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ON THE STABILITY OF COLLOCATED FINITE VOLUME METHOD FOR THE GENERALIZED STOKES PROBLEM

In this paper, a symmetric stabilized collocated formulation of finite volume method is introduced and analyzed for the stationary generalized Stokes problem. This method is based on the lowest-order approximation using piecewise constant functions for both velocity and pressure unknowns. Stabilization is achieved by adding a discrete pressure term to the approximate formulation. The stability and convergence properties are established. Two numerical examples are presented to confirm the stability and accuracy of the proposed method.

Keywords: Stokes problem, inf-sup condition, finite volume methods, stabilized methods.

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Introduction

Finite difference, finite element and finite volume methods are widely used in computational fluid dynamics. Finite difference methods are simple and mass-conservative but lack flexibility for complex geometries. Finite element methods offer flexibility for complex shapes but do not conserve mass at the element level. Furthermore, the finite element approximation spaces for the primitive variables (velocity and pressure) cannot be chosen independently of each other. There is a compatibility condition, commonly called the *inf-sup* (or LBB) condition (see [1]), that needs to be satisfied if the resulting approximation is to be effective. Finite volume methods (FVMs) combine the advantages of both finite difference and finite element methods, offering the flexibility of finite element methods while being as easy to implement as finite difference methods. FVMs are also known as marker and cell methods [2, 3], finite volume element methods [4, 5], cell-centered methods [6], or covolume methods in some literature [3].

FV approximation of generalized Stokes problems is a current research topic. Various FV schemes have been developed by incorporating finite element concepts to achieve a more rigorous FV methodology. Among these new approaches, collocated FV schemes have attracted the attention of CFD researchers for several reasons, including the collocated arrangement of unknowns, computational efficiency, ease of coupling with additional conservation law solvers, local conservation properties, and the ability to construct discrete operators that preserve properties of the continuous problem. Unfortunately, a crucial drawback was observed from the beginning. When applied to incompressible flow problems, collocated FVM leads to *inf-sup* unstable schemes, which are usually handled using a stabilization technique.

In [7], we introduced and analyzed a novel stabilized FVM for the Stokes equations. The proposed method achieves stability by incorporating a pressure jump operator into the discrete formulation. The main goal of this paper is to investigate the mathematical properties of this FVM applied to the generalized Stokes problem. The generalized Stokes problem provides a broader framework by incorporating additional terms that model more complex fluid behaviors, such as heterogeneous material properties and variable density. This generalization leads to a more accurate and comprehensive representation of physical phenomena, particularly in cases where the classical Stokes equations may be too simplistic or inadequate. Regarding the stability issue, we demonstrate a weaker form of the *inf-sup* condition based on the global stabilization term, which ensures the stability of the scheme. Additionally, we establish first-order error estimates in the energy norm. Numerical tests are presented to illustrate the efficiency and effectiveness of the proposed method.

The rest of the paper is organized as follows. In Section 1, the generalized Stokes problem is briefly introduced, along with its weak formulation. Section 2 is devoted to the derivation of the approximate problem. A weaker form of the *inf-sup* condition, which holds for the spaces of interest, is given in Section 3. The core of the paper is Section 4, in which we present a thorough study of the proposed FV scheme, including stability and error estimation results. Numerical results are presented in Section 5. Finally, certain conclusions are drawn in the closing section.

§ 1. The generalized Stokes problem

Let $\Omega \subset \mathbb{R}^d$ $(d = 2$ or $3)$ be an open bounded domain with polygonal or polyhedral boundary $\partial\Omega$. We consider the generalized Stokes problem

$$\begin{aligned} \alpha\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 \quad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \quad & \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\mathbf{u}$ is the fluid velocity, p is the pressure, $\mathbf{f}$ is a given source term, $\mu > 0$ is the kinematic viscosity coefficient, and $\alpha > 0$ is a real parameter that may arise from the time discretization of the evolution term $\frac{\partial \mathbf{u}}{\partial t}$ in the unsteady Stokes equations (cf. [8]).

To derive the weak formulation of the generalized Stokes problem (1.1), we introduce some useful preliminaries and notations. We recall the classical definitions for the Sobolev space $H^m(\Omega) = W^{m,2}(\Omega)$ with the usual norm $\|\cdot\|_{m,\Omega}$ and seminorm $|\cdot|_{m,\Omega}$. In particular, $H^0(\Omega) = L^2(\Omega)$, the space of square integrable functions in Ω with inner product $(\cdot,\cdot)_{0,\Omega}$ and norm $\|\cdot\|_{0,\Omega}$. Let $\mathbf{H}^m(\Omega)$ be the space of vector-valued functions $\mathbf{v} = (v_1, \dots, v_d)$ with components v_i in $H^m(\Omega)$. The norm and seminorm on $\mathbf{H}^m(\Omega)$ are given by

$$\|\mathbf{v}\|_{m,\Omega} = \left(\sum_{i=1}^{d} \big\|v^{(i)}\big\|_{m,\Omega}^2\right)^{1/2} \quad \text{and} \quad |\mathbf{v}|_{m,\Omega} = \left(\sum_{i=1}^{d} \big|v^{(i)}\big|_{m,\Omega}^2\right)^{1/2}.$$

We define the following function spaces for velocity and pressure

$$\mathbf{V} := \mathbf{H}_0^1(\Omega) = \big\{\mathbf{v} \in \mathbf{H}^1(\Omega) \colon \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega\big\},$$

and

$$Q := L_0^2(\Omega) = \Big\{q \in L^2(\Omega) \colon \int_\Omega q\,dx = 0\Big\}.$$

Then, the weak formulation of the generalized Stokes problem (1.1) is given as follows:

$$\begin{aligned} &\text{Find } (\mathbf{u}, p) \in \mathbf{V} \times Q \text{ such that} \\ &\alpha(\mathbf{u},\mathbf{v})_{0,\Omega} + \mu(\nabla\mathbf{u},\nabla\mathbf{v})_{0,\Omega} - (p, \operatorname{div}\mathbf{v})_{0,\Omega} = (\mathbf{f},\mathbf{v})_{0,\Omega} \qquad \forall\, \mathbf{v} \in \mathbf{V}, \\ &\qquad\qquad -(q, \operatorname{div}\mathbf{u})_{0,\Omega} = 0 \qquad \forall\, q \in Q. \end{aligned} \tag{1.2}$$

Note that the second equation in (1.2) has been multiplied by minus one to ensure a symmetric formulation. Furthermore, we can take the right-hand side $\mathbf{f}$ in $\mathbf{L}^2(\Omega)$ so that (1.2) is well-defined.

§ 2. Finite volume formulation

2.1. Spatial discretization and inequalities

We consider an admissible discretization for the FVM given in [7, 9]. In order to construct such a discretization, let us assume that $\mathcal{T}_h$ is a family of regular volumes of Ω. Hence, $\mathcal{T}_h$ is a finite family of disjoint non-empty convex subdomains K of Ω (control volumes), such that K is either a rectangle or a triangle with acute internal angles in the 2D case, and K is a rectangular parallelepiped or a tetrahedron with acute internal angles in the 3D case. Let us denote by $\partial K = \overline{K}\backslash K$ and $|K|$, respectively, the boundary and measure of any $K \in \mathcal{T}_h$.

On the other hand, let us denote by $\mathcal{E}_{int}$ and $\mathcal{E}_{ext}$ the finite sets of volume boundaries σ (edges or faces), with measures $|\sigma|$, which are respectively internal to the domain Ω and on $\partial\Omega$. Furthermore, we set $\mathcal{E} = \mathcal{E}_{int} \cup \mathcal{E}_{ext}$ and assume that, for all $K \in \mathcal{T}_h$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \cup_{\sigma\in\mathcal{E}_K}\overline{\sigma}$.

Finally, $\mathcal{P} = (\mathbf{x}_K)_{K\in\mathcal{T}_h}$ is the family of points of Ω, which are intersections of the perpendicular bisectors of each edge in the 2D or intersections of the lines issued from the center of the face and orthogonal to the face in the 3D case.

The obtained admissible finite volume mesh of Ω is denoted by $\mathcal{D} = (\mathcal{T}_h, \mathcal{E}, \mathcal{P})$.

Note that any internal edge σ separating two control volumes K and L is denoted by $\sigma = K \mid L$, and satisfies the condition:

$$\mathbf{x}_\sigma = [\mathbf{x}_K, \mathbf{x}_L] \cap K \mid L.$$

We denote by d_{KL} the distance between $\mathbf{x}_K$ and $\mathbf{x}_L$, and by $d_{K\sigma}$ the distance between $\mathbf{x}_K$ and $\mathbf{x}_\sigma$.

Furthermore, let h_K be the diameter of the control volume K, and let h denote the mesh size, defined as the maximum of h_K over all $K \in \mathcal{T}_h$, that is,

$$h = \sup_{K\in\mathcal{T}_h} h_K.$$

We shall measure the regularity of the mesh $\mathcal{D}$ through the function $\mathrm{regul}(\mathcal{D})$, defined by

$$\mathrm{regul}(\mathcal{D}) = \inf\left(\left\{\frac{d_{K\sigma}}{h_K}; K \in \mathcal{T}_h, \sigma \in \mathcal{E}_K\right\} \cup \left\{\frac{h_K}{h}; K \in \mathcal{T}_h\right\} \cup \left\{\frac{1}{\mathrm{card}\,(\mathcal{E}_K)}; K \in \mathcal{T}_h\right\}\right),$$

where $\mathrm{card}\,(\mathcal{E}_K)$ is the number of edges (i. e., the cardinality of the set $\mathcal{E}_K$). To ensure sufficient regularity, we assume the existence of a constant $\theta > 0$ such that $\mathrm{regul}\,(\mathcal{D}) > \theta$.

Now, given an admissible mesh $\mathcal{D}$, let us introduce the discrete space $V_h \subset L^2(\Omega)$ of piecewise constant functions on each control volume $K \in \mathcal{T}_h$. Likewise, we also make use of the discrete space

$$Q_h = V_h \cap L^2_0(\Omega).$$

For all $v_h \in V_h$, we denote by $v_{h,K}$ the value (constant) of v_h in any $K \in \mathcal{T}_h$.

For $(v_h, w_h) \in [V_h]^2$, we define the following inner product called "discrete H^1_0 inner product",

$$[v_h, w_h]_\mathcal{D} = \sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} \frac{|\sigma|}{d_{KL}}(v_{h,L} - v_{h,K})(w_{h,L} - w_{h,K}) + \sum_{\substack{\sigma\in\mathcal{E}_{ext}\\(\sigma\in\mathcal{E}_K)}} \frac{|\sigma|}{d_{K\sigma}} v_{h,K} w_{h,K}.$$

A norm in V_h, called "discrete H^1_0 norm", is obtained as follows

$$\|v_h\|_\mathcal{D} = [v_h, v_h]^{1/2}_\mathcal{D}.$$

We also define the following bilinear form

$$\langle v_h, w_h\rangle_\mathcal{D} = \sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} \frac{|\sigma|}{d_{KL}}(v_{h,L} - v_{h,K})(w_{h,L} - w_{h,K}),$$

with the seminorm

$$|v_h|_{\mathcal{D}} = \langle v_h, v_h \rangle_{\mathcal{D}}^{1/2}.$$

These definitions extend naturally to vector valued functions as follows. For $\mathbf{v}_h = (v_h^{(i)})_{i=1,\dots,d} \in \mathbf{V}_h := [V_h]^d$ and $\mathbf{w}_h = (w_h^{(i)})_{i=1,\dots,d} \in \mathbf{V}_h$, we set

$$[\mathbf{v}_h, \mathbf{w}_h]_{\mathcal{D}} = \sum_{i=1}^{d} [v_h^{(i)}, w_h^{(i)}]_{\mathcal{D}}, \qquad \|\mathbf{v}_h\|_{\mathcal{D}} = \left(\sum_{i=1}^{d} [v_h^{(i)}, v_h^{(i)}]_{\mathcal{D}} \right)^{1/2}.$$

Proposition 2.1. *The following discrete Poincaré inequalities hold*

$$\begin{aligned} \|v_h\|_{0,\Omega} &\le \operatorname{diam}(\Omega) \|v_h\|_{\mathcal{D}} \quad \forall v \in V_h, \\ \|v_h\|_{0,\Omega} &\le C(\Omega) |v_h|_{\mathcal{D}} \qquad \forall v \in Q_h, \end{aligned} \tag{2.1}$$

where $C(\Omega)$ depends only on Ω (cf. [9]*).*

As in [10], let us define the interpolation operator $\pi_h \colon L^2(\Omega) \to V_h$ by setting $(\pi_{\mathcal{D}} u)_K = \emptyset_K(\mathbf{x}_K)$ for all $K \in \mathcal{T}_h$ and $u \in L^2(\Omega)$, where $\emptyset_K$ being the orthogonal projection of $L^2(\Omega)$ on $\mathbb{P}_1$.

It has a natural extension to vector-valued functions. We will keep the same notation.

The operator π_h satisfies the following proposition.

Proposition 2.2. *Let $\mathbf{u} \in \mathbf{V}$. Assume that* $\operatorname{regul}(\mathcal{D}) > \theta$. *Then*

$$\|\pi_h \mathbf{u}\|_{\mathcal{D}} \le C |\mathbf{u}|_{1,\Omega}$$

where C only depends on Ω and θ.

2.2. The finite volume scheme

Let $\mathcal{D}$, as defined above, be a discretization of Ω. In order to construct a finite volume scheme, we begin by defining a discrete Laplace operator $\Delta_h \mathbf{u}_h \in V_h$, which is expressed for any $K \in \mathcal{T}_h$ as follows

$$(\Delta_h \mathbf{u}_h)_K = \frac{1}{|K|} \left(\sum_{\sigma = K|L} \frac{|\sigma|}{d_{LK}} (\mathbf{u}_{h,L} - \mathbf{u}_{h,K}) + \sum_{\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K} \frac{|\sigma|}{d_{K\sigma}} (-\mathbf{u}_{h,K}) \right) \qquad \forall \mathbf{u}_h \in \mathbf{V}_h.$$

Next, we consider a discrete divergence operator div_h, mapping $\mathbf{V}_h$ to V_h, which is defined by

$$(\operatorname{div}_h \mathbf{u}_h)_K = \frac{1}{|K|} \sum_{\sigma = K|L} |\sigma| \frac{\mathbf{u}_{h,L} + \mathbf{u}_{h,K}}{2} \cdot \mathbf{n}_\sigma \qquad \forall K \in \mathcal{T}_h,$$

where $\mathbf{n}_\sigma$ is an orthogonal unit vector to σ.

The adjoint of this discrete divergence operator, with respect to the discrete L^2 inner product, defines a discrete gradient ∇_h. Thus, for any $p \in V_h$, we define its discrete gradient $\nabla_h p_h \in \mathbf{V}_h$ by

$$(\nabla_h p_h)_K = \frac{1}{|K|} \sum_{\sigma = K|L} |\sigma| \frac{p_{h,L} - p_{h,K}}{2} \mathbf{n}_\sigma \qquad \forall K \in \mathcal{T}_h.$$

Since for all $K \in \mathcal{T}_h$, $\sum_{\sigma \in \mathcal{E}_K} |\sigma| \mathbf{n}_\sigma = \mathbf{0}$, this discrete gradient can equivalently be expressed as

$$(\nabla_h p_h)_K = \frac{1}{|K|} \left(\sum_{\sigma = K|L} |\sigma| \frac{p_{h,L} + p_{h,K}}{2} \mathbf{n}_\sigma + \sum_{\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K} |\sigma| p_{h,K} \mathbf{n}_\sigma \right).$$

Based on the above notations, the finite volume approximation of (1.2) is defined as:

$$
\begin{aligned}
&\text{Find } (\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \text{ such that} \\
&\alpha(\mathbf{u}_h, \mathbf{v}_h)_{0,\Omega} + \mu[\mathbf{u}_h, \mathbf{v}_h]_{\mathcal{D}} - (p_h, \operatorname{div}_h \mathbf{v}_h)_{0,\Omega} = (\mathbf{f}, \mathbf{v}_h)_{0,\Omega} \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\
&\qquad -(q_h, \operatorname{div}_h \mathbf{u}_h)_{0,\Omega} = 0 \qquad \forall q_h \in Q_h.
\end{aligned}
\tag{2.2}
$$

For a stable and accurate approximation of (2.2), the discrete spaces $\mathbf{V}_h$ and Q_h must satisfy the discrete *inf-sup* condition. That is, there exists a constant $\widetilde{\beta} > 0$ independent of h such that for the pair $\mathbf{V}_h$ and Q_h, we have

$$
\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(p_h, \operatorname{div}_h \mathbf{v}_h)_{0,\Omega}}{\|\mathbf{v}_h\|_{\mathcal{D}}} \geq \widetilde{\beta} \|p_h\|_{0,\Omega}, \qquad \forall p_h \in Q_h. \tag{2.3}
$$

As noted earlier, the finite volume space pair $\mathbf{V}_h$ and Q_h does not satisfy the discrete *inf-sup* condition (2.3).

§ 3. Weak *inf-sup* condition

In this section, we show that the unstable velocity–pressure pair $\mathbf{V}_h$ and Q_h satisfies a weaker form of the *inf-sup* condition, which can be employed in the stabilization procedure. This condition introduces the so-called global jump stabilization term [11]

$$
J(p_h, q_h) = \beta \sum_{K \in \mathcal{T}_h} \int_{\partial K \setminus \partial \Omega} h_{\partial K} [p_h][q_h] \, ds, \tag{3.1}
$$

where $\beta > 0$ is the global stabilization parameter and $[\cdot]$ is the jump operator across interior edges or faces.

Lemma 3.1. *Let $\mathbf{V}_h$ and Q_h be the spaces defined above. Then, there exist positive constants δ_1 and δ_2 independent of h, such that*

$$
\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(p_h, \operatorname{div}_h \mathbf{v}_h)_{0,\Omega}}{\|\mathbf{v}_h\|_{\mathcal{D}}} \geq \delta_1 \|p_h\|_{0,\Omega} - \delta_2 J(p_h, p_h)^{1/2}.
$$

P r o o f. Let $p_h \in Q_h$ be given. We apply a classical property of the divergence operator [1]. Thus, there exists $\mathbf{v} \in \mathbf{V}$ such that

$$
\operatorname{div} \mathbf{v}(\mathbf{x}) = p_h(\mathbf{x}) \ \text{ and } \ \|\mathbf{v}\|_{1,\Omega} \leq C_1 \|p_h\|_{0,\Omega}. \tag{3.2}
$$

We set $\mathbf{v}_h = \pi_h \mathbf{v} \in \mathbf{V}_h$ with

$$
v_{h,K}^{(j)} = \frac{1}{|K|} \int_K v^{(j)}(x) \, dx, \qquad \forall K \in \mathcal{T}_h, \quad j = 1, \ldots, d, \tag{3.3}
$$

and

$$
v_{h,\sigma}^{(j)} = \frac{1}{|\sigma|} \int_\sigma v^{(j)}(x) \, d\gamma(x), \qquad \forall \sigma \in \mathcal{E}, \quad j = 1, \ldots, d. \tag{3.4}
$$

Then, $v_{h,\sigma}^{(j)} = 0$ for all $\sigma \in \mathcal{E}_{ext}$ and $j = 1, \ldots, d$.

Using classical arguments [9, pp. 777–779], we can show the existence of a constant $C_2 > 0$ such that

$$
\forall K \in \mathcal{T}_h, \quad \forall \sigma \in \mathcal{E}_K, \qquad |\mathbf{v}_{h,K} - \mathbf{v}_{h,\sigma}|^2 \leq C_2 \frac{h_K}{|\sigma|} \int_K |\nabla \mathbf{v}(\mathbf{x})|^2 \, d\mathbf{x}. \tag{3.5}
$$

In addition, by the continuity of the interpolation operator π_h (Proposition 2.2), there exists another constant $C_3 > 0$ such that

$$\|\mathbf{v}_h\|_{\mathcal{D}} \leq C_3 \|p_h\|_{0,\Omega}. \tag{3.6}$$

Next,

$$(p_h, \operatorname{div}_h \mathbf{v}_h)_{0,\Omega} = \sum_{K \in \mathcal{T}_h} p_{h,K} \sum_{\sigma = K|L} |\sigma| \frac{(\mathbf{v}_{h,L} + \mathbf{v}_{h,K})}{2} \cdot \mathbf{n}_\sigma = A + B,$$

where

$$\begin{aligned} A &= \sum_{K \in \mathcal{T}_h} p_{h,K} \sum_{\sigma = K|L} |\sigma| \mathbf{v}_{h,\sigma} \cdot \mathbf{n}_\sigma = \sum_{K \in \mathcal{T}_h} p_{h,K} \int_{\partial K} \mathbf{v}(\mathbf{x}) \cdot \mathbf{n}_{\partial K} \, d\gamma(\mathbf{x}) \\ &= \sum_{K \in \mathcal{T}_h} p_{h,K} \int_K \operatorname{div} \mathbf{v}(\mathbf{x}) \, d\mathbf{x} = \|p_h\|_{0,\Omega}^2, \end{aligned}$$

and

$$\begin{aligned} B &= \sum_{K \in \mathcal{T}_h} p_{h,K} \sum_{\sigma = K|L} |\sigma| \left(\frac{\mathbf{v}_{h,L} + \mathbf{v}_{h,K}}{2} - \mathbf{v}_{h,\sigma} \right) \cdot \mathbf{n}_\sigma \\ &= \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} |\sigma| (p_{h,K} - p_{h,L}) \left(\frac{\mathbf{v}_{h,L} + \mathbf{v}_{h,K}}{2} - \mathbf{v}_{h,\sigma} \right) \cdot \mathbf{n}_\sigma. \end{aligned}$$

Applying the Cauchy–Schwarz inequality gives

$$|B|^2 \leq C_4 J(p_h, p_h) \left(\sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \frac{|\sigma|}{h_{\partial K}} \left(\frac{\mathbf{v}_{h,L} + \mathbf{v}_{h,K}}{2} - \mathbf{v}_{h,\sigma} \right)^2 \right).$$

Applying (3.5) and the obvious inequality

$$\left(\frac{\mathbf{v}_{h,L} + \mathbf{v}_{h,K}}{2} - \mathbf{v}_{h,\sigma} \right)^2 \leq \frac{1}{2} \big((\mathbf{v}_{h,K} - \mathbf{v}_{h,\sigma})^2 + (\mathbf{v}_{h,L} - \mathbf{v}_{h,\sigma})^2 \big)$$

yields

$$|B|^2 \leq C_5 J(p_h, p_h) \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \frac{|\sigma|}{h_{\partial K}} \left(\frac{h_K}{|\sigma|} \int_{K \cup L} |\nabla \mathbf{v}(\mathbf{x})|^2 \, d\mathbf{x} \right) \leq C_6 J(p_h, p_h) \|\mathbf{v}\|_{1,\Omega}^2.$$

Using (3.2), we get

$$|B| \leq C_7 J(p_h, p_h)^{1/2} \|p_h\|_{0,\Omega}.$$

Collecting all estimated terms, we obtain

$$\frac{(p_h, \operatorname{div}_h \mathbf{v}_h)_{0,\Omega}}{\|p_h\|_{0,\Omega}} \geq \|p_h\|_{0,\Omega} - C_7 J(p_h, p_h)^{1/2} \tag{3.7}$$

which, combined with relation (3.6), concludes the proof. ☐

§ 4. Stability and convergence analysis

Based on Lemma 3.1, the discrete problem (2.2) can be stabilized by introducing the global jump stabilization term (3.1) into the discrete incompressibility constraint. This leads to the following stabilized scheme:

$$\begin{aligned} &\text{Find } (\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \text{ such that} \\ &\alpha(\mathbf{u}_h, \mathbf{v}_h)_{0,\Omega} + \mu[\mathbf{u}_h, \mathbf{v}_h]_{\mathcal{D}} - (p_h, \operatorname{div}_h \mathbf{v}_h)_{0,\Omega} = (\mathbf{f}, \mathbf{v}_h)_{0,\Omega} \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ &-(q_h, \operatorname{div}_h \mathbf{u}_h)_{0,\Omega} - J(p_h, q_h) = 0 \qquad \forall q_h \in Q_h. \end{aligned} \tag{4.1}$$

The proposed scheme (4.1) may be recast in the following "flux form" obtained by rewriting it for each control volume $K \in \mathcal{T}_h$:

$$\begin{aligned} &\text{Find } (\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \text{ such that} \\ &\alpha \int_K \mathbf{u}_{h,K} - \mu \int_K (\Delta_h \mathbf{u}_h)_K + \int_K (\nabla_h p_h)_K = \int_K \mathbf{f}, \\ &-\int_K (\operatorname{div}_h \mathbf{u}_h)_K - \beta \sum_{\sigma = K|L} \int_\sigma h_\sigma [p_h] = 0. \end{aligned} \tag{4.2}$$

We also need to introduce the following bilinear form on which we will base our FVM

$$B\big[(\mathbf{u}_h, p_h), (\mathbf{v}_h, p_h)\big] = \alpha(\mathbf{u}_h, \mathbf{v}_h)_{0,\Omega} + \mu[\mathbf{u}_h, \mathbf{v}_h]_{\mathcal{D}} - (p_h, \operatorname{div}_h \mathbf{v}_h)_{0,\Omega} - (q_h, \operatorname{div}_h \mathbf{u}_h)_{0,\Omega} - J(p_h, q_h) \tag{4.3}$$

in such a way that our proposed FV formulation reads:

$$\begin{aligned} &\text{Find } (\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \text{ such that} \\ &B\big[(\mathbf{u}_h, p_h), (\mathbf{v}_h, p_h)\big] = (\mathbf{f}, \mathbf{v}_h)_{0,\Omega} \qquad \forall (\mathbf{v}_h, p_h) \in \mathbf{V}_h \times Q_h. \end{aligned} \tag{4.4}$$

We now state the following main *inf-sup* result, which ensures the well-posedness of our FV scheme (4.4).

Theorem 4.1. *There exists a positive constant γ independent of h such that*

$$\sup_{(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h} \frac{B\big[(\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)\big]}{\|\mathbf{v}_h\|_{\mathcal{D}} + \|q_h\|_{0,\Omega}} \geq \gamma\big(\|\mathbf{u}_h\|_{\mathcal{D}} + \|p_h\|_{0,\Omega}\big) \qquad \forall (\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h. \tag{4.5}$$

P r o o f. First, setting $(\mathbf{v}_h, q_h) = (\mathbf{u}_h, -p_h)$ in (4.3) yields

$$B\big[(\mathbf{u}_h, p_h), (\mathbf{u}_h, -p_h)\big] \geq \mu \|\mathbf{u}_h\|_{\mathcal{D}}^2 + J(p_h, p_h). \tag{4.6}$$

Second, for a given arbitrary but fixed $p_h \in Q_h$, let $\mathbf{w}$ and $\mathbf{w}_h$ be the functions that satisfy (3.2)–(3.4). Taking $(\mathbf{v}_h, q_h) = (-\mathbf{w}_h, 0)$ in (4.3) and using the Cauchy–Schwarz inequality, (3.6), and (3.7) yields

$$B\big[(\mathbf{u}_h, p_h), (-\mathbf{w}_h, 0)\big] \geq -C_1 \|\mathbf{u}_h\|_{\mathcal{D}} \|p_h\|_{0,\Omega} + \|p_h\|_{0,\Omega}\big(\|p_h\|_{0,\Omega} - C_2 J(p_h, p_h)^{1/2}\big).$$

By using Young's inequality to the right-hand side and by summing the resulting inequalities, we obtain

$$B\big[(\mathbf{u}_h, p_h), (-\mathbf{w}_h, 0)\big] \geq \frac{-C_1 l_1}{2} \|\mathbf{u}_h\|_{\mathcal{D}}^2 + \left(1 - \frac{C_1}{2l_1} - \frac{C_2 l_2}{2}\right) \|p_h\|_{0,\Omega}^2 - \frac{C_2}{2l_2} J(p_h, p_h), \tag{4.7}$$

where l_1 and l_2 are any positive constants. Choose $l_1 = 2C_1$ and $l_2 = \frac{1}{2C_2}$ in (4.7), then we get

$$B\big[(\mathbf{u}_h, p_h), (-\mathbf{w}_h, 0)\big] \geq -C_1^2\|\mathbf{u}_h\|_{\mathcal{D}}^2 + \frac{1}{2}\|p_h\|_{0,\Omega}^2 - C_2^2 J(p_h, p_h). \tag{4.8}$$

Finally, by setting $(\mathbf{v}_h, q_h) = (\mathbf{u}_h - \lambda\mathbf{w}_h, -p_h)$ in (4.3), where λ is a positive constant, and using (4.6) and (4.8), yields

$$B\big[(\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)\big] \geq (\mu - \lambda C_1^2)\|\mathbf{u}_h\|_{\mathcal{D}}^2 + \frac{\lambda}{2}\|p_h\|_{0,\Omega}^2 + (1 - \lambda C_2^2) J(p_h, p_h).$$

By taking $\lambda = \min\Big\{\frac{\mu}{2C_1^2}, \frac{1}{2C_2^2}\Big\}$, we get

$$B\big[(\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)\big] \geq \frac{\mu}{2}\|\mathbf{u}_h\|_{\mathcal{D}}^2 + \frac{\lambda}{2}\|p_h\|_{0,\Omega}^2 + \frac{1}{2} J(p_h, p_h),$$

which implies that

$$B\big[(\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)\big] \geq C_3\big(\|\mathbf{u}_h\|_{\mathcal{D}} + \|p_h\|_{0,\Omega}\big)^2, \tag{4.9}$$

where $C_3 = \frac{1}{4}\min\{\mu, \lambda\}$.

On the other hand, it is easy to see that

$$\begin{aligned}\|\mathbf{v}_h\|_{\mathcal{D}} + \|p_h\|_{0,\Omega} &= \|\mathbf{u}_h - \lambda\mathbf{w}_h\|_{\mathcal{D}} + \|p_h\|_{0,\Omega}\\ &\leq \|\mathbf{u}_h\|_{\mathcal{D}} + \lambda\|\mathbf{w}_h\|_{\mathcal{D}} + \|p_h\|_{0,\Omega}\\ &\leq \|\mathbf{u}_h\|_{\mathcal{D}} + C_4\|p_h\|_{\mathcal{D}} + \|p_h\|_{0,\Omega}\\ &\leq C_5\big(\|\mathbf{u}_h\|_{\mathcal{D}} + \|p_h\|_{0,\Omega}\big).\end{aligned} \tag{4.10}$$

Finally, combining (4.9) and (4.10) establishes the desired inequality (4.5) with $\gamma = \frac{C_3}{C_5}$. □

Remark 4.1. From (4.5), we can obtain an a priori estimate for $(\mathbf{u}_h, p_h)$ by using the Cauchy–Schwarz inequality and the discrete Poincaré inequalities (2.1) as follows

$$\|\mathbf{u}_h\|_{\mathcal{D}} + \|p_h\|_{0,\Omega} \leq \frac{1}{\gamma} \sup_{(\mathbf{v}_h, q_h)\in\mathbf{V}_h\times Q_h} \frac{(\mathbf{f}, \mathbf{v}_h)_{0,\Omega}}{\|\mathbf{v}_h\|_{\mathcal{D}} + \|q_h\|_{0,\Omega}} \leq C\|\mathbf{f}\|_{0,\Omega}.$$

Next, we state an important result that is required below.

Proposition 4.1. *Let $(\mathbf{u}, p) \in (\mathbf{H}^2(\Omega) \cap \mathbf{V}) \times (H^1(\Omega) \cap Q)$ and $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ be the respective solutions of (1.1) and (4.4). Then, there exists a positive constant C, independent of h, such that*

$$\|\pi_h\mathbf{u} - \mathbf{u}_h\|_{\mathcal{D}} \leq Ch(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}), \tag{4.11}$$

$$J(p_h, p_h)^{\frac{1}{2}} \leq Ch(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}), \tag{4.12}$$

$$\|\pi_h p - p_h\|_{0,\Omega} \leq Ch(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}). \tag{4.13}$$

P r o o f. First, let $(\widehat{\mathbf{u}}_h, \hat{p}_h) \in \mathbf{V}_h \times Q_h$ be defined by $\widehat{\mathbf{u}}_h = \pi_h\mathbf{u}$ and $\hat{p}_h = \pi_h p$. Integrating the first equation of (1.1) over $K \in \mathcal{T}_h$ gives

$$\alpha\int_K \mathbf{u} - \mu\sum_{\sigma\in\mathcal{E}_K}\int_\sigma \nabla\mathbf{u}\cdot\mathbf{n}_\sigma + \sum_{\sigma\in\mathcal{E}_K}\int_\sigma p\mathbf{n}_\sigma = \int_K \mathbf{f}.$$

Now, let us introduce for $K \in \mathcal{T}_h$ the following consistency residuals

$$R_u = \widehat{\mathbf{u}}_{h,K} - \frac{1}{|K|}\int_K \mathbf{u};$$
$$R_\Delta = \begin{cases} \dfrac{\widehat{\mathbf{u}}_{h,L} - \widehat{\mathbf{u}}_{h,K}}{d_{KL}} - \dfrac{1}{|\sigma|}\displaystyle\int_\sigma \nabla\mathbf{u}\cdot\mathbf{n}_\sigma & \text{if } \sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K \quad (\sigma = K \mid L),\\ \dfrac{-\widehat{\mathbf{u}}_{h,K}}{d_{K\sigma}} - \dfrac{1}{|\sigma|}\displaystyle\int_\sigma \nabla\mathbf{u}\cdot\mathbf{n}_\sigma & \text{if } \mathcal{E}_{ext} \cap \mathcal{E}_K; \end{cases}$$
$$R_\nabla = \begin{cases} \dfrac{\hat{p}_{h,L} + \hat{p}_{h,K}}{2} - \dfrac{1}{|\sigma|}\displaystyle\int_\sigma p & \text{if } \sigma \in \mathcal{E}_{int} \cap \mathcal{E}_K \quad (\sigma = K \mid L),\\ \hat{p}_{h,K} - \dfrac{1}{|\sigma|}\displaystyle\int_\sigma p & \text{if } \mathcal{E}_{ext} \cap \mathcal{E}_K. \end{cases}$$

Using these notations and the relation $\sum_{\sigma\in\mathcal{E}_K} |\sigma|\mathbf{n}_\sigma = \mathbf{0}$, we get

$$\alpha|K|\widehat{\mathbf{u}}_{h,K} - \mu\Bigg(\sum_{\sigma=K|L}|\sigma|\frac{\widehat{\mathbf{u}}_{h,L} - \widehat{\mathbf{u}}_{h,K}}{d_{KL}} + \sum_{\sigma\in\mathcal{E}_{ext}\cap\mathcal{E}_K}|\sigma|\frac{-\widehat{\mathbf{u}}_{h,K}}{d_{K\sigma}}\Bigg) + \sum_{\sigma=K|L}|\sigma|\frac{\hat{p}_{h,L} - \hat{p}_{h,K}}{2}\mathbf{n}_\sigma = \\ = \int_K \mathbf{f} + |K|R_K,$$

with

$$R_K = \alpha R_u - \mu\frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K}|\sigma|R_\Delta + \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K}|\sigma|R_\nabla\mathbf{n}_\sigma.$$

Set $\mathbf{e}_h = \widehat{\mathbf{u}}_h - \mathbf{u}_h$ and $\epsilon_h = \hat{p}_h - p_h$. Subtracting the first equation of (4.2) from the above equation, we then get

$$\alpha|K|\mathbf{e}_{h,K} - \mu\sum_{\sigma=K|L}|\sigma|\frac{\mathbf{e}_{h,L} - \mathbf{e}_{h,K}}{d_{KL}} - \mu\sum_{\sigma\in\mathcal{E}_{ext}\cap\mathcal{E}_K}|\sigma|\frac{-\mathbf{e}_{h,K}}{d_{K\sigma}} + \sum_{\sigma=K|L}|\sigma|\frac{\epsilon_{h,L} - \epsilon_{h,K}}{2}\mathbf{n}_\sigma = |K|R_K.$$

For all $\mathbf{v}_h \in \mathbf{V}_h$, we get

$$\alpha(\mathbf{e}_h, \mathbf{v}_h)_{0,\Omega} + \mu[\mathbf{e}_h, \mathbf{v}_h]_{\mathcal{D}} - (\epsilon_h, \operatorname{div}_h \mathbf{v}_h)_{0,\Omega} = (R, \mathbf{v}_h)_{0,\Omega}. \tag{4.14}$$

By setting $\mathbf{v}_h = \mathbf{e}_h$ in the last relation, we get

$$\alpha\|\mathbf{e}_h\|^2_{0,\Omega} + \mu\|\mathbf{e}_h\|^2_{\mathcal{D}} - (\epsilon_h, \operatorname{div}_h \mathbf{e}_h)_{0,\Omega} = (R, \mathbf{e}_h)_{0,\Omega}. \tag{4.15}$$

Now, let us integrate the second equation of (1.1) on $K \in \mathcal{T}_h$. This gives

$$\sum_{\sigma\in\mathcal{E}_K}\int_\sigma \mathbf{u}\cdot\mathbf{n}_\sigma = 0.$$

Since $\mathbf{u}$ vanishes on the boundary of Ω, we obtain

$$\sum_{\sigma=K|L}\frac{|\sigma|}{2}(\widehat{\mathbf{u}}_{h,L} + \widehat{\mathbf{u}}_{h,K})\cdot\mathbf{n}_\sigma = \sum_{\sigma=K|L}|\sigma|R_{\mathrm{div}} \qquad \forall K \in \mathcal{T}_h$$

with

$$R_{\mathrm{div}} = \left(\frac{1}{2}(\widehat{\mathbf{u}}_{h,L} + \widehat{\mathbf{u}}_{h,K}) - \frac{1}{|\sigma|}\int_\sigma \mathbf{u}\right)\cdot\mathbf{n}_\sigma.$$

Then, subtracting the second equation of (4.2) from the above equation gives

$$\sum_{\sigma=K|L} \frac{|\sigma|}{2}(\mathbf{e}_{h,L}+\mathbf{e}_{h,K})\cdot\mathbf{n}_\sigma = \sum_{\sigma=K|L} |\sigma| R_{\mathrm{div}} + \delta \sum_{\sigma=K|L} |\sigma| h_\sigma [p_h].$$

For all $q_h \in Q_h$, this yields

$$(q_h, \mathrm{div}_h\, \mathbf{e}_h)_{0,\Omega} = \sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} |\sigma| R_{\mathrm{div}}(q_{h,K}-q_{h,L}) + J(p_h, q_h),$$

and setting $q_h = \epsilon_h$ in this relation gives

$$(\epsilon_h, \mathrm{div}_h\, \mathbf{e}_h)_{0,\Omega} = \sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} |\sigma| R_{\mathrm{div}}(\epsilon_{h,K}-\epsilon_{h,L}) + J(p_h, \epsilon_h). \tag{4.16}$$

Gathering (4.15) and (4.16), we get

$$\alpha\|\mathbf{e}_h\|_{0,\Omega}^2 + \mu\|\mathbf{e}_h\|_{\mathcal{D}}^2 + J(p_h,p_h) = (R,\mathbf{e}_h)_{0,\Omega} + \sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} |\sigma| R_{\mathrm{div}}(\epsilon_{h,K}-\epsilon_{h,L}) + J(p_h,\hat{p}_h). \tag{4.17}$$

Next, let us study the terms at the right-hand side of (4.17). From [7], we derive the following result

$$(R,\mathbf{e}_h)_{0,\Omega} \le \alpha\|\mathbf{e}_h\|_{0,\Omega}^2 + C_1\frac{h^2}{\varepsilon}\|\mathbf{u}\|_{2,\Omega}^2 + 2\varepsilon\|\mathbf{e}_h\|_{\mathcal{D}}^2 + C_2\frac{h^2}{\varepsilon}\|p\|_{1,\Omega}^2, \tag{4.18}$$

$$\sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} |\sigma| R_{\mathrm{div}}(\epsilon_{h,K}-\epsilon_{h,L}) \le C_3 h^2\left(\frac{1}{\varepsilon}\|\mathbf{u}\|_{2,\Omega}^2 + \varepsilon\|p\|_{1,\Omega}^2\right) + C_4\frac{h^2}{\varepsilon}\|\mathbf{u}\|_{2,\Omega}^2 + \varepsilon J(p_h,p_h), \tag{4.19}$$

$$J(p_h,\hat{p}) \le \varepsilon J(p_h,p_h) + C_5\frac{h^2}{\varepsilon}\|p\|_{1,\Omega}^2, \tag{4.20}$$

where ε is any positive constant.

Gathering (4.18), (4.19) and (4.20) yields the control error inequality

$$(\mu-2\varepsilon)\|\mathbf{e}_h\|_{\mathcal{D}}^2 + (1-2\varepsilon)J(p_h,p_h) \le C_6 h^2 \max\left\{\frac{1}{\varepsilon},\varepsilon\right\}(\|\mathbf{u}\|_{2,\Omega}^2 + \|p\|_{1,\Omega}^2).$$

It is clear that choosing ε sufficiently small, e. g. $\varepsilon < \frac{1}{2}\min\{\mu,1\}$ the latter implies (4.11) and (4.12).

To conclude we need control $\|\epsilon_h\|_{0,\Omega}^2$. For that, we again follow the methodology already used in the proof of Lemma 3.1. Since $\int_\Omega \epsilon_h(\mathbf{x}) = 0$ from $\int_\Omega \hat{p}(\mathbf{x})\,d\mathbf{x} = 0$, let $\mathbf{w}\in\mathbf{V}$ be given such that

$$\mathrm{div}\,\mathbf{w}(\mathbf{x}) = \epsilon_h(\mathbf{x}) \quad \text{and} \quad \|\mathbf{w}\|_{1,\Omega} \le C_7\|\epsilon_h\|_{0,\Omega}.$$

We again define $\mathbf{w}_h$ as in (3.3) and (3.4). In addition, we have

$$\|\mathbf{w}_h\|_{\mathcal{D}} \le C_8\|\epsilon_h\|_{0,\Omega}. \tag{4.21}$$

Like in (3.7), this gives

$$\|\epsilon_h\|_{0,\Omega}^2 \le (\epsilon_h, \mathrm{div}_h\,\mathbf{w}_h)_{0,\Omega} + J(\epsilon_h,\epsilon_h)^{1/2} C_9\|\epsilon_h\|_{0,\Omega}. \tag{4.22}$$

We now use $\mathbf{w}_h$ as a test function in (4.14) to have

$$(\epsilon_h, \operatorname{div}_h \mathbf{w}_h)_{0,\Omega} = \mu[\mathbf{e}_h, \mathbf{w}_h]_{\mathcal{D}} - (R, \mathbf{w}_h)_{0,\Omega}.$$

Taking into account (4.22) and using Young's inequality, (4.18), (4.20), and (4.21), we get

$$\|\epsilon_h\|^2_{0,\Omega} \le \frac{C_{10}}{\varepsilon}\|\mathbf{e}_h\|^2_{\mathcal{D}} + C_{11}\varepsilon\|\epsilon_h\|^2_{0,\Omega} + \frac{C_{12}h^2}{\varepsilon}\left(\|\mathbf{u}\|^2_{2,\Omega} + \|p\|^2_{1,\Omega}\right) + \frac{1}{\varepsilon}J(p_h, p_h).$$

So,

$$(1 - C_{11}\varepsilon)\|\epsilon_h\|^2_{0,\Omega} \le \frac{C_{10}}{\varepsilon}\|\mathbf{e}_h\|^2_{\mathcal{D}} + \frac{C_{12}h^2}{\varepsilon}\left(\|\mathbf{u}\|^2_{2,\Omega} + \|p\|^2_{1,\Omega}\right) + \frac{1}{\varepsilon}J(p_h, p_h). \tag{4.23}$$

Substituting (4.11) and (4.12) into (4.23) gives

$$(1 - C_{11}\varepsilon)\|\epsilon_h\|^2_{0,\Omega} \le \frac{C_{13}h^2}{\varepsilon}\left(\|\mathbf{u}\|^2_{2,\Omega} + \|p\|^2_{1,\Omega}\right).$$

The claim now follows by taking ε sufficiently small, e. g. $\varepsilon < \frac{1}{C_{11}}$. □

The convergence of the proposed stabilization scheme is established by the following error estimate.

Theorem 4.2. *Let* $(\mathbf{u}, p) \in (\mathbf{H}^2(\Omega) \cap \mathbf{V}) \times (H^1(\Omega) \cap Q)$ *and* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *be the respective solutions of* (1.1) *and* (4.4). *Then, there exists a positive constant* C, *independent of* h, *such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathcal{D}} + \|p - p_h\|_{0,\Omega} \le Ch(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}). \tag{4.24}$$

P r o o f. (4.24) is straightforwardly deduced from (4.11) and (4.13) by applying the triangular inequality and a classical interpolation result (cf. [10]). □

§ 5. Numerical tests

This section presents numerical experiments to evaluate the accuracy and efficiency of the proposed stabilized FVM. Two numerical examples are presented to study the accuracy of the discrete solution and the convergence rates for generalized Stokes flows. The computational domain is $\Omega =]0,1[\, \times\,]0,1[$ and the problem (1.1) is to be discretized and solved using uniform partitionings of Ω into $n \times n$ equal squares, where $n = 10, 20, \ldots, 100$. To ensure a unique pressure field, a zero-mean pressure constraint is imposed on Ω. Many fixed values of the stabilization parameter β were considered; however, we present only the representative cases $\beta = 0.1$ and $\beta = 0.01$. In all numerical tests, the kinematic viscosity coefficient μ is set to 1, while the parameter α takes values 10^n for $n \in \{0, 1, 10\}$, covering a wide range of magnitudes. The source term $\mathbf{f}$ is chosen to satisfy the first equation in (1.1).

The velocity and pressure errors are defined as

$$e^{(i)}_{h,K} = u^{(i)}_K(\mathbf{x}_K) - u^{(i)}_{h,K}, \qquad \epsilon_{h,K} = p_K(\mathbf{x}_K) - p_{h,K}.$$

The following discrete error norms are used to investigate convergence rates

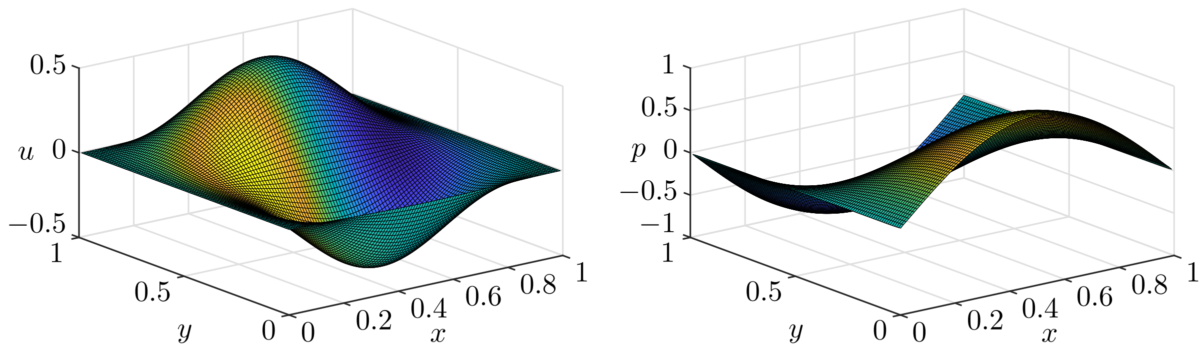
$$\|\mathbf{e}_h\|_{\mathcal{D}} = \sqrt{\sum_{i=1}^{2}\left[e^{(i)}_h, e^{(i)}_h\right]_{\mathcal{D}}} \quad \text{and} \quad \|\epsilon_h\|_{0,\Omega} = \sqrt{\int_\Omega \epsilon_h^2 \, d\Omega}.$$

5.1. Example 1

The first numerical example is as follows (see [12])

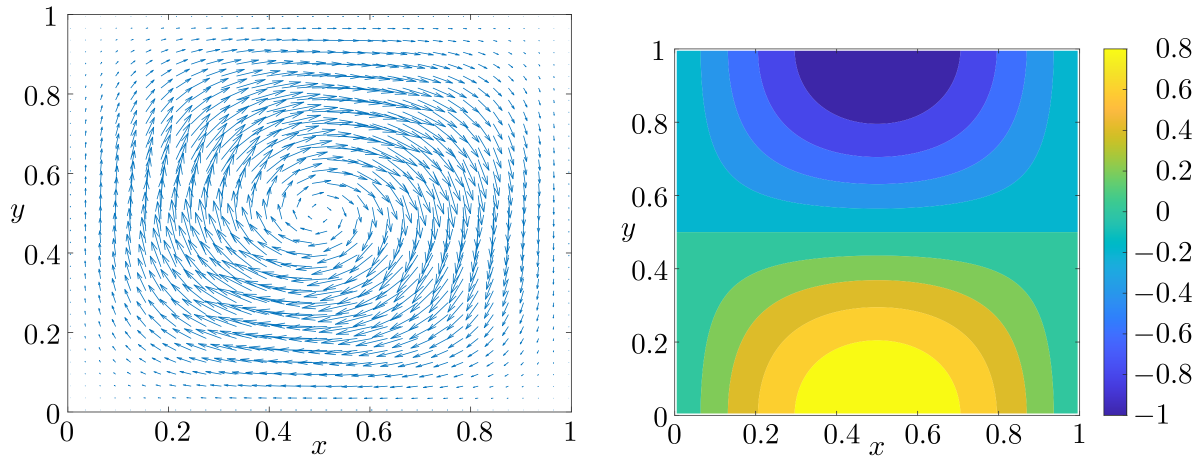
$$\begin{aligned}\mathbf{u}_1(x,y) &= -\sin^2(\pi x)\sin(\pi y)\cos(\pi y),\\ \mathbf{u}_2(x,y) &= \sin(\pi y)\cos(\pi x)\sin^2(\pi y),\\ p(x,y) &= \sin(\pi x)\cos(\pi y).\end{aligned}$$

The elevation of the discrete solution $(\mathbf{u}_h, p_h)$, along with the approximate velocity vectors and pressure isolines, are illustrated in Figures 1 and 2 for $\beta = 0.1$ and $\alpha = 1$. The presented graphs exhibit excellent agreement with the exact solution. The computed convergence rates, shown in Figure 3, indicate that for all considered values of α, the obtained results exceed theoretical predictions, with convergence rates approaching $3/2$ for $\|\mathbf{e}_h\|_{\mathcal{D}}$, and nearly 1 for $\|\epsilon_h\|_{0,\Omega}$. Additionally, the error magnitudes remain relatively stable as α increases, further underscoring the robustness of the numerical method.



(a) Velocity **(b)** Pressure

Fig. 1. Elevation of the velocity and the pressure



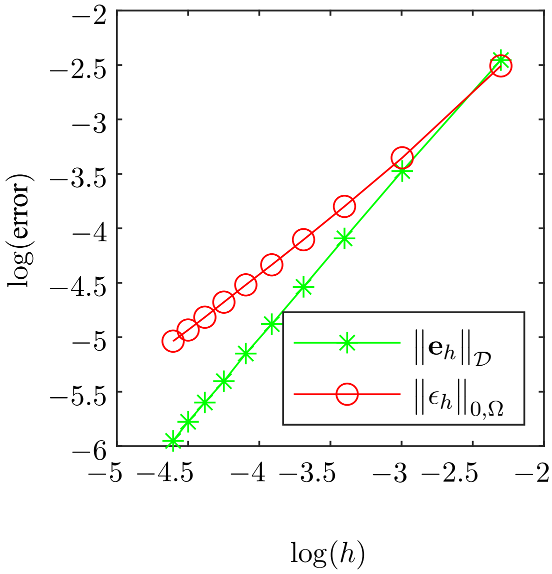
(a) Velocity **(b)** Pressure

Fig. 2. Approximate velocity vectors and pressure isolines

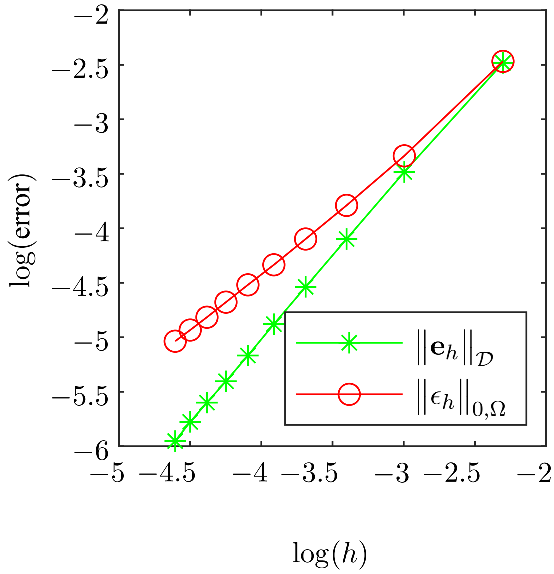
5.2. Example 2

The second numerical example considers the following functions (see [7])

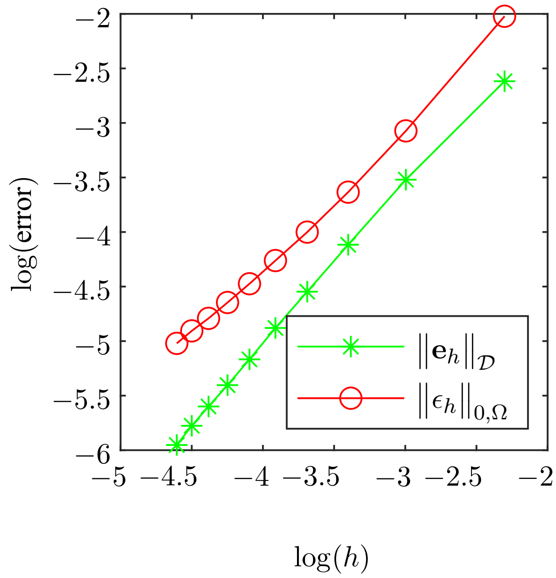
$$\begin{aligned}\mathbf{u}_1(x,y) &= 2000(x-x^2)^2(y-y^2)(1-2y),\\ \mathbf{u}_2(x,y) &= -2000(y-y^2)^2(x-x^2)(1-2x),\\ p(x,y) &= 100(x^2+y^2-2/3).\end{aligned}$$



(a) $\alpha = 1$



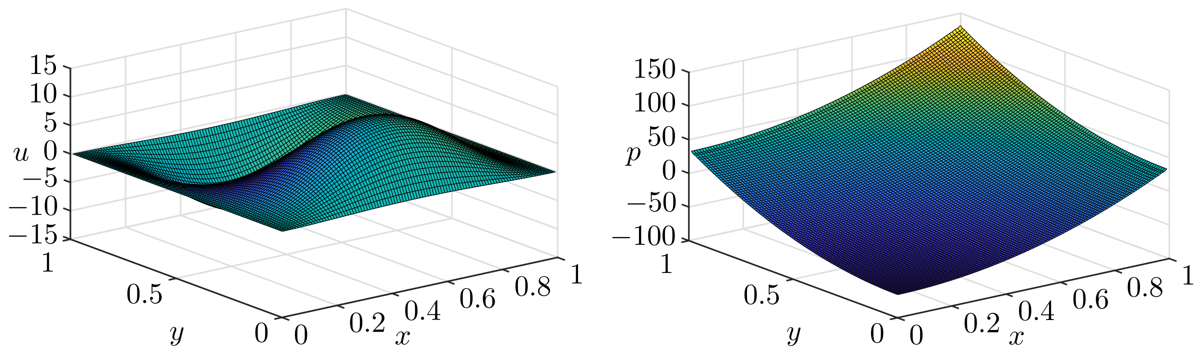
(b) $\alpha = 10$



(c) $\alpha = 100$

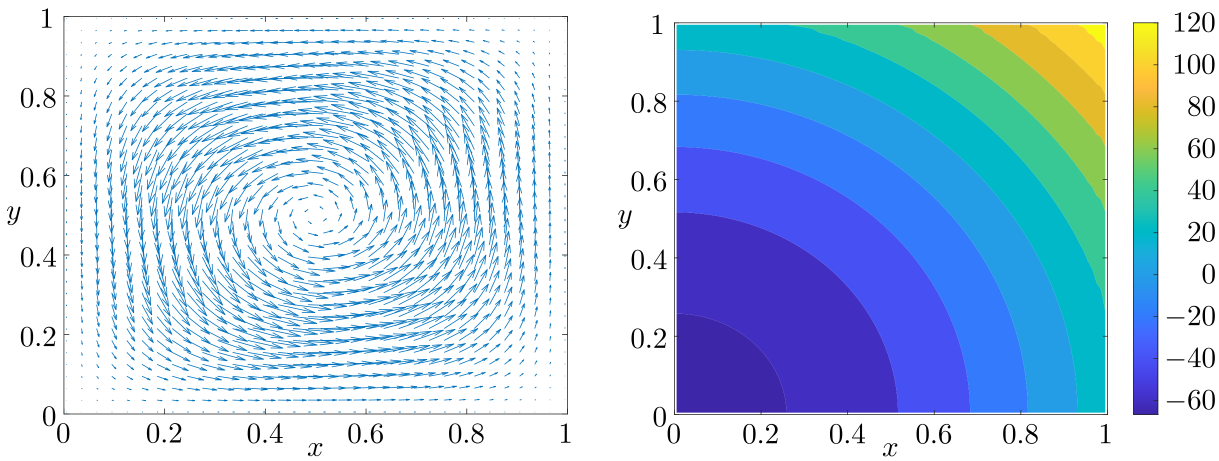
Fig. 3. Convergence history as α increases

For $\beta = 0.01$ and $\alpha = 1$, Figure 4 shows the elevation of the discrete solution, while Figure 5 displays the approximate velocity vectors and pressure isolines, demonstrating the high accuracy of the method. Additionally, the convergence error history, presented in Figure 6, confirms that the observed convergence rates consistently surpass the theoretical predictions for all considered values. Here again, we observe that the error magnitudes remain relatively stable as α increases.



(a) Velocity **(b)** Pressure

Fig. 4. Elevation of the velocity and the pressure



(a) Velocity **(b)** Pressure

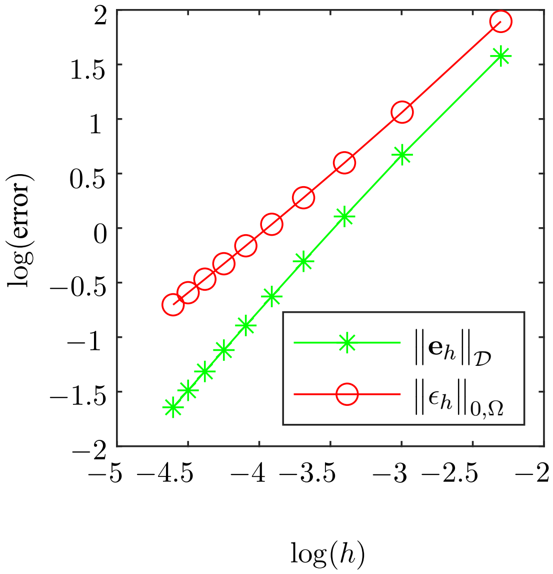
Fig. 5. Approximate velocity vectors and pressure isolines

5.3. Sensitivity with respect to the stabilization parameter

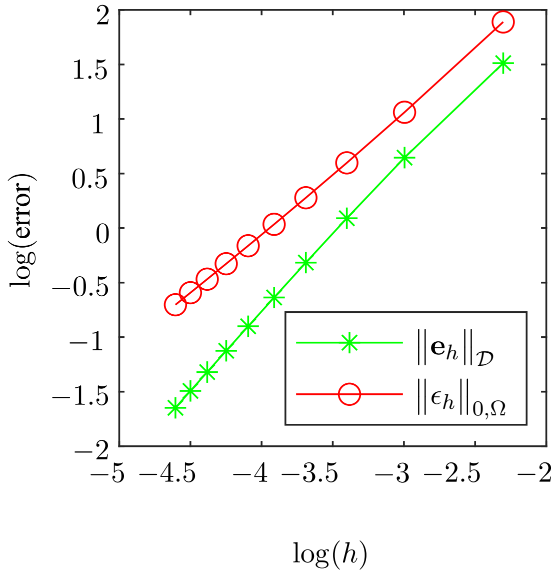
We present a graph illustrating how convergence is affected by the stabilization parameter β for a 100×100 grid. Figure 7 shows the variation of velocity and pressure errors in the discrete H_0^1 norm and the $L_2(\Omega)$ norm, respectively, as functions of β. For simplicity, we fix $\alpha = 1$. It is observed that as β increases, the errors also increase. Furthermore, the results in Figure 7 suggest that a good value for β lies between 10^{-2} and 10^{0}. Notably, the numerical errors in velocity and pressure remain relatively stable even as β varies across several orders of magnitude, demonstrating the robustness of the method.

Conclusion

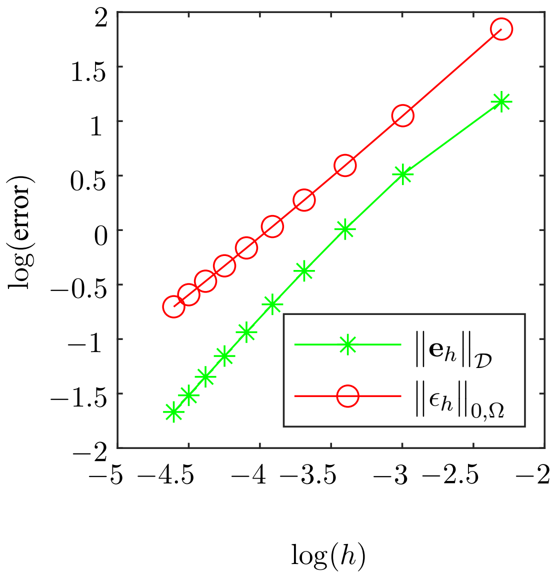
In this paper, we introduced and analyzed a symmetric stabilized collocated FVM for solving the generalized Stokes problem. The proposed method is based on a low-order approximation that employs piecewise constant functions for both velocity and pressure. Stability was achieved by incorporating a discrete pressure stabilization term into the formulation. Through rigorous mathematical analysis, we demonstrated the well-posedness and convergence of the method. A key contribution of this work is the proof that the method satisfies a weaker form of the *inf-sup* condition, which ensures the stability of the discrete system.



(a) $\alpha = 1$

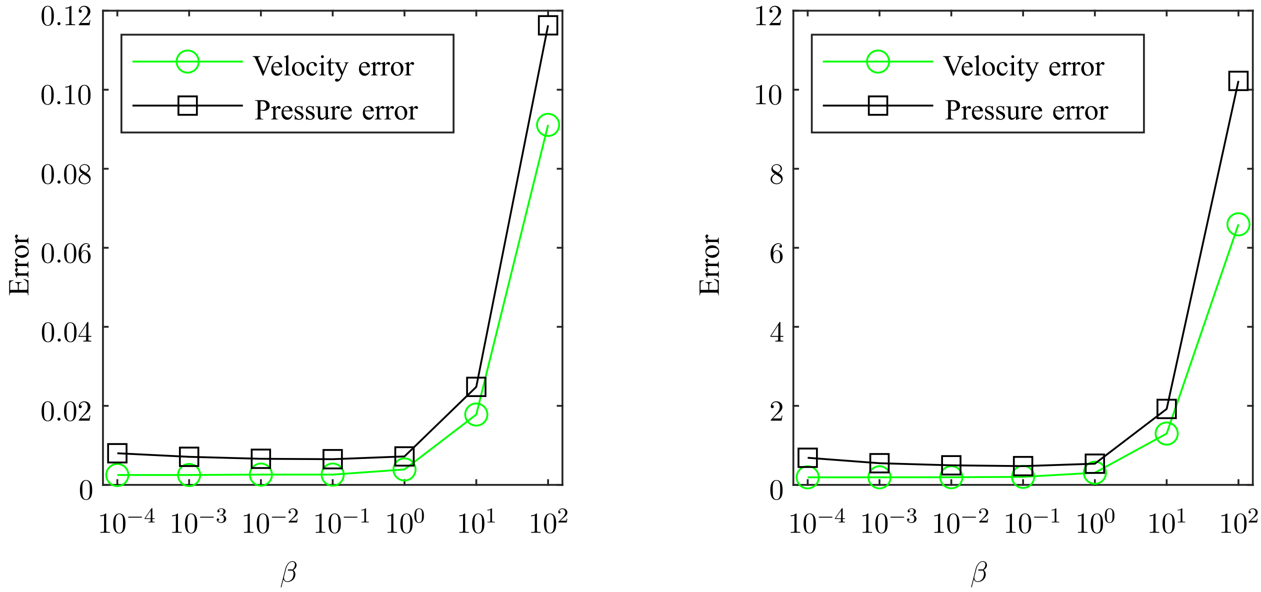


(b) $\alpha = 10$



(c) $\alpha = 100$

Fig. 6. Convergence history as α increases



(a) Sensitivity of errors for Example 1 **(b)** Sensitivity of errors for Example 2

Fig. 7. Sensitivity of errors with respect to β

Additionally, we derived first-order error estimates in the energy norm, confirming the theoretical convergence rate of the method. The numerical results demonstrate the stability and accuracy of the proposed method. Notably, the observed convergence rates in our simulations exceeded the theoretical a priori estimates derived in Section 3. While similar behavior has been reported for the Stokes problem [6, 7, 10], its theoretical justification remains an open question. We hypothesize that this enhanced performance may be due to factors such as the regularity of the mesh and the specific structure of the stabilization term, which not only ensures stability but also reduces numerical oscillations in the pressure field, thereby improving accuracy. This dual role of the stabilization term-ensuring stability and enhancing convergence by mitigating spurious pressure oscillations-highlights its importance in collocated FVMs. Future research directions include optimizing the performance of the method through an informed selection of the stabilization parameter β, extending the method to time-dependent problems, and applying it to nonlinear flow models and three-dimensional domains. These extensions will further demonstrate the versatility and applicability of the proposed stabilized collocated FVM in a wide range of fluid dynamics problems.

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А. Букабаше

Об устойчивости коллокационного метода конечных объемов для обобщенной задачи Стокса

Ключевые слова: задача Стокса, условие инф-суп, методы конечных объемов, стабилизированные методы.

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В данной работе представлена и проанализирована симметричная стабилизированная коллокационная формулировка метода конечных объемов для стационарной обобщенной задачи Стокса. Этот метод основан на аппроксимации наинизшего порядка (кусочно-постоянные функции) для обеих неизвестных величин: скорости и давления. Стабилизация достигается за счет добавления в формулировку дискретного слагаемого, связанного с давлением. Установлены свойства устойчивости и сходимости метода. В заключение представлены два численных примера, подтверждающие устойчивость и точность предложенного метода.

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