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MATHEMATICS 2025. Vol. 35. Issue 2. Pp. 198–214.

MSC2020: 53C30, 53C43, 22E15

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CHARACTERIZING BI-HARMONIC HOMOMORPHISMS IN THREE-DIMENSIONAL UNIMODULAR LIE GROUPS

This study is dedicated to the classification of bi-harmonic homomorphisms $\varphi\colon (G,g) \to (H,h)$, where G and H represent connected and simply connected three-dimensional unimodular Lie groups, while g and h denote left invariant Riemannian metrics.

Keywords: bi-harmonic homomorphisms, unimodular Riemannian Lie groups, left invariant Riemannian metrics.

DOI: 10.35634/vm250203

Introduction

The theory of harmonic maps into Lie groups has garnered significant attention among mathematicians, particularly in the context of homomorphisms in compact Lie groups, as evidenced by the works of various researchers (for example, see [1,4]). This interest extends to harmonic maps into Lie groups themselves (see [9]), as well as harmonic inner automorphisms within compact connected semi-simple Lie groups (see [8]). Furthermore, intensive investigations have been conducted on harmonic and bi-harmonic homomorphisms between Riemannian Lie groups equipped with left invariant Riemannian metrics (see [3]).

The research presented in this work draws inspiration from the paper by [2], wherein the author undertook the classification of harmonic and biharmonic maps $\varphi\colon (G,g_1) \to (G,g_2)$ where G represents a non-abelian, connected, and simply connected three-dimensional unimodular Lie group, φ is a homomorphism of Lie groups, and g_1 and g_2 denote two left invariant Riemannian metrics. It is worth noting that G is classified as unimodular if and only if $|\det \mathrm{ad}_x| = 1$ for all $x \in G$, which is equivalent to $\mathrm{tr}\,\mathrm{ad}_X = 0$ for all X in its Lie algebra $\mathfrak{g}$, and, consequently, $\mathfrak{g}$ being unimodular.

In a previous study [10], we explored the classification of harmonic homomorphisms between distinct non-abelian, connected, and simply connected three-dimensional unimodular Lie groups. In this paper, we extend our investigation to encompass the classification, up to a conjugation by automorphism of Lie groups, of bi-harmonic homomorphisms. These bi-harmonic homomorphisms are defined between two different non-abelian, connected, and simply connected three-dimensional unimodular Lie groups $\varphi\colon (G,g) \to (H,h)$, where g and h represent two left invariant Riemannian metrics on G and H, respectively.

§ 1. Preliminaries

Let $\varphi\colon (M,g) \to (N,h)$ be a smooth map between two Riemannian manifolds of dimensions m and n respectively (for more details see [5]). We denote by ∇^M and ∇^N the Levi-Civita connections associated with g and h respectively. Additionally, we introduce the vector bundle $T^\varphi N$ over M, which is the pull-back of TN by φ. Notably, $T^\varphi N$ is a Euclidean vector bundle, and the tangent map of φ, denoted as $d\varphi\colon TM \to T^\varphi N$, is a bundle homomorphism. Furthermore, $T^\varphi N$ carries a connection, denoted as ∇^φ, which is the pull-back of ∇^N by φ. There also exists a connection on the vector bundle $\mathrm{End}(TM, T^\varphi N)$ defined as

$$(\nabla_X A)(Y) = \nabla^\varphi_X A(Y) - A(\nabla^M_X Y), \quad X, Y \in \Gamma(TM), \quad A \in \Gamma(\mathrm{End}(TM, T^\varphi N)).$$

The map φ is termed bi-harmonic (see [7]) if it represents a critical point of the bi-energy functional $E_2(\varphi)$, defined as:

$$E_2(\varphi) = \int_M |\tau(\varphi)|^2 \, dv_g,$$

where $\tau(\varphi)$ is the tension field of φ given by:

$$\tau(\varphi) = \operatorname{tr} \nabla d\varphi = \sum_{i=1}^{m} (\nabla_{e_i} d\varphi) e_i,$$

and $(e_i)_{i=1}^m$ is a local frame of orthonormal vector fields.

The corresponding Euler–Lagrange equation for the bi-energy functional involves the vanishing of the bi-tension field:

$$\begin{aligned} \tau_2(\varphi) &= -\operatorname{tr}_g(\nabla)^2 \tau(\varphi) - \operatorname{tr}_g R^N(\tau(\varphi), d\varphi) d\varphi \\ &= -\left[\nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i} \tau(\varphi) - \nabla^{\varphi}_{\nabla^M_{e_i} e_i} \tau(\varphi)\right] - R^N\big(\tau(\varphi), d\varphi(e_i)\big) d\varphi(e_i) \end{aligned}$$

where R^N represents the curvature of ∇^N defined as:

$$R^N(X, Y) = \nabla^N_X \nabla^N_Y - \nabla^N_Y \nabla^N_X - \nabla^N_{[X,Y]}.$$

Now, let (G, g) be a Riemannian Lie group, which is a Lie group endowed with a left-invariant Riemannian metric. If $\mathfrak{g} = T_e G$ is its Lie algebra and $\langle , \rangle_{\mathfrak{g}} = g(e)$, then there exists a unique bilinear map $A\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Levi-Civita product associated with $(\mathfrak{g}, \langle , \rangle_{\mathfrak{g}})$. It is defined by the formula:

$$2\langle A_u v, w \rangle_{\mathfrak{g}} = \langle [u, v]^{\mathfrak{g}}, w \rangle_{\mathfrak{g}} + \langle [w, u]^{\mathfrak{g}}, v \rangle_{\mathfrak{g}} + \langle [w, v]^{\mathfrak{g}}, u \rangle_{\mathfrak{g}}.$$

The map A is entirely determined by two key properties:

(1) for any $u, v \in \mathfrak{g}$, $A_u v - A_v u = [u, v]^{\mathfrak{g}}$;

(2) for any $u, v, w \in \mathfrak{g}$, $\langle A_u v, w \rangle_{\mathfrak{g}} + \langle v, A_u w \rangle_{\mathfrak{g}} = 0$.

If we denote by u^ℓ the left-invariant vector field on G associated with $u \in \mathfrak{g}$, then the Levi-Civita connection associated with (G, g) satisfies $\nabla_{u^\ell} v^\ell = (A_u v)^\ell$. The pair $(\mathfrak{g}, \langle , \rangle_{\mathfrak{g}})$ also defines a vector denoted as $U^{\mathfrak{g}}$ given by:

$$\langle U^{\mathfrak{g}}, v \rangle_{\mathfrak{g}} = \operatorname{tr}(\operatorname{ad}_v), \quad \text{for any } \ v \in \mathfrak{g}.$$

It is straightforward to observe that for any orthonormal basis $(e_i)_{i=1}^m$ of $\mathfrak{g}$:

$$U^{\mathfrak{g}} = \sum_{i=1}^{m} A_{e_i} e_i.$$

Moreover, note that $\mathfrak{g}$ is classified as unimodular if and only if $U^{\mathfrak{g}} = 0$.

Now, consider a Lie group homomorphism $\varphi\colon (G, g) \to (H, h)$ between two Riemannian Lie groups. The differential $\xi\colon \mathfrak{g} \to \mathfrak{h}$ of φ at e is a Lie algebra homomorphism. There exists a left action of G on $\Gamma(T^\varphi H)$ defined as:

$$(a.X)(b) = T_{\varphi(ab)} L_{\varphi(a^{-1})} X(ab), \quad a, b \in G, \quad X \in \Gamma(T^\varphi H).$$

A section X of $T^{\varphi}H$ is considered left invariant if, for any $a \in G$, $a.X = X$. For any left invariant section X, it holds that $X(a) = (X(e))^{\ell}(\varphi(a))$. Consequently, the space of left invariant sections is isomorphic to the Lie algebra $\mathfrak{h}$. As both g and h are left invariant due to φ being a Lie group homomorphism, it is evident that $\tau(\varphi)$ and $\tau_2(\varphi)$ are also left invariant. Thus, φ is harmonic (or biharmonic) if and only if $\tau(\varphi)(e) = 0$ (or $\tau_2(\varphi)(e) = 0$). Additionally, it can be shown that (see [3]):

$$\tau(\xi) := \tau(\varphi)(e) = U^{\xi} - \xi(U^{\mathfrak{g}}),$$

where

$$\tau_2(\xi) := \tau_2(\varphi)(e) = -\sum_{i=1}^{n}\big(B_{\xi(e_i)}B_{\xi(e_i)}\tau(\xi) + \mathrm{K}^H(\tau(\xi),\xi(e_i))\xi(e_i)\big) + B_{\xi(U^g)}\tau(\xi),$$

and K^H is the curvature of B given by

$$K^H(u,v) = [B_u, B_v] - B_{[u,v]},$$

with

$$U^{\xi} = \sum_{i=1}^{m} B_{\xi(e_i)}\xi(e_i),$$

Here, B represents the Levi-Civita product associated with $(\mathfrak{h}, \langle,\rangle_{\mathfrak{h}})$, and $(e_i)_{i=1}^m$ is an orthonormal basis of $\mathfrak{g}$. Thus, φ is harmonic if and only if $\tau(\xi) = 0$, where $\xi\colon \mathfrak{g} \to \mathfrak{h}$ is the differential of φ at e (see [10, Proposition 2.1]). Additionally, let $\xi\colon (\mathfrak{g}, \langle,\rangle_{\mathfrak{g}}) \to (\mathfrak{h}, \langle,\rangle_{\mathfrak{h}})$ be a homomorphism between unimodular Euclidean Lie algebras. The following formulas were established in [3]:

$$\begin{aligned}\langle\tau(\xi), X\rangle_{\mathfrak{h}} &= \mathrm{tr}_{\mathfrak{g}}(\xi^* \circ \mathrm{ad}_X \circ \xi) \quad \forall X \in \mathfrak{h},\\ \langle\tau_2(\xi), X\rangle_{\mathfrak{h}} &= \mathrm{tr}_{\mathfrak{g}}(\xi^* \circ (\mathrm{ad}_X + \mathrm{ad}_X^*) \circ \mathrm{ad}_{\tau(\xi)} \circ \xi) - \langle[X,\tau(\xi)],\tau(\xi)\rangle_h \quad \forall X \in \mathfrak{h},\end{aligned} \tag{1.1}$$

where $\xi^*\colon \mathfrak{h} \to \mathfrak{g}$ is given by

$$\langle\xi^* U, V\rangle_{\mathfrak{g}} = \langle U, \xi V\rangle_{\mathfrak{h}}, \quad \text{for } V \in \mathfrak{g} \text{ and } U \in \mathfrak{h}. \tag{1.2}$$

§ 2. Riemannian three-dimensional unimodular Lie groups

In the realm of non-abelian unitary Lie groups that are both connected and simply connected, there exist five noteworthy examples, as referenced in [10]. However, this study narrows its focus to a subset of three such groups: the nilpotent Lie group (often referred to as the Heisenberg group), the solvable Lie groups denoted as Sol, and the universal covering group $\widetilde{E}_0(2)$ of the connected component of the Euclidean group. For a more comprehensive understanding of these specific groups and their properties, one may refer to the detailed exposition in [6].

Definition 2.1 (The Heisenberg group Nil). The nilpotent Lie group, commonly referred to as the Heisenberg group and denoted as Nil, possesses a Lie algebra that we shall denote as $\mathfrak{n}$.

$$\mathrm{Nil} = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \text{ with } a, b, c \in \mathbb{R} \right\}$$

and

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, \text{ with } x, y, z \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{n}$ possesses a basis denoted as X, Y, Z, with the following matrix representations:

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Among these, the only non-vanishing Lie bracket is given by $[X, Y] = Z$.

Proposition 2.1 (see [6]). *Any left-invariant metric on the* Nil *manifold is equivalent, up to automorphism, to a metric whose associated matrix has the following form:*

$$\langle\ ,\ \rangle_{\mathfrak{n}} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \textit{where} \quad \rho > 0. \tag{2.1}$$

Definition 2.2 (The solvable Lie group Sol). The solvable Lie group Sol, whose Lie algebra is denoted by $\mathfrak{sol}$, can be defined as follows:

(1) Lie Algebra $\mathfrak{sol}$: The Lie algebra $\mathfrak{sol}$ is given by $\mathfrak{sol} = \mathbb{R}^2 \times_\iota \mathbb{R}$, where the action $\iota(t)$ is defined as: $\iota(t) = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$. We can choose a basis X, Y, Z for $\mathfrak{sol}$ as follows:

$$X = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0\right), \quad Y = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0\right) \quad \text{and} \quad Z = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 1\right).$$

The non-vanishing Lie brackets are $[Z, X] = X$ and $[Y, Z] = Y$.

(2) Lie Group Sol: The Lie group associated with the solvable Lie algebra $\mathfrak{sol}$ is the solvable Lie group Sol, which can be represented as the semi-direct product $\mathbb{R}^2 \rtimes_\Theta \mathbb{R}$. Here, $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ via the action $\Theta(t)$ defined as: $\Theta(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. In summary, the solvable Lie group Sol is defined through its Lie algebra $\mathfrak{sol}$, and it is realized as a semi-direct product of $\mathbb{R}^2$ and $\mathbb{R}$ with specific actions $\iota(t)$ and $\Theta(t)$.

The following proposition describes the equivalence of left-invariant metrics on the solvable Lie group Sol and provides two specific forms for the associated metric matrices.

Proposition 2.2 (see [6], Metric Equivalence on Solvable Lie Group Sol). *Any left-invariant metric on the solvable Lie group* $\mathrm{Sol} = \mathbb{R}^2 \rtimes_\Theta \mathbb{R}$ *is equivalent, up to automorphism, to a metric whose associated matrix is of one of the following forms:*

$$\langle\ ,\ \rangle_{\mathfrak{sol}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix}, \quad \textit{where} \quad \nu > 0, \tag{2.2}$$

$$\langle\ ,\ \rangle_{\mathfrak{sol}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}, \quad \textit{where} \quad \nu > 0 \ \textit{and} \ \mu > 1. \tag{2.3}$$

Definition 2.3 (The solvable Lie group $\widetilde{E}_0(2)$). The solvable Lie group denoted as $\widetilde{E}_0(2)$ is defined as the universal covering group of $E_0(2)$, where $E_0(2)$ is itself a Lie group with the Lie algebra $\mathfrak{e}_0(2)$. The Lie algebra $\mathfrak{e}_0(2)$ is given by:

$$\mathfrak{e}_0(2) = \mathbb{R}^2 \rtimes \mathfrak{sol}(2).$$

In this Lie algebra, a basis X, Y, Z is chosen, where:

$$X = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right), \quad Y = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right), \quad Z = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right).$$

The non-vanishing Lie brackets in this algebra are given by:

$$[Z, X] = Y, \quad [Y, Z] = X.$$

The group $E_0(2)$ is not simply connected, and the unique simply connected Lie group corresponding to the Lie algebra $\mathfrak{e}_0(2)$ is the universal covering group $\widetilde{E}_0(2)$. This group $\widetilde{E}_0(2)$ can be represented as a semi-direct product $\mathbb{C} \rtimes \mathbb{R}$, where group elements are of the form (z, t) with the operation:

$$(z, t).(z', t') = (z + z'e^{2i\pi t}, t + t').$$

This group operation has a faithful matrix representation in $GL(3, \mathbb{C})$ given by:

$$(z, t) \mapsto \begin{pmatrix} e^{2i\pi t} & z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{pmatrix}.$$

Here, z is a complex number, and t is a real number.

Proposition 2.3 (see [6], Invariant Metrics on $\widetilde{E}_0(2)$). *Any left-invariant metric on the Lie group $\widetilde{E}_0(2)$ is, up to automorphism, equivalent to a metric whose associated matrix has the following form:*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \varrho & 0 \\ 0 & 0 & \sigma \end{pmatrix}. \tag{2.4}$$

Here, σ is a positive real number ($\sigma > 0$), and ϱ is a real number satisfying $0 < \varrho \leq 1$.

§ 3. Bi-harmonic homomorphisms

3.1. Bi-harmonic homomorphisms between Sol and Nil

In this subsection, we present a comprehensive classification of bi-harmonic homomorphisms between the Lie algebras $\mathfrak{sol}$ equipped with the left-invariant metric defined in (2.2) or (2.3) and $\mathfrak{n}$ equipped with the left-invariant metric defined in (2.1).

Proposition 3.1 (see [10, Theorem 4.1]). *Any homomorphism from $\mathfrak{sol}$ to $\mathfrak{n}$ is conjugate to $\xi\colon \mathfrak{sol} \to \mathfrak{n}$, where*

$$\xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix} \quad \textit{with} \quad a, b, c \in \mathbb{R}. \tag{3.1}$$

Let $\{E, F, H\}$ be the basis of algebra $\mathfrak{n}$, where $[H, E] = [H, F] = 0$, $[E, F] = H$ and

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Theorem 3.1. *Let $\xi\colon \mathfrak{sol} \to \mathfrak{n}$ be a homomorphism defined in (3.1), $\mathfrak{sol}$ be a Lie algebra equipped with the left invariant metric defined in (2.2) or (2.3), and $\mathfrak{n}$ be equipped with the left invariant metric defined in (2.1). Then, we have*

$$\tau_2(\xi) = \frac{\rho c(a^2+b^2)}{\nu^2}[bE - aF].$$

P r o o f. Recalling [10, Theorem 4.2], we have the following relationship:

$$\tau(\xi) = \frac{bc}{\nu}E - \frac{ac}{\nu}F.$$

Now, let's compute the adjoint representations. We have

$$\mathrm{ad}_{\tau(\xi)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{ac}{\nu} & \frac{bc}{\nu} & 0 \end{pmatrix}, \quad \mathrm{ad}_E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{ad}_F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}_H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The adjoint representations in their dual spaces are given by:

$$\mathrm{ad}^*_E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}^*_F = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{ad}^*_H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, using formula (1.2), where $U \in \mathfrak{n}$ and $V \in \mathfrak{sol}$, we get

$$\xi^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \rho a & \rho b & c \end{pmatrix}.$$

Next, applying formula (1.1), we perform a straightforward calculation for each basis element:

$$\langle \tau_2(\xi), E\rangle_{\mathfrak{n}} = \mathrm{tr}(\xi^* \circ (\mathrm{ad}_E + \mathrm{ad}^*_E) \circ \mathrm{ad}_{\tau(\xi)} \circ \xi) - \langle [E, \tau(\xi)], \tau(\xi)\rangle_2 = \frac{\rho bc(a^2+b^2)}{\nu},$$
$$\langle \tau_2(\xi), F\rangle_{\mathfrak{n}} = \mathrm{tr}(\xi^* \circ (\mathrm{ad}_F + \mathrm{ad}^*_F) \circ \mathrm{ad}_{\tau(\xi)} \circ \xi) - \langle [F, \tau(\xi)], \tau(\xi)\rangle_2 = \frac{-\rho ac(a^2+b^2)}{\nu},$$

and

$$\langle \tau_2(\xi), H\rangle_{\mathfrak{n}} = \mathrm{tr}(\xi^* \circ (\mathrm{ad}_H + \mathrm{ad}^*_H) \circ \mathrm{ad}_{\tau(\xi)} \circ \xi) - \langle [H, \tau(\xi)], \tau(\xi)\rangle_2 = 0.$$

This concludes the proof. □

Corollary 3.1. *A homomorphism $\xi\colon \mathfrak{sol} \to \mathfrak{n}$ is bi-harmonic if and only if it is one of the following forms:*

$$\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proposition 3.2 (see [10]). *Any homomorphism from* $\mathfrak{n}$ *to* $\mathfrak{sol}$ *is conjugate to one of the following forms:*

$\xi_1\colon \mathfrak{sol} \to \mathfrak{n}$ *with*

$$\xi_1 = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.2}$$

where a *and* b *are real numbers;*

$\xi_2\colon \mathfrak{sol} \to \mathfrak{n}$ *with*

$$\xi_2 = \begin{pmatrix} a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.3}$$

where a_1*,* a_2*,* b_1*, and* b_2 *are real numbers.*

Theorem 3.2. *Let* $\xi_1\colon \mathfrak{n} \to \mathfrak{sol}$ *be a homomorphism defined in* (3.2)*, such that* a *and* b *are real numbers, and the Lie algebra* $\mathfrak{sol}$ *is equipped with the left invariant metric defined in formula* (2.2)*. Then we have*

$$\tau_2(\xi_1) = 2(a^2 + b^2)(a^2 - b^2)Z.$$

P r o o f. Recalling [10, Theorem 4.4], we have the following relationship:

$$\tau(\xi_1) = (a^2 - b^2)Z.$$

Now, let's compute the adjoint representations:

$$\mathrm{ad}_{\tau(\xi_1)} = \begin{pmatrix} (a^2 - b^2) & 0 & 0 \\ 0 & -(a^2 - b^2) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{ad}_X = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}_Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{ad}_Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The adjoint representations in their dual spaces are given by:

$$\mathrm{ad}^*_X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}^*_Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{ad}^*_Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, using formula (1.2) with $V \in \mathfrak{n}$ and $U \in \mathfrak{sol}$, we get

$$\xi_1^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & b & 0 \end{pmatrix}.$$

Using formula (1.1), a simple calculation gives us

$$\langle \tau_2(\xi_1), X \rangle_{\mathfrak{n}} = \mathrm{tr}(\xi^* \circ (\mathrm{ad}_X + \mathrm{ad}^*_X) \circ \mathrm{ad}_{\tau(\xi_1)} \circ \xi_1) - \langle [X, \tau(\xi_1)], \tau(\xi_1) \rangle_2 = 0,$$
$$\langle \tau_2(\xi_1), Y \rangle_{\mathfrak{n}} = \mathrm{tr}(\xi^* \circ (\mathrm{ad}_Y + \mathrm{ad}^*_Y) \circ \mathrm{ad}_{\tau(\xi_1)} \circ \xi_1) - \langle [Y, \tau(\xi_1)], \tau(\xi_1) \rangle_2 = 0,$$

and

$$\langle \tau_2(\xi_1), Z \rangle_{\mathfrak{n}} = \mathrm{tr}(\xi^* \circ (\mathrm{ad}_Z + \mathrm{ad}^*_Z) \circ \mathrm{ad}_{\tau(\xi_1)} \circ \xi_1) - \langle [Z, \tau(\xi_1)], \tau(\xi_1) \rangle_2 = 2(a^2 + b^2)(a^2 - b^2).$$

This concludes the proof. □

Corollary 3.2. *Let $\xi_1 \colon \mathfrak{n} \to \mathfrak{sol}$ be a homomorphism defined in (3.2) such that a and b are real numbers, and the Lie algebra $\mathfrak{sol}$ be equipped with the left invariant metric defined in formula (2.2). Then, ξ_1 is bi-harmonic if and only if $a = b$ or $a = -b$.*

Theorem 3.3. *Let $\xi_1 \colon \mathfrak{n} \to \mathfrak{sol}$ be a homomorphism defined in (3.2), where $a, b \in \mathbb{R}$ and the Lie algebra $\mathfrak{sol}$ be equipped with the left invariant metric defined in formula (2.3). Then*

$$\tau_2(\xi_1) = (a^2 - \mu b^2)(a^2 - b^2 + 2ab + \mu b^2(1 + \mu))Z.$$

P r o o f. Using Theorem 4.5 from [10], we have:

$$\tau(\xi_1) = (a^2 - \mu b^2)Z,$$

We also have the representation matrices of the adjoint operators:

$$\mathrm{ad}_{\tau(\xi_1)} = \begin{pmatrix} (a^2-\mu b^2) & 0 & 0 \\ 0 & -(a^2-\mu b^2) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\mathrm{ad}_X = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}_Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{ad}_Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, the adjoint operators of the matrices are given by:

$$\mathrm{ad}^*_X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}^*_Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \mu & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{ad}^*_Z = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -\mu & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By using formula (1.2) with $V \in \mathfrak{n}$ and $U \in \mathfrak{sol}$, we obtain:

$$\xi_1^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a+b & a+\mu b & 0 \end{pmatrix}.$$

Applying formula (1.1), a straightforward calculation gives us:

$$\langle \tau_2(\xi_1), X\rangle_{\mathfrak{n}} = \mathrm{tr}(\xi^* \circ (\mathrm{ad}_X + \mathrm{ad}^*_X) \circ \mathrm{ad}_{\tau(\xi_1)} \circ \xi_1) - \langle [X, \tau(\xi_1)], \tau(\xi_1)\rangle_2 = 0,$$
$$\langle \tau_2(\xi_1), Y\rangle_{\mathfrak{n}} = \mathrm{tr}(\xi^* \circ (\mathrm{ad}_Y + \mathrm{ad}^*_Y) \circ \mathrm{ad}_{\tau(\xi_1)} \circ \xi_1) - \langle [Y, \tau(\xi_1)], \tau(\xi_1)\rangle_2 = 0,$$

and

$$\begin{aligned}\langle \tau_2(\xi_1), Z\rangle_{\mathfrak{n}} &= \mathrm{tr}(\xi^* \circ (\mathrm{ad}_Z + \mathrm{ad}^*_Z) \circ \mathrm{ad}_{\tau(\xi_1)} \circ \xi_1) - \langle [Z, \tau(\xi_1)], \tau(\xi_1)\rangle_2 \\ &= (a^2 - \mu b^2)(a^2 - b^2 + 2ab + \mu b^2(1+\mu)).\end{aligned}$$

Thus, the proof is complete. □

Corollary 3.3. *Let $\xi_1 \colon \mathfrak{n} \to \mathfrak{sol}$ be a homomorphism defined in (3.2), with $a, b \in \mathbb{R}$, and the Lie algebra $\mathfrak{sol}$ be equipped with the left invariant metric defined in formula (2.3). Then, ξ_1 is bi-harmonic if and only if:*

$$a = b = 0, \quad or \quad a = b\sqrt{\mu}, \quad or \quad a = -b\sqrt{\mu}.$$

Theorem 3.4. *Let $\xi_2\colon \mathfrak{n} \to \mathfrak{sol}$ be a homomorphism defined in (3.3), with $a_i, b_i \in \mathbb{R}$, and the Lie algebra $\mathfrak{sol}$ is equipped with the left invariant metric defined in formula (2.2). Then, we have the following:*

$$\langle \tau_2(\xi_2), X\rangle_{\mathfrak{n}} = 0,$$
$$\langle \tau_2(\xi_2), Y\rangle_{\mathfrak{n}} = 0,$$

and

$$\begin{aligned}\langle \tau_2(\xi_2), Z\rangle_{\mathfrak{n}} &= \operatorname{tr}(\xi^* \circ (\operatorname{ad}_Z + \operatorname{ad}_Z^*) \circ \operatorname{ad}_{\tau(\xi_2)} \circ \xi_1) - \langle [Z, \tau(\xi_1)], \tau(\xi_1)\rangle_2 \\ &= \frac{1}{\rho}[(a_1^2 + a_2^2) + (b_1^2 + b_2^2)]\tau(\xi_2).\end{aligned}$$

Thus, ξ_2 is bi-harmonic if and only if it is harmonic.

P r o o f. In [10, Theorem 4.4], we have the following result:

$$\tau(\xi_2) = \frac{(a_1^2 + a_2^2) - (b_1^2 + b_2^2)}{\rho} Z.$$

The representation matrices of the adjoint operators are as follows:

$$\operatorname{ad}_{\tau(\xi_2)} = \begin{pmatrix} \dfrac{(a_1^2 + a_2^2) - (b_1^2 + b_2^2)}{\rho} & 0 & 0 \\ 0 & -\dfrac{(a_1^2 + a_2^2) - (b_1^2 + b_2^2)}{\rho} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\operatorname{ad}_X = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \operatorname{ad}_Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \operatorname{ad}_Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Furthermore,

$$\operatorname{ad}_X^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \operatorname{ad}_Y^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad \operatorname{ad}_Z^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By using the formula (1.2), where $V \in \mathfrak{n}$ and $U \in \mathfrak{sol}$, we get

$$\xi_2^* = \begin{pmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using formula (1.1), a simple calculation gives us

$$\langle \tau_2(\xi_2), X\rangle_{\mathfrak{n}} = \operatorname{tr}(\xi^* \circ (\operatorname{ad}_X + \operatorname{ad}_X^*) \circ \operatorname{ad}_{\tau(\xi_1)} \circ \xi_1) - \langle [X, \tau(\xi_1)], \tau(\xi_1)\rangle_2 = 0,$$
$$\langle \tau_2(\xi_2), Y\rangle_{\mathfrak{n}} = \operatorname{tr}(\xi^* \circ (\operatorname{ad}_Y + \operatorname{ad}_Y^*) \circ \operatorname{ad}_{\tau(\xi_1)} \circ \xi_1) - \langle [Y, \tau(\xi_1)], \tau(\xi_1)\rangle_2 = 0,$$

and

$$\begin{aligned}\langle \tau_2(\xi_2), Z\rangle_{\mathfrak{n}} &= \operatorname{tr}(\xi^* \circ (\operatorname{ad}_Z + \operatorname{ad}_Z^*) \circ \operatorname{ad}_{\tau(\xi_2)} \circ \xi_1) - \langle [Z, \tau(\xi_1)], \tau(\xi_1)\rangle_2 \\ &= \frac{1}{\rho}[(a_1^2 + a_2^2) + (b_1^2 + b_2^2)]\tau(\xi_2).\end{aligned}$$

□

Theorem 3.5. *Let $\xi_2\colon \mathfrak{n} \to \mathfrak{sol}$ be a homomorphism defined in (3.3), with a_1, a_2, b_1, b_2 being real numbers, and the Lie algebra $\mathfrak{sol}$ be equipped with the left-invariant metric defined in formula (2.3). Then*

$$\tau_2(\xi_2) = \frac{1}{\rho}\big[a_1^2 + a_2^2 + 2a_1b_1 + 2a_2b_2 + (\mu^2+\mu-1)(b_1^2+b_2^2)\big]\tau(\xi_2).$$

Hence, ξ_2 is bi-harmonic if and only if it is harmonic.

P r o o f. In Theorem 4.5 from [10], we have the following expression for $\tau(\xi_2)$:

$$\tau(\xi_2) = \frac{(a_1^2-\mu b_1^2)+(a_2^2-\mu b_2^2)}{\rho}Z.$$

The adjoint representations of the basis elements in $\mathfrak{sol}$ are as follows:

$$\mathrm{ad}_{\tau(\xi_2)} = \begin{pmatrix} \frac{(a_1^2-\mu b_1^2)+(a_2^2-\mu b_2^2)}{\rho} & 0 & 0 \\ 0 & -\frac{(a_1^2-\mu b_1^2)+(a_2^2-\mu b_2^2)}{\rho} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{ad}_X = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}_Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{ad}_Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Furthermore,

$$\mathrm{ad}_X^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}_Y^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \mu & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{ad}_Z^* = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -\mu & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using formula (1.2), where $V \in \mathfrak{n}$ and $U \in \mathfrak{sol}$, we get

$$\xi_2^* = \begin{pmatrix} a_1+b_1 & a_1+\mu b_1 & 0 \\ a_2+b_2 & a_2+\mu b_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, using formula (1.1), we can calculate the components:

$$\langle \tau_2(\xi_2), X\rangle_{\mathfrak{n}} = \mathrm{tr}(\xi^*\circ(\mathrm{ad}_X+\mathrm{ad}_X^*)\circ \mathrm{ad}_{\tau(\xi_1)}\circ\xi_1) - \langle [X,\tau(\xi_1)],\tau(\xi_1)\rangle_2 = 0,$$
$$\langle \tau_2(\xi_2), Y\rangle_{\mathfrak{n}} = \mathrm{tr}(\xi^*\circ(\mathrm{ad}_Y+\mathrm{ad}_Y^*)\circ \mathrm{ad}_{\tau(\xi_1)}\circ\xi_1) - \langle [Y,\tau(\xi_1)],\tau(\xi_1)\rangle_2 = 0,$$

and

$$\begin{aligned}\langle \tau_2(\xi_2), Z\rangle_{\mathfrak{n}} &= \mathrm{tr}(\xi^*\circ(\mathrm{ad}_Z+\mathrm{ad}_Z^*)\circ \mathrm{ad}_{\tau(\xi_2)}\circ\xi_1) - \langle [Z,\tau(\xi_1)],\tau(\xi_1)\rangle_2 \\ &= \frac{1}{\rho}\big[a_1(a_1+(2-\mu)b_1) + a_2(a_2+(2-\mu)b_2) - b_1((1-\mu^2)b_1 - \mu(b_1+a_1)) \\ &\quad - b_2((1-\mu^2)b_2-\mu(b_2+a_2))\big]\tau(\xi_2) \\ &= \frac{1}{\rho}\big[a_1^2+a_2^2+2a_1b_1+2a_2b_2+(\mu^2+\mu-1)(b_1^2+b_2^2)\big]\tau(\xi_2).\end{aligned}$$

Hence, we have shown that $\langle \tau_2(\xi_2), X\rangle_{\mathfrak{n}} = 0$ and $\langle \tau_2(\xi_2), Y\rangle_{\mathfrak{n}} = 0$, and we have computed $\langle \tau_2(\xi_2), Z\rangle_{\mathfrak{n}}$ as above. This completes the proof. □

3.2. Bi-harmonic homomorphisms between Sol and $\mathfrak{e}_0(2)$

The following result gives a classification of bi-harmonic homomorphisms between $\mathfrak{sol}$ equipped with the left invariant metric defined in (2.2) or (2.3) and $\mathfrak{e}_0(2)$ equipped with the left invariant metric defined in (2.4).

Let $\{A, B, C\}$ be the basis of algebra $\mathfrak{e}_0(2)$, where $[A, B] = 0$, $[A, C] = -B$, $[B, C] = A$, and

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Theorem 3.6. *Let $\xi\colon \mathfrak{sol} \to \mathfrak{n}$ be a homomorphism defined in (3.1), and $\mathfrak{sol}$ be equipped with the left invariant metric defined in (2.2) or (2.3). Then,*

$$\tau_2(\xi) = \frac{1}{\nu^2}\big[\varrho bc(a^2(\varrho + \sigma - 1) - \sigma\varrho(a^2 + c^2))A + ac(\sigma c^2 + (\varrho - \sigma)(\varrho - 1)b^2)B + \\ + (\varrho - 1)ab\big(a^2(\varrho - 1) + 2\varrho c^2\big)C\big]$$

P r o o f. In [10, Theorem 5.2], we have

$$\tau(\xi) = \frac{1}{\nu}\big(-\varrho bcA + acB + (\varrho - 1)abC\big).$$

The adjoint representations of the basis elements and the tension field $\tau(\xi)$ in $\mathfrak{e}_0(2)$ are as follows:

$$\mathrm{ad}_{\tau(\xi)} = \begin{pmatrix} 0 & -\dfrac{(\varrho-1)ab}{\nu} & \dfrac{ac}{\nu} \\ \dfrac{(\varrho-1)ab}{\nu} & 0 & \dfrac{\varrho bc}{\nu} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{ad}_A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}_B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{ad}_C = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So,

$$\mathrm{ad}^*_A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\varrho & 0 \end{pmatrix}, \quad \mathrm{ad}^*_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{ad}^*_C = \begin{pmatrix} 0 & \varrho & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using formula (1.2), where $U \in \mathfrak{e}_0(2)$ and $V \in \mathfrak{sol}$, we get

$$\xi^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & \varrho b & \sigma c \end{pmatrix}.$$

Using formula (1.1), we have

$$\begin{aligned} \langle \tau_2(\xi), A\rangle_{\mathfrak{n}} &= \mathrm{tr}(\xi^* \circ (\mathrm{ad}_A + \mathrm{ad}^*_A) \circ \mathrm{ad}_{\tau(\xi)} \circ \xi) - \langle [A, \tau(\xi)], \tau(\xi)\rangle_2 \\ &= \frac{\varrho bc}{\nu^2}\big(a^2(\varrho + \sigma - 1) - \sigma\varrho(a^2 + c^2)\big), \\ \langle \tau_2(\xi), B\rangle_{\mathfrak{n}} &= \mathrm{tr}(\xi^* \circ (\mathrm{ad}_B + \mathrm{ad}^*_B) \circ \mathrm{ad}_{\tau(\xi)} \circ \xi) - \langle [B, \tau(\xi)], \tau(\xi)\rangle_2 \\ &= \frac{ac}{\nu^2}\big(\sigma c^2 + (\varrho - \sigma)(\varrho - 1)b^2\big), \end{aligned}$$

and

$$\langle \tau_2(\xi), C\rangle_{\mathfrak{n}} = \operatorname{tr}(\xi^* \circ (\operatorname{ad}_C + \operatorname{ad}_C^*) \circ \operatorname{ad}_{\tau(\xi)} \circ \xi) - \langle [C, \tau(\xi)], \tau(\xi)\rangle_2$$
$$= \frac{(\varrho - 1)ab}{\nu^2}\big(a^2(\varrho - 1) + 2\varrho c^2\big).$$

This concludes the proof. □

Corollary 3.4. *Let* $\xi\colon \mathfrak{sol} \to \mathfrak{e}_0(2)$ *be a homomorphism defined in* (3.1)*, and* $\mathfrak{sol}$ *be equipped with the left invariant metric defined in* (2.2) *or* (2.3)*. Then,* ξ *is bi-harmonic if one of the following conditions holds:*

(1) $a = 0$ *and either* $b = 0$ *or* $c = 0$;

(2) $b = 0$ *and either* $a = 0$ *or* $c = 0$;

(3) $c = 0$ *and either* $a = 0$, $b = 0$, *or* $\varrho = 1$.

Theorem 3.7. *Let* $\xi\colon \mathfrak{e}_0(2) \to \mathfrak{sol}$ *be a homomorphism defined in* (3.1)*, and* $\mathfrak{sol}$ *be equipped with the left invariant metric defined in* (2.2)*. Then,*

$$\tau_2(\xi) = \frac{c}{\nu^2}\big[(a^2-b^2)(a-\nu b)+\nu bc^2\big]X+\frac{bc}{\nu^2}\big[(a^2-b^2)(1-\nu)+\nu c^2\big]Y+\frac{2(a^2-b^2)}{\nu^2}\big[a^2+b^2+2c^2\big]Z.$$

P r o o f. In [10, Theorem 5.4], we have

$$\tau(\xi) = \frac{1}{\sigma}\big(-acX + bcY + (a^2 - b^2)Z\big).$$

The adjoint representations of the basis elements and the tension field $\tau(\xi)$ in $\mathfrak{e}_0(2)$ are as follows:

$$\operatorname{ad}_{\tau(\xi)} = \begin{pmatrix} \frac{(a^2-b^2)}{\sigma} & 0 & \frac{ac}{\sigma} \\ 0 & -\frac{(a^2-b^2)}{\sigma} & \frac{bc}{\sigma} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\operatorname{ad}_X = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \operatorname{ad}_Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \operatorname{ad}_Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So,

$$\operatorname{ad}_X^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \operatorname{ad}_Y^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad \operatorname{ad}_Z^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using formula (1.2), where $U \in \mathfrak{e}_0(2)$ and $V \in \mathfrak{sol}$, we get

$$\xi^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & b & \nu c \end{pmatrix}.$$

Using formula (1.1), a direct calculation gives us

$$\begin{aligned}
\langle \tau_2(\xi), X\rangle_{\mathfrak{n}} &= \operatorname{tr}(\xi^* \circ (\mathrm{ad}_X + \mathrm{ad}_X^*) \circ \mathrm{ad}_{\tau(\xi)} \circ \xi) - \langle [X, \tau(\xi)], \tau(\xi)\rangle_2 \\
&= \frac{c}{\nu^2}\big[(a^2-b^2)(a-\nu b) + \nu b c^2\big], \\
\langle \tau_2(\xi), Y\rangle_{\mathfrak{n}} &= \operatorname{tr}(\xi^* \circ (\mathrm{ad}_Y + \mathrm{ad}_Y^*) \circ \mathrm{ad}_{\tau(\xi)} \circ \xi) - \langle [Y, \tau(\xi)], \tau(\xi)\rangle_2 \\
&= \frac{bc}{\nu^2}\big[(a^2-b^2)(1-\nu) + \nu c^2\big], \\
\langle \tau_2(\xi), Z\rangle_{\mathfrak{n}} &= \operatorname{tr}(\xi^* \circ (\mathrm{ad}_Z + \mathrm{ad}_Z^*) \circ \mathrm{ad}_{\tau(\xi)} \circ \xi) - \langle [Z, \tau(\xi)], \tau(\xi)\rangle_2 \\
&= \frac{2(a^2-b^2)}{\nu^2}\big[a^2 + b^2 + 2c^2\big].
\end{aligned}$$

This concludes the proof. □

Corollary 3.5. *Let $\xi\colon \mathfrak{e}_0(2) \to \mathfrak{sol}$ be a homomorphism defined in (3.1), and $\mathfrak{sol}$ equipped with the left invariant metric defined in (2.2), then ξ is bi-harmonic if one of the following conditions holds:*

(1) $\nu = 1$ *and* $c = 0$, *and* $a = \pm b$;

(2) $\nu = 1$ *and* $a = b = 0$;

(3) $c = 0$ *and* $a = \pm b$.

Theorem 3.8. *Let $\xi\colon \mathfrak{e}_0(2) \to \mathfrak{sol}$ be a homomorphism defined in (3.1), and $\mathfrak{sol}$ be equipped with the left invariant metric defined in (2.3). Then,*

$$\begin{aligned}
\tau_2(\xi) &= \frac{c}{\nu^2}\big(\mu^2 b^3 + b(ba - b^2 - c^2)\mu c - (a+b)(b^2\mu + c^2\nu)\big)X \\
&+ \frac{\mu bc}{\nu^2}\big[b^2 + \mu^2 + \mu\nu(a^2+c^2) - b(a+b)\big]Y \\
&+ \frac{1}{\nu^2}\big(-b(b^3+c^3)\mu^3 + b^2\big(b(a-b) - 2c^2\big)\mu^2 + b^2(a+b)(b-2a)\mu + 3(a+b)^2c^2\big)Z.
\end{aligned}$$

P r o o f. In [10, Theorem 5.5], we recall that

$$\tau(\xi) = \frac{1}{\sigma}\big(-(a+b)cX + \mu bcY - \mu b^2 Z\big).$$

The adjoint representations of the basis elements and the tension field $\tau(\xi)$ in $\mathfrak{e}_0(2)$ are as follows:

$$\mathrm{ad}_{\tau(\xi)} = \begin{pmatrix} \dfrac{(a^2-b^2)}{\sigma} & 0 & \dfrac{ac}{\sigma} \\ 0 & -\dfrac{(a^2-b^2)}{\sigma} & \dfrac{bc}{\sigma} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{ad}_X = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}_Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{ad}_Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So,

$$\mathrm{ad}_X^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix}, \quad \mathrm{ad}_Y^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mu & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{ad}_Z^* = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -\mu & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using formula (1.2), where $U \in \mathfrak{e}_0(2)$ and $V \in \mathfrak{sol}$, we get

$$\xi^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & b & \nu c \end{pmatrix}.$$

Using formula (1.1), we obtain

$$\begin{aligned}
\langle \tau_2(\xi), X\rangle_{\mathfrak{n}} &= \operatorname{tr}(\xi^* \circ (\mathrm{ad}_X + \mathrm{ad}_X^*) \circ \mathrm{ad}_{\tau(\xi)} \circ \xi) - \langle [X, \tau(\xi)], \tau(\xi)\rangle_2 \\
&= \frac{c}{\nu^2}\big(\mu^2 b^3 + b(ba - b^2 - c^2)\mu c - (a+b)(b^2\mu + c^2\nu)\big), \\
\langle \tau_2(\xi), Y\rangle_{\mathfrak{n}} &= \operatorname{tr}(\xi^* \circ (\mathrm{ad}_Y + \mathrm{ad}_Y^*) \circ \mathrm{ad}_{\tau(\xi)} \circ \xi) - \langle [Y, \tau(\xi)], \tau(\xi)\rangle_2 \\
&= \frac{\mu bc}{\nu^2}\big(b^2 + \mu^2 + \mu\nu(a^2 + c^2) - b(a+b)\big),
\end{aligned}$$

and

$$\begin{aligned}
\langle \tau_2(\xi), Z\rangle_{\mathfrak{n}} &= \operatorname{tr}(\xi^* \circ (\mathrm{ad}_Z + \mathrm{ad}_Z^*) \circ \mathrm{ad}_{\tau(\xi)} \circ \xi) - \langle [Z, \tau(\xi)], \tau(\xi)\rangle_2 \\
&= \frac{1}{\nu^2}\big(-b(b^3 + c^3)\mu^3 + b^2\big(b(a-b) - 2c^2\big)\mu^2 + b^2(a+b)(b-2a)\mu + 3(a+b)^2c^2\big).
\end{aligned}$$

This concludes the proof. □

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Received 20.01.2025
Accepted 30.04.2025

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Citation: K. Zegga, A. Zagane, W. Merchela. Characterizing bi-harmonic homomorphisms in three-dimensional unimodular Lie groups, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, 2025, vol. 35, issue 2, pp. 198–214.

МАТЕМАТИКА 2025. Т. 35. Вып. 2. С. 198–214.

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Характеризация бигармонических гомоморфизмов в трехмерных унимодулярных группах Ли

Ключевые слова: бигармонические гомоморфизмы, унимодулярные римановы группы Ли, левоинвариантные римановы метрики.

УДК 514.7

DOI: 10.35634/vm250203

Данное исследование посвящено классификации бигармонических гомоморфизмов $\varphi\colon (G,g) \to (H,h)$, где G и H представляют связные и односвязные трехмерные унимодулярные группы Ли, а g и h обозначают левоинвариантные римановы метрики.

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Поступила в редакцию 20.01.2025
Принята к публикации 30.04.2025

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Цитирование: К. Зегга, А. Загане, В. Мерчела. Характеризация бигармонических гомоморфизмов в трехмерных унимодулярных группах Ли // Вестник Удмуртского университета. Математика. Механика. Компьютерные науки. 2025. Т. 35. Вып. 2. С. 198–214.