}$. Consider the Bochner–Hua Luogeng type integral

$$F^\pm = \int\limits_V \frac{f(U)}{\det^{mn}\left(I^{(m)} - \langle Z, U\rangle\right)} d\mu, \; Z \in \mathbb{B}^\pm_{m,n}.$$

By virtue of the theorem 3, we have $F^+ \in H^2(\mathbb{B}_{m,n})$, $F^- \in H^2\left(\mathbb{B}^-_{m,n}\right)$ and

$$F^+(Z)\big|_V + (-1)^{mn} F^-(Z)\big|_V = f(Z)|_V.$$

Then

$$(-1)^{mn} F^-(Z)\big|_V = \left[F(Z) - F^+(Z)\right]_V,$$

therefore, by the [29] wedge tip theorem, the function

$$\Phi(Z) = \begin{cases} (-1)^{mn} F^-(Z), & Z \in \mathbb{B}^-_{m,n}, \\ F(z) - F^+(Z), & Z \in \mathbb{B}_{m,n}, \end{cases}$$

is holomorphic on the set $\mathbb{B}_{m,n} \cup \mathbb{B}^-_{m,n} \cup V$. This suggests that $F^-(Z)$ continues holomorphically to $\mathbb{B}_{m,n}$.

Sufficiency. Let $F^-(Z)$ extends to a holomorphic function $g(Z)$ in $\mathbb{B}_{m,n}$. Then, by the Theorem 3, the function

$$h(Z) = (-1)^{mn} g(Z) + F^+(Z)$$

is holomorphic in $\mathbb{B}_{m,n}$ and $h(Z)|_V = f(Z)|_V$. □

Remark 2. For $n = 1$, the Theorem 6 completely coincides with the Theorem 2 from [4], and for $m = n = 1$, the Theorem 6 is L. A. Aizenberg's theorem [3, § 1].

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Голоморфное продолжение в матричный шар функций, заданных на куске его остова

Ключевые слова: матричной шар, граница Шилова, интеграл Бохнера–Хуа Ло-кена, пространство Харди, голоморфное продолжение, ортонормальная система.

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Вопрос о возможности голоморфного продолжения в область функций, заданных на всей границе этой области, достаточно хорошо изучен. Представляет интерес задача описания функций, заданных на части границы, которые могут быть голоморфно продолжены в фиксированную область. В статье переформулируем рассматриваемую задачу: *При выполнении каких условий можно голоморфно продолжить в матричный шар, функции заданных на части остова?* Описаны области, в которые голоморфно продолжается интеграл типа Бохнера–Хуа Ло-кена для матричного шара. Получен основной результат нашей работы — критерий голоморфного продолжения в матричной шар функций, заданных на части остова матричного шара. Кратко излагаются доказательства нескольких основных результатов. Приводятся некоторые недавние достижения. Сформулированы нерешенные задачи. Результаты, полученные в этой статье, являются общими случаями результатов Л. А. Айзенберга, А. М. Кытманова, Г. Худайберганова.

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