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ON SOME CONDITIONS FOR THE EXISTENCE OF A HOLOMORPHIC CONTINUATION OF FUNCTIONS IN A BALL

The paper considers the Cauchy–Fantappiè integral representation, which is close to the Bochner–Martinelli integral representation, and the kernel of which consists of derivatives of the fundamental solution of the Laplace equation. The aim of the work is to study the properties of this integral representation for integrable functions. Namely, the paper considers an integral (integral operator) with this kernel for integrable functions f on the boundary S of the unit ball B. Iterations of the integral of this integral operator of the order k are considered. We prove that they converge to a function holomorphic in B as $k \to \infty$.

Keywords: Bochner–Martinelli integral representation, Cauchy–Fantappiè integral representation, ball, iterations of the integral operator, holomorphic continuation of functions into a ball.

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Introduction

The method of integral representations is one of the main constructive methods in the study of holomorphic functions on several complex variables. The Bochner–Martinelli integral representation plays a particularly important role in multidimensional complex analysis (see, for example, monographs [1–4]). Its kernel is universal (independent of the type of domain) and quite simple. It has many properties of the Cauchy kernel on a complex plane, with the exception of holomorphicity. The Bochner–Martinelli integral is considered in detail in the monograph [5]. This integral is also closely related to classical potential theory (see, for example, [6]). It is shown in [5, Ch. 1] that it is an analog of the double layer potential. The Bochner–Martinelli integral plays a particularly important role in the analytic continuation of functions of various classes of smoothness (see [5, 7]).

Close to the Bochner–Martinelli representation is the Cauchy–Fantappiè integral representation, the kernel of which consists of derivatives of the fundamental solution of the Laplace equation. The aim of the work is to study the properties of this integral representation for integrable functions. Namely, the paper considers an integral (integral operator) with this kernel for integrable functions f on the boundary S of the unit ball B. Iterations of the integral of this integral operator of order k are considered. We prove that they converge to a function holomorphic in B as $k \to \infty$.

§ 1. Preliminary information

Consider the n-dimensional complex space $\mathbb{C}^n$, $n > 1$, of variables $z = (z_1, \ldots, z_n)$, $z_j = x_j + ix_{n+j}$, where x_j are real numbers, $j = 1, \ldots, n$. We introduce the module of the vector $|z| = \sqrt{|z_1|^2 + \ldots + |z_n|^2}$ and the differential forms $dz = dz_1 \wedge \ldots \wedge dz_n$ and $d\bar{z} = d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_n$, and also $dz[k] = dz_1 \wedge \ldots \wedge dz_{k-1} \wedge dz_{k+1} \wedge \ldots \wedge dz_n$. The topology in $\mathbb{C}^n$ is defined by the metric $|z - w|$.

Let B be a unit ball in $\mathbb{C}^n$ with boundary $\partial B = S$. This means that $B = \{z \in \mathbb{C}^n \colon \rho(z) < 0\}$, where $\rho(z) = |z|^2 - 1$. We introduce the 'complex' guiding cosines in the ball

$$\rho_k(z) = \frac{1}{|\mathrm{grad}\,\rho|}\frac{\partial \rho}{\partial z_k} = \bar{z}_k, \quad \rho_{\bar{k}}(z) = \frac{1}{|\mathrm{grad}\,\rho|}\frac{\partial \rho}{\partial \bar{z}_k} = z_k, \quad k = 1, \ldots, n. \tag{1.1}$$

We denote the Sobolev space, $s \in \mathbb{N}$, as $\mathcal{W}_2^s(B)$. Recall that this space consists of functions $f \in \mathcal{L}^2(B)$ for which all derivatives $\partial^\alpha f$ up to the order of s belong to $\mathcal{L}^2(B)$, where

$$\partial^\alpha f = \frac{\partial^{\|\alpha\|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n} \partial \bar{z}_1^{\alpha_{n+1}} \cdots \partial \bar{z}_n^{\alpha_{2n}}},$$

and $\alpha = (\alpha_1, \ldots, \alpha_{2n})$, $\|\alpha\| = \alpha_1 + \ldots, \alpha_{2n}$.

In the space $\mathcal{L}^2(B)$, the scalar product $(f, h)_{\mathcal{L}^2(B)}$ of functions from $\mathcal{L}^2(B)$ is given by the integral

$$(f, h)_{\mathcal{L}^2(B)} = \int_B f \cdot \bar{h} \, dv,$$

where dv is a volume element on B. Then the scalar product of the functions $f, h \in \mathcal{W}_2^s(B)$ is given by the formula

$$(f, h)_{\mathcal{W}_2^s(B)} = \sum_{\|\alpha\| \leqslant s} (\partial^\alpha f, \partial^\alpha h)_{\mathcal{L}^2(B)},$$

and the norm in $\mathcal{W}_2^s(B)$ has the form

$$\|f\|_{\mathcal{W}_2^s(B)} = \sqrt{(f, f)_{\mathcal{W}_2^s(B)}}.$$

In the space $\mathcal{L}^2(S)$ the scalar product $(f, h)_{\mathcal{L}^2(S)}$ of functions from $\mathcal{L}^2(S)$ is given by the integral

$$(f, h)_{\mathcal{L}^2(S)} = \int_S f \cdot \bar{h} \, d\sigma,$$

where $d\sigma$ is a normalized Lebesgue measure on S.

Consider the space $\mathcal{W}_2^{s+\lambda}(S)$ for $0 < \lambda \leqslant 1$. It consists of functions $f \in \mathcal{W}_2^s(S)$ for which

$$\int_S \int_S \sum_{\|\alpha\| = s} \frac{|\partial^\alpha f(z) - \partial^\alpha f(\zeta)|^2}{|\zeta - z|^{2n+2\lambda-1]}} \, d\sigma(\zeta) \, d\sigma(z) < \infty.$$

We will use the following properties of these spaces (see [8]).

1. The restriction of the function $f \in \mathcal{W}_2^s(B)$ to S belongs to the space $\mathcal{W}_2^{s-1/2}(S)$ and restriction operator is continuous.

2. If we denote the subspace of harmonic functions from $\mathcal{W}_2^s(B)$ by $\mathcal{G}_2^s(B)$, then the restriction operator from $\mathcal{G}_2^s(B)$ to $\mathcal{W}_2^{s-1/2}(S)$ is a linear topological isomorphism. And the following decomposition

$$\mathcal{W}_2^s(B) = \mathcal{G}_2^s(B) + \mathcal{N}_2^s(B)$$

is also valid where the space $\mathcal{N}_2^s(B)$ consists of functions $\mathcal{W}_2^s(B)$ equal to 0 on S.

3. Embedding theorems imply that there exists a compact continuous embedding of $\mathcal{W}_2^s(S)$ into $\mathcal{C}^k(S)$ for $s > n + k - \frac{1}{2}$.

We also recall that in the ball the Poisson's kernel for harmonic functions has the form

$$P(\zeta, z) = \frac{(n-1)!}{2\pi^n} \frac{1-|z|^2}{|\zeta - z|^{2n}}, \quad \zeta \in S, \quad z \in B.$$

Consider the Bochner–Martinelli kernel $U(\zeta, z)$, which is an exterior differential form of type $(n, n-1)$ of the form (see, for example, [5, Ch. 1], [7, Ch. 1])

$$U(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{n} (-1)^{k-1} \frac{\bar\zeta_k - \bar z_k}{|\zeta - z|^{2n}} \, d\bar\zeta[k] \wedge d\zeta.$$

This kernel plays an important role in multidimensional complex analysis (see, for example, [1–5]). It is a closed differential form with harmonic coefficients.

Let $g(\zeta, z)$ be the fundamental solution of the Laplace equation, i. e.,

$$g(\zeta, z) = -\frac{(n-2)!}{(2\pi i)^n} \frac{1}{|\zeta - z|^{2n-2}}, \quad n > 1,$$

then

$$g(\zeta, z) = -\frac{P(\zeta, z)|\zeta - z|^2}{(n-1)2^{n-1} i^n (1-|z|^2)} \tag{1.2}$$

and

$$U(\zeta, z) = \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial g}{\partial \zeta_k} \, d\bar\zeta[k] \wedge d\zeta.$$

For the function $f \in \mathcal{L}^2(S)$, we introduce the Bochner–Martinelli integral (integral operator)

$$M[f](z) = \int_S f(\zeta) U(\zeta, z), \quad z \notin S,$$

as well as the simple layer potential (integral operator)

$$\Phi[f](z) = -i^n 2^{n-1} \int_S f(\zeta) g(\zeta, z) \, d\sigma(\zeta) = \frac{(n-2)!}{2\pi^n} \int_S f(\zeta) \frac{d\sigma}{|\zeta - z|^{2n-2}}, \quad z \notin S.$$

It is clear that $M[f]$, $\Phi[f]$ are harmonic functions outside of S.

We will denote the Bochner–Martinelli integral inside the ball B by $M^+(f)$, and the Bochner–Martinelli integral outside $\bar B$ by $M^-(f)$. Similarly, we will denote the functions Φ^+ and Φ^-.

The jump theorems are well known for the Bochner–Martinelli integral (see, for example, [5, 7]). Here is one of them.

Theorem 1.1. *Let $f \in \mathcal{C}^1(S)$, then the function $M^+(f)$ extends continuously to the closure of the ball B, and the function $M^-(f)$ extends continuously to $\mathbb{C}^n \setminus B$ and the equality*

$$M^+[f] - M^-[f] = f \quad on \quad S$$

is fulfilled.

From Lemma 3.5 of [5, Ch. 1] we have

$$d\bar\zeta[k] \wedge d\zeta\big|_S = (-1)^{k-1} 2^{n-1} i^n \rho_{\bar k} d\sigma,$$

then using the guiding cosines (1.1), we get

$$U(\zeta, z)\big|_S = \frac{(n-1)!}{2\pi^n} \sum_{k=1}^{n} \frac{\zeta_k (\bar\zeta_k - \bar z_k) \, d\sigma(\zeta)}{|\zeta - z|^{2n}}.$$

If we denote $\langle \zeta, z \rangle = \zeta_1 z_1 + \cdots + \zeta_n z_n$, then

$$U(\zeta, z)\big|_S = \frac{(n-1)!}{2\pi^n} \cdot \frac{1 - \langle \zeta, \bar z \rangle}{|\zeta - z|^{2n}} \, d\sigma. \tag{1.3}$$

§ 2. Setting of the problems

In the following, we will consider one Cauchy–Fantappiè integral representation that arises when considering the following differential condition (see [5, § 23]).

Let $f \in \mathcal{C}^1(\bar{B})$ and $w(\cdot) = \sum_{k=1}^{n} w_k \dfrac{\partial\,\cdot}{\partial \bar{z}_k}$, $w_k \in \mathcal{C}^1(S)$, $k = 1, \ldots, n$, in addition, $w(\rho) \neq 0$ on S, i. e., the vector field w does not lie in the complex tangent space $T^c_z(S)$ for any point $z \in S$. Let us formulate the following problem (see [5, § 23]). (In [5, § 23] it is formulated for any bounded domain D.)

Problem 2.1. Let $f \in \mathcal{C}^1(\bar{B})$ and f be a harmonic in B. If

$$\bar{w}(f) = \sum_{k=1}^{n} \bar{w}_k \frac{\partial f}{\partial \bar{z}_k} = 0 \quad \text{on} \quad S, \tag{2.1}$$

then will f be holomorphic in B?

Unlike the tangent Cauchy–Riemann conditions, in the problem 2.1 it is required that the action of a non-tangent vector field $\bar{w}$ vanishes. This problem is an analog of the problem with an oblique derivative for real-valued harmonic functions.

If the condition $w(\rho) \neq 0$ is not satisfied, then it is easy to give an example when the condition (2.1) is fulfilled, but the function f will not be holomorphic in B (see [5, § 23]).

For some very special cases, the problem 2.1 is solved in [5, § 23].

This problem can be reformulated as follows (see [5, § 23]. Let $f \in \mathcal{C}^1(\bar{B})$ and f be harmonic in B, and the differential form

$$\mu_f = \sum_{k=1}^{n} (-1)^{n+k-1} \frac{\partial f}{\partial \bar{\zeta}_k}\, d\zeta[k] \wedge d\bar{\zeta}.$$

Problem 2.2. If

$$\mu_f\big|_S = \sum_{k>l} a_{k,l}(z)\, df \wedge d\bar{z}[l,k] \wedge dz\big|_S, \tag{2.2}$$

where $a_{k,l}$ are some smooth functions on S, $k, l = 1, \ldots, n$, and $d\bar{z}[l,k]$ is obtained from the differential form $d\bar{z}$ by throwing away the differentials $d\bar{z}_l$ and $d\bar{z}_k$, then will f be holomorphic in B?

The problem 2.2 is related to the problem of holomorphicity of functions represented by the Bochner–Martinelli integral (see [5, § 15]) (in this case, all functions $a_{kl} = 0$).

Recall Green's formula (in complex form) for the function f [5, Corollary 1.2].

Theorem 2.1 (Green's formula). *Let D be a bounded domain with a piecewise smooth boundary, the function f be harmonic in D and $f \in \mathcal{C}^1(\overline{D})$, then*

$$\int_{\partial D} f(\zeta) U(\zeta, z) - \int_{\partial D} g(\zeta, z) \mu_f = \begin{cases} f(z), & z \in D, \\ 0, & z \notin \overline{D}. \end{cases} \tag{2.3}$$

Using the equality (2.2), we get

$$f(z) = \int_S f(\zeta) U(\zeta, z) - \int_S g(\zeta, z) \sum_{k>l} a_{k,l}(\zeta)\, df \wedge d\bar{\zeta}[l,k] \wedge d\zeta, \quad z \in B.$$

Using Stokes' and Green's formulas (2.3), it is obtained in [5, § 23] that for functions $f \in \mathcal{C}^1(\bar{B})$ being harmonic in B the equality (2.2) is equivalent to the condition

$$f(z) = \int\limits_S f(\zeta)U(\zeta,z) + \int\limits_S f(\zeta)\sum_{k>l} d\big(a_{k,l}(\zeta)g(\zeta,z)\big)\wedge d\bar{\zeta}[l,k]\wedge d\zeta, \quad z\in B.$$

The second integral (integral operator) is denoted by

$$G[f](z) = \int\limits_S f(\zeta)\sum_{k>l} d\big(a_{k,l}(\zeta)g(\zeta,z)\big)\wedge d\bar{\zeta}[l,k]\wedge d\zeta, \quad z\notin S.$$

Note that the integral $G[f](z)$ is a harmonic function outside of S. In $\mathbb{C}^1$ the integral $G[f](z)=0$.

Introducing the kernel

$$W(\zeta,z) = \sum_{k>l} d\big(a_{k,l}(\zeta)g(\zeta,z)\big)\wedge d\bar{\zeta}[l,k]\wedge d\zeta,$$

we obtain that the integral representation

$$f(z) = \int\limits_S f(\zeta)\big(U(\zeta,z)+W(\zeta,z)\big), \quad z\in B, \tag{2.4}$$

is valid for holomorphic functions f.

Thus, the problem 2.2 turns into the following problem.

Problem 2.3. Let a function f of class $\mathcal{C}(\bar{B})$ satisfy the equality (2.4) in the ball B. Will f be holomorphic in B? (see [5, § 23].)

In the monograph [5, § 23], this problem is solved with a positive answer for a ball if all functions $a_{k,l}(z)$ are holomorphic. In this paper, we will study the properties of the integral with the kernel $U(\zeta,z)+W(\zeta,z)$, calculate its iterations and find their limit.

Let us denote the integral operator $M+G$ by Q

$$Q[f](z) = \int\limits_S f(\zeta)\big(U(\zeta,z)+W(\zeta,z)\big), \quad z\notin S. \tag{2.5}$$

The integral $Q[f](z)$ is a harmonic function outside of S.

We show that the integral representation (2.4) is the Cauchy–Fantappiè integral representation. Let us recall the form of the Cauchy–Fantappiè representation obtained by Leray in [9, 10] (see also the monograph [4, Ch. 1]).

Let D be a bounded domain with a smooth boundary, and a continuously differentiable vector function $\eta(\zeta,z) = \big(\eta_1(\zeta,z),\dots,\eta_n(\zeta,z)\big)$ is defined for a point $z\in D$ on ∂D such that

$$\sum_{k=1}^n (\zeta_k - z_k)\eta_k(\zeta,z) \neq 0, \quad \zeta\in\partial D.$$

Theorem 2.2. *For every function $f\in\mathcal{C}(\bar{D})$ that is holomorphic in D, it satisfies the equation*

$$f(z) = \frac{(n-1)!}{(2\pi i)^n}\int\limits_{\partial D} f(\zeta)\,\omega\big(z-\zeta,\eta(\zeta,z)\big), \quad z\in D, \tag{2.6}$$

where

$$\omega\big(\zeta - z,\eta(\zeta,z)\big) = \frac{\sum\limits_{k=1}^n (-1)^{k-1}\eta_k d\eta[k]\wedge d\zeta}{(\eta,\zeta-z)^n}.$$

Leray called the integral representation (2.6) the Cauchy–Fantappiè integral representation and its kernel — the Cauchy–Fantappiè kernel.

For $k > l$, consider the differential form

$$U(\zeta, z) + d\big(a(\zeta)g(\zeta, z)\big) \wedge d\bar{\zeta}[l, k] \wedge d\zeta$$

for some smooth function $a(\zeta)$ on S. It is also the kernel of the integral representation for holomorphic functions.

Introduce a vector function

$$\eta(\zeta, z) = (\bar{\zeta}_1 - \bar{z}_1, \dots, \bar{\zeta}_l + (-1)^{k+l}\frac{\partial(ag)}{\partial\bar{\zeta}_k} - \bar{z}_l, \dots, \bar{\zeta}_k + (-1)^{k+l-1}\frac{\partial(ag)}{\partial\bar{\zeta}_l} - \bar{z}_k, \dots, \bar{\zeta}_n - \bar{z}_n).$$

Then

$$(\eta, \zeta - z) = |\zeta - z|^2 + (-1)^{k+l}|\zeta - z|^2\left((\zeta_l - z_l)\frac{\partial a}{\partial\bar{\zeta}_k} - (\zeta_k - z_k)\frac{\partial a}{\partial\bar{\zeta}_l}\right) \neq 0$$

for sufficiently small derivatives of the function $a(\zeta)$. If, for example, the function $a(\zeta)$ is holomorphic then

$$(\eta, \zeta - z) = |\zeta - z|^2.$$

The vector function η can be normalized to the vector function η^* so that $\eta^*(\zeta - z) = |\zeta - z|^2$. Then it is clear that for the vector function η^* the Cauchy–Fantappiè kernel will coincide with the kernel

$$U(\zeta, z) + d\big(a(\zeta)g(\zeta, z)\big) \wedge d\bar{\zeta}[l, k] \wedge d\zeta.$$

For the differential form $U(\zeta, z) + W(\zeta, z)$, the reasoning is similar. Thus, we have

Lemma 2.1. *The differential form*

$$U(\zeta, z) + W(\zeta, z)$$

is the Cauchy–Fantappiè kernel.

§ 3. Auxiliary results

Let us express the Bochner–Martinelli kernel $U(\zeta, z)$ and the kernel $W(\zeta, z)$ in terms of the Poisson kernel.

Lemma 3.1. *The following equalities are true*

$$U(\zeta, z)\big|_S = \frac{1 - \langle\zeta, \bar{z}\rangle}{1 - |z|^2} P(\zeta, z)\, d\sigma(\zeta), \quad \zeta \in S, \quad z \in B,$$

and also

$$dg(\zeta, z) \wedge d\bar{\zeta}[l, k] \wedge d\zeta\big|_S = (-1)^{k+l} P(\zeta, z)\frac{\zeta_k z_l - \zeta_l z_k}{1 - |z|^2}\, d\sigma(\zeta), \quad \zeta \in S, \quad z \in B, \quad l < k.$$

P r o o f. The first equality easily follows from the form of the Poisson kernel and the formula (1.3).

Since

$$dg(\zeta,z)\wedge d\bar\zeta[l,k]\wedge d\zeta=(-1)^k\frac{\partial g}{\partial\bar\zeta_k}\,d\bar\zeta[l]\wedge d\zeta+(-1)^{l-1}\frac{\partial g}{\partial\bar\zeta_l}\,d\bar\zeta[k]\wedge d\zeta,$$

again using the guiding cosines (1.1), Lemma 3.5 from [5], and formulas for derivatives $\dfrac{\partial g(\zeta,z)}{\partial\bar\zeta_k}=\dfrac{(n-1)!(\zeta_k-z_k)}{(2\pi i)^n|\zeta-z|^{2n}}$, we get

$$dg(\zeta,z)\wedge d\bar\zeta[l,k]\wedge d\zeta\big|_S=2^{n-1}i^n(-1)^{k+l-1}\left(\frac{\partial g}{\partial\bar\zeta_k}\rho_{\bar l}-\frac{\partial g}{\partial\bar\zeta_l}\rho_{\bar k}\right)d\sigma=$$
$$=2^{n-1}i^n(-1)^{k+l}\left(\zeta_k\frac{\partial g}{\partial\bar\zeta_l}-\zeta_l\frac{\partial g}{\partial\bar\zeta_k}\right)d\sigma=\frac{(-1)^{k+l}(n-1)!}{2\pi^n}\left(\zeta_k\frac{\zeta_l-z_l}{|\zeta-z|^{2n}}-\zeta_l\frac{\zeta_k-z_k}{|\zeta-z|^{2n}}\right)d\sigma=$$
$$=\frac{(-1)^{k+l}(n-1)!}{2\pi^n}\frac{\zeta_l z_k-\zeta_k z_l}{|\zeta-z|^{2n}}\,d\sigma.$$

Thus, since $\dfrac{1}{|\zeta-z|^{2n}}=\dfrac{2\pi^nP(\zeta,z)}{(n-1)!(1-|z|^2)}$, then

$$dg(\zeta,z)\wedge d\bar\zeta[l,k]\wedge d\zeta\big|_S=(-1)^{k+l}P(\zeta,z)\cdot\frac{\zeta_l z_k-\zeta_k z_l}{1-|z|^2}\,d\sigma.\qquad\square$$

Transform the kernel $W(\zeta,z)$ as follows:

$$W(\zeta,z)=\sum_{k>l}d(a_{k,l}(\zeta)g(\zeta,z))\wedge d\bar\zeta[l,k]\wedge d\zeta=$$
$$=\sum_{k>l}a_{k,l}(\zeta)dg(\zeta,z)\wedge d\bar\zeta[l,k]\wedge d\zeta+\sum_{k>l}g(\zeta,z))da_{k,l}(\zeta)\wedge d\bar\zeta[l,k]\wedge d\zeta.\tag{3.1}$$

Then, by Lemma 3.1,

$$\sum_{k>l}a_{k,l}(\zeta)dg(\zeta,z))\wedge d\bar\zeta[l,k]\wedge d\zeta\big|_S=\sum_{k>l}(-1)^{k+l}a_{k,l}(\zeta)P(\zeta,z)\cdot\frac{\zeta_l z_k-\zeta_k z_l}{1-|z|^2}\,d\sigma.$$

At the same time, similarly to the previous step and using the formula (1.2), we get

$$\sum_{k>l}g(\zeta,z))da_{k,l}(\zeta)\wedge d\bar\zeta[l,k]\wedge d\zeta\big|_S=g(\zeta,z)\sum_{k>l}da_{k,l}(\zeta)\wedge d\bar\zeta[l,k]\wedge d\zeta\big|_S=$$
$$=g(\zeta,z)\sum_{k>l}2^{n-1}i^n(-1)^{k+l}\left(\zeta_k\frac{\partial a_{k,l}}{\partial\bar\zeta_l}-\zeta_l\frac{\partial a_{k,l}}{\partial\bar\zeta_k}\right)d\sigma=$$
$$=-\frac{P(\zeta,z)|\zeta-z|^2}{(n-1)(1-|z|^2)}\sum_{k>l}(-1)^{k+l}\left(\zeta_k\frac{\partial a_{k,l}}{\partial\bar\zeta_l}-\zeta_l\frac{\partial a_{k,l}}{\partial\bar\zeta_k}\right)d\sigma=\tag{3.2}$$
$$=P(\zeta,z)\frac{1+|z|^2-\langle\bar\zeta,z\rangle-\langle\zeta,\bar z\rangle}{(n-1)(1-|z|^2)}\sum_{k>l}(-1)^{k+l}\left(\zeta_l\frac{\partial a_{k,l}}{\partial\bar\zeta_k}-\zeta_k\frac{\partial a_{k,l}}{\partial\bar\zeta_l}\right)d\sigma.$$

So the following statement is true.

Lemma 3.2. *The restriction of the kernel $W(\zeta,z)$ onto the sphere S is expressed by the formula*

$$W(\zeta,z)\big|_S=P(\zeta,z)\sum_{k>l}(-1)^{k+l}a_{k,l}(\zeta)\frac{\zeta_l z_k-\zeta_k z_l}{1-|z|^2}\,d\sigma+$$
$$+P(\zeta,z)\frac{1+|z|^2-\langle\bar\zeta,z\rangle-\langle\zeta,\bar z\rangle>}{(n-1)(1-|z|^2)}\sum_{k>l}(-1)^{k+l}\left(\zeta_l\frac{\partial a_{k,l}}{\partial\bar\zeta_k}-\zeta_k\frac{\partial a_{k,l}}{\partial\bar\zeta_l}\right)d\sigma.$$

§ 4. Homogeneous harmonic polynomials

We write the Laplace operator Δ in the following form:

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial z_k \partial \bar{z}_k} = \frac{1}{4} \sum_{k=1}^{n} \left(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial x_{n+k}^2} \right).$$

Consider the set of homogeneous harmonic polynomials $\{P_{s,t}(z)\}$, $s, t \in \mathbb{N} \cup \{0\}$, of the form

$$P_{s,t}(z) = \sum_{\|\alpha\|=s,\ \|\beta\|=t} a_{\alpha,\beta} z^\alpha \bar{z}^\beta,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ are multi-indices, $z^\alpha = z_1^{\alpha_1} \cdot \ldots \cdot z_n^{\alpha_n}$, $\bar{z}^\beta = \bar{z}_1^{\beta_1} \cdot \ldots \cdot \bar{z}_n^{\beta_n}$ are monomials, and $\|\alpha\| = \alpha_1 + \ldots + \alpha_n$, $\|\beta\| = \beta_1 + \ldots + \beta_n$. The polynomials $P_{s,t}(z)$ are harmonic homogeneous polynomials of degree s in z and degree t in $\bar{z}$. We denote the set of all such polynomials by $\mathcal{P}_{s,t}$.

In the $\mathbb{R}^n$ space, they are discussed in sufficient detail in S. L. Sobolev's monograph [11, Ch. 11] and are called ball valve there. Here we take into account the complex structure and therefore consider the homogeneity of the polynomial separately in z and in $\bar{z}$ in accordance with [5, Ch. 1].

Since the set of harmonic polynomials is dense in $\mathcal{L}^2(S)$ (even in $\mathcal{L}^1(S)$) (see, for example, [11, Ch. 11]), $\bigcup_{s,t} \mathcal{P}_{s,t}$ is dense in $\mathcal{L}^2(S)$. Moreover, the scalar products $(P_{s,t}, P_{l,m})_{\mathcal{L}^2(S)} = 0$ if $s \neq l$ or $t \neq m$ (see, for example, [5, § 5]). Therefore, we can always choose an orthonormal basis in the space $\mathcal{L}^2(S)$ from the polynomials $P_{s,t}$.

Let us denote by $Pr_{\mathcal{O}}$ the operator of projection from the space $\mathcal{W}_2^s(B)$ onto the subspace of holomorphic functions in $\mathcal{W}_2^s(B)$. The following statement is proved in [12, 13], as well as in [5, Ch. 1].

Theorem 4.1. *For homogeneous harmonic polynomials $P_{s,t}$, the equality*

$$M[P_{s,t}](z) = \frac{n+s-1}{n+s+t-1} P_{s,t}(z)$$

holds.

Thus, the polynomials $P_{s,t}$ are eigenfunctions of the Bochner–Martinelli operator. Every rational number in the interval $(0, 1]$ is an eigenvalue of the operator M of infinite multiplicity.

Moreover, in [12] and [13] the following statement is proved.

Theorem 4.2. *The property*

$$M^k[f] \to Pr_{\mathcal{O}}[f] \ \ as \ \ k \to \infty$$

holds in the strong operator topology of the space $\mathcal{W}_2^s(B)$, $s \geqslant 1$.

In particular,

$$M^k[f] \to Pr_{\mathcal{O}}[f] \ \ \text{as} \ \ k \to \infty$$

according to the norm of space $\mathcal{W}_2^s(B)$.

Let us calculate the action of the operator G on the polynomials $P_{s,t}$. Based on the formulas (3.1) and (3.2) for the kernel W of the operator G, the action of the operator G is the sum of the actions of the operators $G_{k,l}$ of the form

$$G_{k,l}[\varphi](z) = \int_S \varphi(\zeta) dg(\zeta, z) \wedge d\bar{\zeta}[l, k] \wedge d\zeta$$

and the operator Φ.

Consider the integral (integral operator)

$$G_{k,l}[P_{s,t}](z) = \int\limits_S P_{s,t}(\zeta)\, dg(\zeta, z) \wedge d\bar{\zeta}[l,k] \wedge d\zeta.$$

Obviously, it is a harmonic function outside of S. Then, from Lemma 3.1, we get

$$G_{k,l}[P_{s,t}](z) = (-1)^{k+l} \int\limits_S P_{s,t}(\zeta) P(\zeta, z) \frac{\zeta_l z_k - \zeta_k z_l}{1-|z|^2}\, d\sigma(\zeta), \quad z \in B.$$

To calculate this integral, we need to harmonically extend the functions $\zeta_k P_{s,t}(\zeta)$ into the ball B. It is not difficult to verify that this continuation is given by the function

$$h(\zeta) = \zeta_k P_{s,t}(\zeta) + \frac{1-|\zeta|^2}{n+s+t-1} \frac{\partial P_{s,t}(\zeta)}{\partial \bar{\zeta}_k}. \tag{4.1}$$

Indeed,

$$\Delta h = \sum_{m=1}^{n} \frac{\partial^2 h}{\partial \zeta_m \partial \bar{\zeta}_m} =$$
$$= \frac{\partial P_{s,t}}{\partial \bar{\zeta}_k} - \frac{n}{n+s+t-1} \frac{\partial P_{s,t}}{\partial \bar{\zeta}_k} - \frac{1}{n+s+t-1} \sum_{m=1}^{n} \zeta_m \frac{\partial}{\partial \zeta_m} \left(\frac{\partial P_{s,t}}{\partial \bar{\zeta}_k} \right) -$$
$$- \frac{1}{n+s+t-1} \sum_{m=1}^{n} \bar{\zeta}_m \frac{\partial}{\partial \bar{\zeta}_m} \left(\frac{\partial P_{s,t}}{\partial \bar{\zeta}_k} \right) =$$
$$= \frac{\partial P_{s,t}}{\partial \bar{\zeta}_k} - \frac{n}{n+s+t-1} \frac{\partial P_{s,t}}{\partial \bar{\zeta}_k} - \frac{s}{n+s+t-1} \frac{\partial P_{s,t}}{\partial \bar{\zeta}_k} - \frac{t-1}{n+s+t-1} \frac{\partial P_{s,t}}{\partial \bar{\zeta}_k} = 0.$$

Here we used the formula

$$\Delta(u \cdot v) = v \cdot \Delta u + u \cdot \Delta v + \sum_{m=1}^{n} \left(\frac{\partial u}{\partial \bar{z}_m} \cdot \frac{\partial v}{\partial z_m} + \frac{\partial u}{\partial z_m} \cdot \frac{\partial v}{\partial \bar{z}_m} \right),$$

and also the homogeneity of the polynomials $P_{s,t}$ in ζ and $\bar{\zeta}$.

Thus, we have

$$G_{k,l}[P_{s,t}](z) = \frac{(-1)^{k+l}}{1-|z|^2} \int\limits_S P(\zeta, z) \Bigg((\zeta_l z_k - \zeta_k z_l) P_{s,t}(\zeta) +$$
$$+ \frac{(1-|\zeta|^2) z_k}{n+s+t-1} \frac{\partial P_{s,t}(\zeta)}{\partial \bar{\zeta}_l} - \frac{(1-|\zeta|^2) z_l}{n+s+t-1} \frac{\partial P_{s,t}(\zeta)}{\partial \bar{\zeta}_k} \Bigg) d\sigma =$$
$$= \frac{(-1)^{k+l}}{1-|z|^2} \cdot \frac{(1-|z|^2)}{n+s+t-1} \left(z_k \frac{\partial P_{s,t}(z)}{\partial \bar{\zeta}_l} - z_l \frac{\partial P_{s,t}(z)}{\partial \bar{\zeta}_k} \right) =$$
$$= \frac{(-1)^{k+l}}{n+s+t-1} \left(z_k \frac{\partial P_{s,t}(z)}{\partial \bar{\zeta}_l} - z_l \frac{\partial P_{s,t}(z)}{\partial \bar{\zeta}_k} \right).$$

Hence, we get the statement.

Lemma 4.1. *The equality*

$$G_{k,l}[P_{s,t}](z) = \frac{(-1)^{k+l}}{n+s+t-1}\left(z_k \frac{\partial P_{s,t}(z)}{\partial \bar{\zeta}_l} - z_l \frac{\partial P_{s,t}(z)}{\partial \bar{\zeta}_k}\right).$$

holds.

Note that on the right side of this equation there is a homogeneous harmonic polynomial of degree $s+1$ in z and degree $t-1$ in $\bar{z}$.

If we consistently apply the operator $G_{k,l}$ to the polynomials $P_{s,t}(z)$ (i. e., consider iterations of the operator $G_{k,l}$), we get that $G^q_{k,l}[P_{s,t}] = 0$ for $q > t$, and for $q = t$ the iteration $G^t_{k,l}[P_{s,t}]$ is a holomorphic polynomial of degree $s+t$.

Now calculate $\Phi[P_{s,t}]$. Using the formula (1.2), we get

$$\Phi[P_{s,t}](z) = -\int_S P_{s,t}(\zeta) g(\zeta, z)\, d\sigma(\zeta) =$$
$$= \frac{1}{(n-1)(1-|z|^2)} \int_S P_{s,t}(\zeta) P(\zeta, z) |\zeta - z|^2\, d\sigma(\zeta) =$$
$$= \frac{1}{(n-1)(1-|z|^2)} \int_S P_{s,t}(\zeta) \cdot P(\zeta, z)(1 + |z|^2 - \langle \zeta, \bar{z}\rangle - \langle \bar{\zeta}, z\rangle)\, d\sigma.$$

Applying the formula (4.1) and a similar one for conjugate derivatives, we obtain that

$$\Phi[P_{s,t}](z) = \frac{1}{(n-1)(1-|z|^2)} \Bigg[\big(1+|z|^2\big) P_{s,t}(z) - 2|z|^2 P_{s,t}(z) -$$
$$- \frac{1-|z|^2}{n+t+s-1} \sum_{k=1}^{n} \left(z_k \frac{\partial P_{s,t}(z)}{\partial z_k} + \bar{z}_k \frac{\partial P_{s,t}(z)}{\partial \bar{z}_k} \right) \Bigg] =$$
$$= \frac{1}{(n-1)(1-|z|^2)} \left((1-|z^2|) P_{s,t}(z) - (1-|z|^2) \frac{s+t}{n+s+t-1} P_{s,t}(z) \right) =$$
$$= \frac{1}{(n+s+t-1)} P_{s,t}(z).$$

Thus, we obtain the statement

Lemma 4.2. *The simple layer potential is calculated by the formula*

$$\Phi[P_{s,t}](z) = \frac{1}{(n+s+t-1)} P_{s,t}(z).$$

So the homogeneous harmonic polynomials $P_{s,t}$ are eigenfunctions of the operator Φ.

Proposition 4.1. *For functions $f \in \mathcal{W}^l_2(B)$, the sequence $\Phi^k[f] \to 0$ according to the norm of space $\mathcal{W}^l_2(B)$ as $k \to \infty$.*

§ 5. Main results

Consider the integral

$$Q[f](z) = M[f](z) + G[f](z), \quad z \in B.$$

Using Lemmas 3.1 and 3.2, we obtain that

$$Q[f](z) =$$
$$= \int_S f(\zeta)\frac{1-\langle\zeta,\bar z\rangle}{1-|z|^2}P(\zeta,z)\,d\sigma(\zeta) + \int_S f(\zeta)P(\zeta,z)\sum_{k>l}(-1)^{k+l}a_{k,l}(\zeta)\frac{\zeta_l z_k-\zeta_k z_l}{1-|z|^2}\,d\sigma +$$
$$+\int_S f(\zeta)P(\zeta,z)\frac{1+|z|^2-\langle\bar\zeta,z\rangle-\langle\zeta,\bar z\rangle}{(n-1)(1-|z|^2)}\sum_{k>l}(-1)^{k+l}\left(\zeta_l\frac{\partial a_{k,l}}{\partial\bar\zeta_k}-\zeta_k\frac{\partial a_{k,l}}{\partial\bar\zeta_l}\right)d\sigma.$$

For holomorphic functions f, we obtain that $Q[f]=f$.

Let now the function $f\in\mathcal{W}_2^1(B)$ be non-holomorphic, i. e., we assume that $t>0$ in the decomposition of the function f into a series of polynomials $P_{s,t}$. Decomposing the integrand functions into series of polynomials $P_{s,t}$, we obtain

$$f(\zeta)=\sum_{s,t}P_{s,t}(\zeta),\qquad f(\zeta)a_{k,l}(\zeta)=\sum_{s,t}P^{k,l}_{s,t}(\zeta),$$
$$f(\zeta)\sum_{k>l}(-1)^{k+l}\left(\zeta_l\frac{\partial a_{k,l}}{\partial\bar\zeta_k}-\zeta_k\frac{\partial a_{k,l}}{\partial\bar\zeta_l}\right)=\sum_{s,t}\tilde P_{s,t}(\zeta).$$

Moreover, there are no holomorphic components in these series, i. e., all $t>0$.

From Theorem 4.1 and Lemmas 4.1 and 4.2, we obtain

$$Q[f](z)=\sum_{s,t}\left(\frac{n+s-1}{n+s+t-1}P_{s,t}(z)+\sum_{k>l}\frac{(-1)^{k+l}}{n+s+t-1}\left(z_k\frac{\partial P^{k,l}_{s,t}(z)}{\partial\bar z_l}-z_l\frac{\partial P^{k,l}_{s,t}(z)}{\partial\bar z_k}\right)-\right.$$
$$\left.-\frac{1}{(n+s+t-1)i^n2^{n-1}}\tilde P_{s,t}(z)\right). \qquad (5.1)$$

In particular, let $f=P_{s,t}$, and $a_{k,l}(\zeta)$ be homogeneous harmonic polynomials of some degrees, then their product (of the total degree m) by S can be represented by the Gauss formula (see, for example, [11, Ch. 11]) as a restriction by S of a homogeneous harmonic polynomial of the form

$$P_{s,t}(z)\cdot a_{k,l}(z)=\psi(z)=\sum_{p\geqslant 0}Z_{m-2p}(z),\qquad z\in S,$$

where

$$Z_{m-2p}(z)=\frac{(m-2p+n-1)}{p!(m+n-p-1)!}\sum_{j\geqslant 0}(-1)^j\frac{(m-j-2p+n-2)!}{j!}\cdot|z|^{2j}\Delta^{j+p}\psi(z).$$

A similar formula is also valid for the expression

$$P_{s,t}(z)\sum_{k>l}(-1)^{k+l}\left(z_l\frac{\partial a_{k,l}}{\partial\bar z_k}-\zeta_k\frac{\partial a_{k,l}}{\partial\bar\zeta_l}\right).$$

It should be noted here that the total degree of this polynomial is still equal to m, and the expression

$$\sum_{k>l}(-1)^{k+l}\left(z_l\frac{\partial a_{k,l}}{\partial\bar z_k}-\zeta_k\frac{\partial a_{k,l}}{\partial\bar\zeta_l}\right)$$

is a homogeneous harmonic polynomial.

Thus, the formula (5.1) applied to the function $P_{s,t}$ shows that $Q[P_{s,t}](z)$ is the sum of the same number of homogeneous harmonic polynomials of the previous form multiplied by a number strictly less than 1.

If we consistently apply the operator Q to the polynomial $P_{s,t}$ (i. e., considering the iterations $Q^m[P_{s,t}]$, $t > 0$), we get that $Q^m[P_{s,t}] \to 0$ as $l \to \infty$, since the coefficients of the resulting polynomials tend to zero.

If $a_{k,l}(z)$ are not homogeneous harmonic polynomials, then we can decompose them into a system of functions $\{P_{s,t}\}$ and apply the resulting statements to each term.

Therefore, the property is valid.

Lemma 5.1. *The equality*

$$Q^l[P_{s,0}](z) = P_{s,0}(z)$$

holds for the polynomial $P_{s,0}$, and if $t > 0$, then $Q^l[P_{s,t}](z) \to 0$ as $l \to \infty$ in the metric $\mathcal{G}_2^s(B)$.

From Lemma 5.1, we obtain the statements.

Theorem 5.1. *Let function $f \in \mathcal{G}_2^s(B)$. The properties are true:*

(1) $Q^m[f] = f$ *for any* $m \in \mathbb{N}$, *if* f *is holomorphic in* B;

(2) $Q^m[f] \to 0$ *as* $m \to \infty$, *if the decomposition* f *has all* $P_{s,0} = 0$.

Theorem 5.2. *For function $f \in \mathcal{G}_2^s(B)$, the property*

$$Q^m[f] \to Pr_{\mathcal{O}}[f] \quad \text{as} \quad m \to \infty$$

holds in the topology of space $\mathcal{G}_2^s(B)$.

As a consequence, we have

Corollary 5.1. *Let B be a unit ball, the function $f \in \mathcal{G}_2^s(B)$, the functions $a_{l,k} \in \mathcal{C}^1(S)$, $k, l = 1, \ldots, n$, and the condition* (2.4) *be fulfilled in the ball B* (*i. e., $Q[f] = f$ in B*)*. Then, the function f is holomorphic in B.*

P r o o f. Since $Q[f] = f$ in B, then $Q^2[f] = Q[Q[f]] = Q[f]$, etc. We obtain that $Q^k[f] = f$ in B. On the other hand, by Theorem 5.2, iterations of $Q^k[f]$ tend to the holomorphic function $\tilde{f}$ in B. □

Thus, Problem 2.3 is solved positively.

Corollary 5.2. *Let B be a unit ball, the function $f \in \mathcal{C}(S)$, the functions $a_{l,k} \in \mathcal{C}^1(S)$, $k, l = 1, \ldots, n$, and the condition*

$$Q[f](z) = 0 \ \text{ outside the ball } \ \bar{B},$$

be fulfilled. Then, the function f extends holomorphically into B as the function $F \in \mathcal{C}(\bar{B})$.

P r o o f. From the formula (2.5), we have

$$Q[f](z) = M[f](z) + G[f](z) = \int_S f(\zeta)\big(U(\zeta, z) + W(\zeta, z)\big), \quad z \notin S.$$

The jump of the integral $M[f]$ is equal to f (see Theorem 1.1), i. e., $M^+[f](z)-M^-[f](z) = f(z)$ on S. The jump of the integral $G[f]$ is 0, since it is the potential of a simple layer (see, for example, [6]).

From this, we get that if $Q^-[f](z) = 0$ outside the closure of the ball B, then $Q^+[f](z)$ is the harmonic continuation of the function f in the ball B. By the previous corollary, $Q^+[f](z)$ is holomorphic in B. □

Corollary 5.3. *Let B be a unit ball, the function $f \in \mathcal{C}(\bar{B})$, the functions $a_{l,k} \in \mathcal{C}^1(S)$, $k, l = 1, \ldots, n$ and for some natural number s_0 the condition*

$$Q^{s_0}[f](z) = f(z), \quad z \in B,$$

be fulfilled. Then, f is holomorphic in B.

P r o o f. Consider the iterations of $Q^{ks_0}[f](z)$. By Theorem 5.2, they tend to a holomorphic function in B. On the other hand, they are all equal to $f(z)$. □

In conclusion, we present the theorem.

Theorem 5.3. *Let $f \in \mathcal{C}^1(\bar{B})$ and f be harmonic in B, $w(\rho) \neq 0$ on ∂B. If*

$$\bar{w}(f) = \sum_{k=1}^{n} \bar{w}_k \frac{\partial f}{\partial \bar{z}_k} = 0 \ \text{ on } \ \partial B,$$

then f is holomorphic in B.

Thus, Problem 2.1 is solved positively for smooth functions.

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О некоторых условиях существования голоморфного продолжения функций в шар

Ключевые слова: интегральное представление Бохнера–Мартинелли, интегральное представление Коши–Фантаппье, шар, итерации интегрального оператора, голоморфное продолжение функций в шар.

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В работе рассмотрено близкое к интегральному представлению Бохнера–Мартинелли интегральное представление Коши–Фантаппье, ядро которого состоит из производных фундаментального решения уравнения Лапласа. Целью работы является исследование свойств этого интегрального представления для интегрируемых функций. А именно, в работе рассматривается интеграл (интегральный оператор) с этим ядром для интегрируемых функций f на границе S единичного шара B. Рассмотрены итерации интеграла данного интегрального оператора порядка k. Доказано, что они сходятся к функции, голоморфной в B, при $k \to \infty$.

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