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THE VERTEX LOCAL ANTIMAGICNESS FOR KNÖDEL AND FIBONACCI GRAPHS

In this article, we present the vertex local anti-magic chromatic number for some Knödel graphs $\mathcal{G}$ and Fibonacci graphs; disjoint union of Knödel graphs; and the join graphs $\mathcal{G}\vee\mathcal{H}$, where $\mathcal{H} \in \{O_s, K_s, C_s, K_{s,\ell}\}$.

Keywords: vertex local anti-magic labeling, complete graphs, trivial graph, complete bipartite graphs, Knödel graphs, Fibonacci graphs, join graphs.

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Introduction

In this work, G is a finite and simple graph. Let P_s, C_s, K_s and O_s be a path, a cycle, a complete graph and a trivial graph on s vertices respectively. Let $K_{r,s}$ be a complete bipartite graph of order $r + s$. A magic square involves arranging the integers in a square so that the sums of each row, column and diagonal are equal. The word anti-magic refers to the graphs relationship with magic labelings and magic squares. The terms in the context of graph theory, with more specific notions such as anti-magic labeling and local anti-magic labeling, particularly refer to certain labelings of the edges or vertices in a graph. Both concepts define the assignment of labels, usually integers, on the edges or vertices in such a way that some conditions are met, while they are different in details and restrictions.

Anti-magic labeling only requires that the sum of the labels incident to each vertex be distinct across all vertices.

The local anti-magic labeling requires that the sums of the edge labels incident to adjacent vertices be distinct from one another, adding a condition on adjacent vertices.

The local anti-magic labeling is a more restrictive property and may not be possible for certain graphs that could otherwise have an anti-magic labeling.

Let $f\colon E(G) \to \{1, 2, \ldots, |E(G)|\}$ be a bijection. For $y \in V(G)$, the weight $wt(y) = \sum_{e\in E(y)} f(e)$, where $E(y)$ is the set of edges incident to y. If $wt(y) \neq wt(z)$ for any two different vertices y and $z \in V(G)$, then f is called an *anti-magic labeling* [1] of G. It was introduced, in [1], by Hartsfield and Ringel who posed the conjecture: *any connected graph* $G(\neq K_2)$ *is anti-magic*. Also, they showed that cycles, paths, complete graphs, and wheels are anti-magic. In [2], Cranston obtained anti-magic labeling for $s(> 3)$-regular bipartite graph by using matchings of graphs. Zhang and Sun [3] provided that: if G is an anti-magic regular graph, then the Cartesian product $G\square H$ admits anti-magic. The recent survey of graph labeling can be found in [4].

Vertex local anti-magic coloring (in short VLAC) is addressed in graph theory to investigate and understand specific aspects of graphs, specifically those related to magic and antimagic labeling. A bijection $f\colon E(G) \to \{1, 2, \ldots, |E(G)|\}$ such that the induced vertex labeling $f^+\colon V \to \mathbb{N}$, given by $f^+(y) = \sum f(e)$, has the property that any pair of adjecent vertices have different colors. The number of different induced vertex colors with f is represented by $c(f)$, and is called the color number of f. The *vertex local anti-magic chromatic number of* G, denoted by $\chi_{v\ell a}(G)$, is min $\{c(f)\colon f$ is a *vertex local anti-magic coloring* of $G\}$ [5]. We can see many variations of anti-magic labeling of a graph G. One of the variations is the vertex local anti-magic chromatic number of G introduced by Arumugam et al. in [5], also they determined the specific value of vertex local anti-magic chromatic number of paths, cycles, wheels, complete graphs and bipartite graphs. Also, they posed the following conjectures.

Conjecture 0.1 (see [5]). All connected graphs except K_2 are vertex local anti-magic.

Remark 0.1 (see [5]). For any graph G, $\chi_{v\ell a}(G) \geq \chi(G)$, where $\chi(G)$ is the chromatic number of G.

Local anti-magic coloring is a concept in graph theory that helps understand properties of graphs and their structures through unique labeling schemes. It has practical applications in graph design, coding theory, and communication graphs. The study of labelings presents mathematical challenges, and researchers often pursue the existence or non-existence of such labelings for different graph classes. It can be used for scheduling tasks or resources, preventing conflicts and optimizing resource allocation. This also aids in error detection and correction in coding theory. Combinatorial optimization problems can also relate to local anti-magic labeling. For any integer $x < y$, $[x, y]$ represent the set of integers between x and y.

In 2017, Arumugam et al. [5] investigated the vertex local anti-magic chromatic number for a complete bipartite graph $K_{r,s}$, whenever $r, s \geq 2$ and $r \equiv s \pmod 2$ and they gave problems for leftover cases. Also, they proved lower bound for every tree T with s leaves, $\chi_{v\ell a}(T) \geq s+1$.

In 2020, Lau et al. [6] completely calculated the vertex local anti-magic chromatic number for complete bipartite graphs.

Theorem 0.1 (see [5,6]). *For $s \geq r \geq 1$ and $s \geq 2$,*

$$\chi_{v\ell a}(K_{r,s}) = \begin{cases} s+1, & \text{if } s > r = 1, \\ 2, & \text{if } s > r \geq 2 \text{ and } r \equiv s \pmod 2, \\ 3, & \text{otherwise.} \end{cases}$$

In 2020, Premalatha et al. [9] showed that $\chi_{v\ell a}(T) = s+2$, T is a tree with s pendent vertices, and posed the following

Problem 0.1 (see [9]). Characterize trees T with s leaves, $\chi_{v\ell a}(T) = s+1$ or $s+2$.

Very recently, Lau et al. [10] gave partial solutions of the above problem ℓ-leg spider graph has $\ell + 1 \leq \chi_{v\ell a} \leq \ell + 2$. Also, they provided a partial answer to the classification of s-pendant trees T with $\chi_{v\ell a}(T)$ equaling either $s+1$ or $s+2$. Later, Baca et al. [11] verified the above problem for every complete full t-ary tree of s leaves having vertex local anti-magic chromatic number $s+1$. For some of the trees discussed by various authors, the vertex local anti-magic chromatic number was either $s+1$ or $s+2$. Also, the characterization of the vertex local anti-magic chromatic number of trees is still open.

In 2023, Lau et al. [7] found the vertex local anti-magic chromatic number of even regular circulant bipartite graphs join with cycle and trivial graph. Also, they posed the following conjecture and verified its partial results:

Conjecture 0.2. Let H be a circulant graph.

(a) If $s \geq 1$, then $\chi_{v\ell a}(H \vee O_s) = \chi(H) + 1$.

(b) If $s \geq 3$, then $\chi_{v\ell a}(H \vee C_s) = \chi(H) + \chi(C_s)$.

In 2023, Lau and Shiu [8] gave a sufficient condition for a graph with one pendant to have $\chi_{v\ell a}(G) \geq 3$. Also, they proved Theorem 0.2, considered the circulant graph of order $2r$ with odd lengths $1, \ell_1, \ell_2, \ldots, \ell_t$, where $\gcd(\ell_i, 2r) = 1$, $1 \leq i \leq t$.

Theorem 0.2 (see [8]). *Let $1 < \ell_1, \ell_2, \ldots, \ell_s < r$, and, for every integer $j \in [1, s]$, $\gcd(\ell_j, 2r) = 1$. Then, $\chi_{v\ell a}(C(2r; \{1, \ell_1, \ldots, \ell_s\})) = 3$.*

Moreover, they discussed bipartite and tripartite graphs G with $\chi_{v\ell a}(G) = 3$. In addition, they provided some interesting open problems:

Problem 0.2 (see [8]). Verify the vertex local anti-magic chromatic number for

$$C_r(1, \ell_1, \ell_2, \ldots, \ell_s),$$

where r is odd.

Problem 0.3 (see [8]). Identify the vertex local anti-magic chromatic number for bipartite graphs equaling 2 or 3.

Problem 0.4 (see [8]). Identify the vertex local anti-magic chromatic number for complete tripartite graphs equaling 3 or 4.

In 2024, Uma and Rajasekaran [12] proved the Theorem 0.3, which is a generalization of Theorem 0.2 [8]. In Theorem 0.3, the circulant graph of order $2r$ with arbitrary odd lengths ℓ_1, $\ell_2, \ldots, \ell_s$ ($\ell_1 < \ell_2 < \cdots < \ell_s$) is considered, where $\gcd(\ell_i, 2r)$ is either 1 or d.

Theorem 0.3 (see [12]). *Let $\ell_1 < \ell_2 < \ldots < \ell_s < r$, and, for every integer $j \in [1, s]$, $\gcd(\ell_j, 2r) = 1$ or d. Then, $\chi_{v\ell a}(C(2r; \{\ell_1, \ell_2, \ldots, \ell_s\})) = 3$.*

Various authors discussed some bipartite graphs whose vertex local anti-magic chromatic number is either 2 or 3. Also, the characterization of the vertex local anti-magic chromatic number of bipartite graphs remains open. Hence, in this article, we are interested in calculating the vertex local anti-magic chromatic number of some special bipartite graphs as Knödel and the Fibonacci graphs which are non isomorphic to any other bipartite graphs.

Since the birth of the Internet, our world has become a global village where almost all commercial, social, private, public, and research and development networks fall under the umbrella of the Internet. Fast and reliable dissemination of information is the central issue of all types of real networks, such as ad-hoc, wireless, satellite communications, supercomputers, Internet, cloud-based infrastructure. Much effort, money, and time have been spent on improving the information dissemination. Two ways exist to resolve this issue: compressing the amount of data that is transferred and minimizing delay in the transmission of information. The approaches found to be received well for using the latter methodology either design algorithms that are efficient or establish robust network architectures at an optimal level of information diffusion time. Network architecture means the logical as well as the structural layout of the network. Regular network architectures provide the platform to implement powerful algorithms related to routing, broadcasting and parallel as well as distributed computing [13].

Compared to all of the networks, Knödel graph is the only network that can be designed for any even number of nodes. Moreover, the degree of every node in Knödel graph on n nodes can be any value between 2 and $\lfloor \log_2(n) \rfloor$. When the degree of Knödel graph is 2, then it becomes the well-known cycle. When the degree is equal to $\lfloor \log_2(n) \rfloor$, then Knödel graph is a broadcast and gossip graph, in which the main communication tasks can be performed, theoretically in minimum possible time. The above properties make the Knödel graph the largest possible unique interconnection network, which could be sparse (when degree is constant) or dense (when degree is logarithmic of n). This way Knödel graph can be suitable for all possible applications based on communication time, network design or implementation cost.

The Knödel graph $W_{\Delta,n}$ is a regular graph of even order n and degree Δ, where $1 \leq \Delta \leq \lfloor \log_2(n) \rfloor$. It was introduced by Austrian mathematician Walter Knödel for $\Delta = \lfloor \log_2(n) \rfloor$, in 1975 and was used in an optimal gossiping algorithm [14]. This graph is considered as rather unique because it is the smallest gossip graph. The main properties that need to be considered when dealing with the Knödel graph is gossiping and broadcasting. Start with gossiping, if there is a group of n persons gossiping, then every single person has a chunk of information and tries to communicate it to other with the help of binary cells and if every call lasts a specified period of time, and the final aim is to find in total how much time is taken before every involved person knows the entire information.

In a similar way, when the data is broadcast at time $t = 0$, 2^0 person has the data; at time $t = 1$, 2^1 people know the data; and at time $t = 2$, 2^2 people know the data. Finally, total of n people in the system have the needed least time. In this situation, individuals are denoted to as vertices. Knödel proposed a two-way mode that allows two nodes in a call to exchange information in a single round. David et al. [16] studied a one-way mode, where information can only go in one direction during a call between x and y. This means that only y can get information from x or x can receive information from y but not both. In [14], it was proved that gossiping in a two-way mode requires at least $\lceil \log_2(n) \rceil$ rounds for even n. However, Knödel graphs permit for gossiping in $\lceil \log_2(n) \rceil$ rounds. Similarly, it was proved in [16] that, for even n, gossiping cannot be executed in less than $\lceil \log_\rho(n) \rceil$ rounds in the one-way mode, where $\rho = (1 + \sqrt{5})/2$. However, there are graphs, known as Fibonacci graphs, that permit gossiping to be executed in that number of rounds.

For smaller Δ, the family of Knödel graphs has been defined formally by Fraigniaud and Peters [15]. Since 1994 a lot of research has been done on Knödel graphs, especially because some subfamilies of the Knödel graph tend to have good properties in terms of broadcasting and gossiping [17]. Many graphs introduced as minimum broadcast (resp. gossip) graphs, such as in [18–20], were in fact isomorphic to Knödel graphs [21].

Knödel graphs [17] and *Fibonacci graphs* [22] are bipartite graphs G of $2n$ vertices. Each partition has n vertices labeled from 0 to $n - 1$. The *Knödel graph* on $n \geq 2$ vertices and of maximum degree $1 \leq \Delta \leq \lfloor \log_2(n) \rfloor$, is represented by $W_{\Delta,n}$. The vertices of $W_{\Delta,n}$ are the pairs (i, j), for $i = 1, 2$ and $0 \leq j \leq (n - 2)/2$. For each j, $0 \leq j \leq (n - 2)/2$, there is an edge between vertex $(1, j)$ and each vertex $(2, (j + 2^k - 1) \pmod{n/2})$, for $k = 0, 1, \ldots, \Delta - 1$.

The *Fibonacci graph* [22], on $n \geq 2$, and of maximum degree Δ, where $1 \leq \Delta \leq k$; where $k = F^{-1}(n) - 1$, is represented by $F_{\Delta,n}$. The vertices of $F_{\Delta,n}$ are couples (i, j), for $i = 1, 2$ and $0 \leq j \leq (n - 2)/2$. For every j, $0 \leq j \leq (n - 2)/2$, there is an edge between vertex $(1, j)$ and each vertex $(2, (j + F(k + 1) - 1) \pmod{n/2})$, for $k = 0, 1, 2, \ldots, \Delta - 1$, where $F(k)$ denotes the kth Fibonacci number ($F(0) = F(1) = 1$, and $F(k) = F(k - 1) + F(k - 2)$ for $k \geq 2$ and $F^{-1}(n) - 1$ represents the number k for which $F(k) \leq n < F(k + 1)$). See Fig. 1 and Fig. 2, they show the examples of a Knödel graph and a Fibonacci graph. Specifically, there exist graphs that are not isomorphic to Knödel graph (resp. Fibonacci graph), and permit gossiping in the 2-way mode (resp. 1-way mode) in an optimal manner.

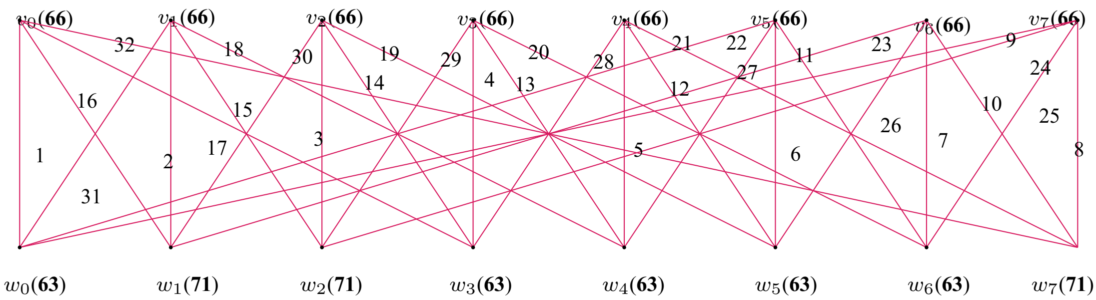

Fig. 1. $\chi_{v\ell a}(W_{4,16}) = 3$

The *join graph* $G \vee H$ [23] of two graphs G and H, is defined as follows: $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{xy\colon x \in V(G),\ y \in V(H)\}$. For more results

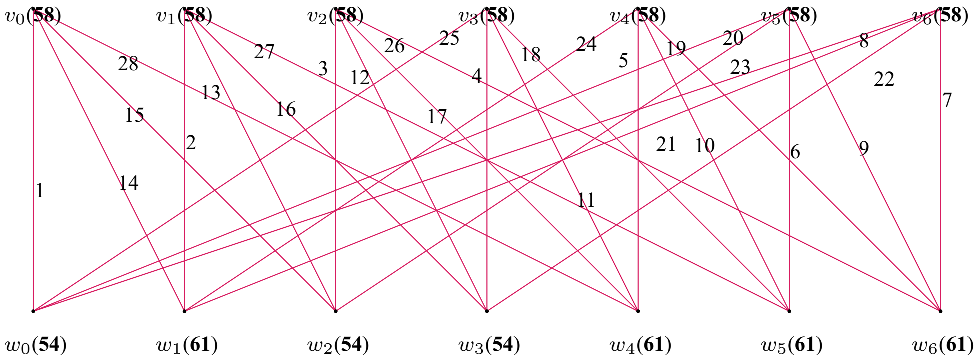

Fig. 2. $\chi_{v\ell a}(F_{4,14}) = 3$

on the vertex local anti-magic chromatic number of join graphs, see refs. [24–27]. Terminologies and notations are not described here and can be found in [23].

The results of this article support Conjecture 0.1, Problem 0.3 and Conjecture 0.2. In the following Sections we concentrate on the vertex local anti-magic chromatic number of Knödel graphs and Fibonacci graphs. Also, we provide the vertex local anti-magic chromatic number of disjoint union Knödel graphs. Additionally, we discuss the vertex local anti-magic chromatic number of some join graphs. The Theorems listed below will be applied.

Theorem 0.4 (see [27]). *Let H be a graph of size e. Assume there is a vertex local anti-magic chromatic number of H inducing a two-coloring of H with colors a and b where $a < b$. Let A and B be the numbers of vertices of colors a and b, respectively. Then, H is a bipartite graph whose sizes of parts are A and B with $A > B$, and $Aa = bB = e(e + 1)/2$.*

Theorem 0.5 (see [26]). *Let G be a graph of order p and size e. Let $s \geq 2$ and $p \equiv s \pmod 2$. Suppose G admits a vertex local anti-magic k-coloring h. Then, $\chi_{v\ell a}(G \vee O_s) \leq k + 1$ if either $s - p \geq 0$ or $p - s \geq 2$ and $2h^+(u) \neq s(p^2 - ps - 2e - 1) + p(1 - s^2)$ for every $u \in V(G)$.*

§ 1. Results

For simplicity, we re-lable the vertices of a Knödel graph as follows: the vertices $\{(1, 0), (1, 1), \ldots, (1, (n - 2)/2)\}$ are labelled as $V = \{v_0, v_1, \ldots, v_{(n-2)/2}\}$, while the other vertices $\{(2, 0), (2, 1), \ldots, (2, (n - 2)/2)\}$ are labelled as $W = \{w_0, w_1, \ldots, w_{(n-2)/2}\}$. Then two vertices of v_i and w_j are adjacent $\Leftrightarrow j \in \{i + 2^0 - 1, i + 2^1 - 1, \ldots, i + 2^{d-1} - 1\}$ $(j - i \in \{2^0 - 1, 2^1 - 1, \ldots, 2^{d-1} - 1\})$. Here $V \cup W$ is the vertex set of $W_{d,n}$ and we use subscript addition modulo $n/2$. Clearly, $W_{2,4} \cong C_4$ and $W_{2,6} \cong C_6$. Since $s \geq 3, \chi_{v\ell a}(C_s) = 3$ [5]. Hence, $\chi_{v\ell a}(W_{2,4}) = \chi_{v\ell a}(W_{2,6}) = 3$.

For $i \in \{0, 1, \ldots, 2^{d-1} - 1\}$, *if $d \geq 2$ is even* and $n = 2^d + 2i$, then the Knödel graph $W_{d,n}$ *is even regular*; *if $d > 1$ is odd* and $n = 2^d + 2i$, then the Knödel graph $W_{d,n}$ *is odd regular*.

In the following Subsections 1.1, 1.2 and 1.3, the Knödel graph $W_{d,n} = \mathcal{G}$, where both $n \geq 8$ and $d > 2$ $(1 \leq d \leq \lfloor \log_2(n) \rfloor)$ are even.

1.1. Knödel graphs

In this Subsection, we find the vertex local anti-magic chromatic number of Knödel graphs.

Theorem 1.1. *For the Knödel graph $\mathcal{G}$, $\chi_{v\ell a}(\mathcal{G}) = 3$.*

P r o o f. Let $V(\mathcal{G}) = \{v_i,\ w_j : 0 \leq i,\ j \leq (n - 2)/2\}$ and

$$E(\mathcal{G}) = \{v_i w_j : 0 \leq i \leq (n - 2)/2, \quad j \in \{i + 2^0 - 1, i + 2^1 - 1, \ldots, i + 2^{d-1} - 1\}\}.$$

Clearly, $|V(\mathcal{G})| = n$ and $|E(\mathcal{G})| = nd/2$. Define $\varphi\colon E(\mathcal{G}) \to \{1, 2, \dots, nd/2\}$ as follows. For $i \in \{0, 1, \dots, n/2 - 1\}$, $j \in \{i - 1 + 2^k\colon 0 \le i \le n/2 - 1 \text{ and } 0 \le k \le d - 1\}$,

$$\varphi(v_i w_j) = \begin{cases} \dfrac{n(k+1) - 2i}{2}, & \text{if } k \in \{1, 3, \dots, d-1\}, \\ \dfrac{2(i+1) + nk}{2}, & \text{if } k \in \{0, 2, \dots, d-2\}. \end{cases}$$

Thus, φ is a VLAC of $\mathcal{G}$ with vertex colors

$$\varphi^+(v_i) = d(nd+2)/4 \ \text{ for } \ i \in \{0, 1, \dots, (n-2)/2\}. \tag{1.1}$$

For $j \in \{0, 1, \dots, n/2 - 1\}$,

$$\varphi^+(w_j) = \begin{cases} \dfrac{d(nd+2)}{4} + \dfrac{2^d - 1}{3}, & \text{if } 2^k - 1 \le j \le 2^{k+1} - 2 \text{ and } k \in \{1, 3, \dots, d-1\}, \\ \dfrac{d(nd+2)}{4} - \dfrac{n}{2} + \dfrac{2^d - 1}{3}, & \text{otherwise.} \end{cases} \tag{1.2}$$

Hence, φ induces a proper vertex coloring φ^+ of $\mathcal{G}$ with 3 colors and $\chi_{v\ell a}(\mathcal{G}) \le 3$. By Theorem 0.4, $\chi_{v\ell a}(\mathcal{G}) \ge 3$. Thus, $\chi_{v\ell a}(\mathcal{G}) = 3$ (for example, see $\chi_{v\ell a}(W_{4,16})$ in Fig. 1). ☐

1.2. Union of Knödel graphs

In this Subsection we discuss the vertex local anti-magic chromatic number for disjoint union of Knödel graphs.

Remark 1.1. For $r \in \mathbb{N}$, $\chi_{v\ell a}(rH) \ge \chi_{v\ell a}(H)$.

For $r \in \mathbb{N}$, the disjoint union of r copies of $\mathcal{G}$ is a disconnected graph which is denoted by $r\mathcal{G}$ with $V(r\mathcal{G}) = \{v_i^\ell, w_j^\ell\colon 0 \le i, j \le (n-2)/2, 1 \le \ell \le r\}$ and $E(r\mathcal{G}) = \{v_i^\ell w_j^\ell\colon 1 \le \ell \le r, 0 \le i \le (n-2)/2,\ j \in \{i + 2^0 - 1, i + 2^1 - 1, \dots, i + 2^{d-1} - 1\}\}$.

Theorem 1.2. *For $r \in \mathbb{N}$, $\chi_{v\ell a}(r\mathcal{G}) = 3$.*

P r o o f. Define $\varphi\colon E(r\mathcal{G}) \to \{1, 2, \dots, nrd/2\}$ as follows. For $i \in \{0, 1, \dots, (n-2)/2\}$, $j \in \{i + 2^k - 1\colon 0 \le i \le (n-2)/2 \text{ and } 0 \le k \le d-1\}$, and $1 \le \ell \le r$,

$$\varphi(v_i^\ell w_j^\ell) = \begin{cases} \dfrac{n\big(r(k+1) - (\ell - 1)\big) - 2i}{2}, & \text{if } k \in \{1, 3, \dots, d-1\}, \\ \dfrac{n\big(rk + \ell - 1\big) + 2(i+1)}{2}, & \text{if } k \in \{0, 2, \dots, d-2\}. \end{cases}$$

Thus, φ is a VLAC of $r\mathcal{G}$ with vertex colors as follows: $\varphi^+(v_i^\ell) = d(nrd+2)/4$, $i \in \{0, 1, \dots, \dots, n/2 - 1\}$. For $j \in \{0, 1, \dots, n/2 - 1\}$ and $1 \le \ell \le r$,

$$\varphi^+(w_j^\ell) = \begin{cases} \dfrac{d(nrd+2)}{4} + \dfrac{2^d - 1}{3}, & \text{if } 2^k - 1 \le j \le 2^{k+1} - 2 \text{ and } k \in \{1, 3, \dots, d-1\}, \\ \dfrac{d(nrd+2)}{4} + \dfrac{2^d - 1}{3} - \dfrac{n}{2}, & \text{otherwise.} \end{cases}$$

Hence, φ induces a proper vertex coloring φ^+ of $r\mathcal{G}$ with 3 colors and $\chi_{v\ell a}(r\mathcal{G}) \le 3$. By Remark 1.1, $\chi_{v\ell a}(r\mathcal{G}) \ge 3$. Thus, $\chi_{v\ell a}(r\mathcal{G}) = 3$. ☐

1.3. Knödel graphs join with some graphs

This Subsection describes the vertex local anti-magic chromatic number of Knödel graph join with trivial graph (isolated vertices), complete graphs, cycles and complete bipartite graphs.

Let O_s, $s \geq 1$, be the trivial graph with $V(O_s) = \{z_t \colon 1 \leq t \leq s\}$ and let $\mathcal{G} \vee O_s$ be the join graph. Clearly, $V(\mathcal{G} \vee O_s) = V(\mathcal{G}) \cup V(O_s)$ and $E(\mathcal{G} \vee O_s) = E(\mathcal{G}) \cup \big\{v_i z_t, w_j z_t \colon 0 \leq i \leq (n-2)/2,\ 1 \leq t \leq s,\ \text{and}\ 2^k - 1 \leq j \leq 2^{k+1} - 2 \leq (n-2)/2,\ \text{where}\ k \in \{0, 1, \dots, d-1\}\big\}$. Note that, $|V(\mathcal{G} \vee O_s)| = n + s$ and $|E(\mathcal{G} \vee O_s)| = nd/2 + ns$. Hereafter, $p = n + s$ and $q = nd/2 + ns$.

Lemma 1.1. *For $s \geq 2$ even, $\chi_{v\ell a}(\mathcal{G} \vee O_s) = 3$.*

P r o o f. Let $\vartheta \colon E(\mathcal{G} \vee O_s) \to [1, q]$ be an edge labeling of $\mathcal{G} \vee O_s$ and let φ be the vertex local anti-magic chromatic number of $\mathcal{G}$ defined by Theorem 1.1. First we label the edges of $\mathcal{G}$ by using $[1, nd/2]$ labels such that $\vartheta(e) = \varphi(e)$ for all $e \in E(\mathcal{G})$.

Next label the edges of $v_i z_t$, as shown in the table below:

	z_1	z_2	z_3	z_4	$\dots$	z_{s-2}	z_{s-1}	z_s
v_0	$\frac{nd}{2} + ns$	$\frac{nd}{2} + n(s-1) + 1$	$\frac{nd}{2} + n(s-1)$	$\frac{nd}{2} + n(s-2) + 1$	$\dots$	$\frac{n(d-s+2)}{2} + ns + 1$	$\frac{n(d-s+2)}{2} + ns$	$\frac{n(d-s)}{2} + ns + 1$
v_1	$\frac{nd}{2} + ns - 1$	$\frac{nd}{2} + n(s-1) + 2$	$\frac{nd}{2} + n(s-1) - 1$	$\frac{nd}{2} + n(s-2) + 2$	$\dots$	$\frac{n(d-s+2)}{2} + ns + 2$	$\frac{n(d-s+2)}{2} + ns - 1$	$\frac{n(d-s)}{2} + ns + 2$
v_2	$\frac{nd}{2} + ns - 2$	$\frac{nd}{2} + n(s-1) + 3$	$\frac{nd}{2} + n(s-1) - 2$	$\frac{nd}{2} + n(s-2) + 3$	$\dots$	$\frac{n(d-s+2)}{2} + ns + 3$	$\frac{n(d-s+2)}{2} + ns - 2$	$\frac{n(d-s)}{2} + ns + 3$
v_3	$\frac{nd}{2} + ns - 3$	$\frac{nd}{2} + n(s-1) + 4$	$\frac{nd}{2} + n(s-1) - 3$	$\frac{nd}{2} + n(s-2) + 4$	$\dots$	$\frac{n(d-s+2)}{2} + ns + 4$	$\frac{n(d-s+2)}{2} + ns - 3$	$\frac{n(d-s)d}{2} + ns + 4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$v_{\frac{n-6}{2}}$	$\frac{n(d-1)}{2} + ns + 3$	$\frac{n(d-1)}{2} + ns - 2$	$\frac{n(d-3)}{2} + ns + 3$	$\frac{n(d-3)}{2} + ns - 2$	$\dots$	$\frac{n(d-s+3)}{2} + ns - 2$	$\frac{n(d-s+1)}{2} + ns + 3$	$\frac{n(d-s+1)}{2} + ns - 2$
$v_{\frac{n-4}{2}}$	$\frac{n(d-1)}{2} + ns + 2$	$\frac{n(d-1)}{2} + ns - 1$	$\frac{n(d-3)}{2} + ns + 2$	$\frac{n(d-3)}{2} + ns - 1$	$\dots$	$\frac{n(d-s+3)}{2} + ns - 1$	$\frac{n(d-s+1)}{2} + ns + 2$	$\frac{n(d-s+1)}{2} + ns - 1$
$v_{\frac{n-2}{2}}$	$\frac{n(d-1)}{2} + ns + 1$	$\frac{n(d-1)}{2} + ns$	$\frac{n(d-3)}{2} + ns + 1$	$\frac{n(d-3)}{2} + ns$	$\dots$	$\frac{n(d-s+3)}{2} + ns$	$\frac{n(d-s+1)}{2} + ns + 1$	$\frac{n(d-s+1)}{2} + ns$

Table 1. The edge labeling of $v_i z_t$

From the above table, the first column is the series of numbers in $[1 + q - n/2, q]$ in reverse natural order; the second column is the series of numbers in $[1 + q - n, q - n/2]$ in natural order; the third column is the series of numbers $[1 + q - 3n/2, q - n]$ in reverse natural order; the fourth column is the series of numbers in $[1 + q - 2n, q - 3n/2]$ in natural order; the fifth column is the series of numbers in $[1 + q - 5n/2, q - 2n]$ in reverse natural order; continuing the above process we obtain that odd columns are the series of numbers in reverse natural order and even columns are the series of numbers in natural order. Finally, $(s-2)$th column is the series of numbers $[q + 1 - n(s-2)/2, q - n(s-3)/2]$ in natural order; then $(s-1)$th column is the series of numbers $[q + 1 - n(s-1)/2, q - n(s-2)/2]$ in natural order; and sth column is the series of numbers $[q + 1 - sn/2, q - n(s-1)/2]$ in natural order. Easily, we see that each row sum is

$$g^+(v_i) = f^+(v_i) + \frac{ns(2d+3s)+2s}{4} = \frac{nd+2}{4} + \frac{ns(2d+3s)+2s}{4}$$
$$= \frac{nd + ns(2d+3s) + 2(s+1)}{4}.$$

Next we label the edges of $w_j z_1$ as follows:

$$\vartheta^+(w_jz_1) = \begin{cases} \dfrac{nd+2}{2} + (j-2^k+1) + \displaystyle\sum_{i=1}^{(k-1)/2} 2^{2i-1}, & \text{if } k \in \{1,3,\dots,d-1\}, \\ \dfrac{n(d+1)-2^d+4}{2} + (j-2^k+1) + 2\displaystyle\sum_{i=0}^{(d-4)/2} 4^i + \sum_{i=0}^{(k-2)/2} 2^{2i}, & \\ & \text{if } k \in \{0,2,\dots,d-2\}. \end{cases}$$

Moreover, we label the edges of $w_j z_t$, $t \in \{2,3,\dots,s\}$ and $k \in \{0,2,\dots,d-2\}$,

$$\vartheta^+(w_jz_t) = \begin{cases} \dfrac{n(d+t)}{2} - (j-2^k+1) - \displaystyle\sum_{i=0}^{(k-2)/2} 2^{2i}, & \text{if } t \text{ is even}, \\ \dfrac{n(d+t-1)+2}{2} + (j-2^k+1) + \displaystyle\sum_{i=0}^{(k-2)/2} 2^{2i}, & \text{if } t \text{ is odd}. \end{cases}$$

For $k \in \{1,3,\dots,d-1\}$,

$$\vartheta^+(w_jz_t) = \begin{cases} \dfrac{n(d+t)}{2} - (j-2^k+1) - \displaystyle\sum_{i=0}^{d-2/2} 4^i - \sum_{i=1}^{(k-1)/2} 2^{2i-1}, & \text{if } t \text{ is even}, \\ \dfrac{n(d+t-1)+2}{2} + (j-2^k+1) + \displaystyle\sum_{i=0}^{(d-2)/2} 4^i + \sum_{i=1}^{(k-1)/2} 2^{2i-1}, & \text{if } t \text{ is odd}. \end{cases}$$

Hence, the induced vertex colors of ϑ are as follows:

$$\vartheta^+(v_i) = \varphi^+(v_i) + \frac{ns(2d+3s)+2s}{4} \quad \text{for } 0 \le i \le \frac{n-2}{2}, \tag{1.3}$$

$$\vartheta^+(w_j) = \varphi^+(w_j) + \frac{p+2}{2} + \frac{ns(s+2d)}{4} - 2^{d-1} + 2\sum_{i=0}^{(d-4)/2} 4^i \quad \text{for } k \in \{0,2,\dots,d-2\}, \tag{1.4}$$

$$\vartheta^+(w_j) = \varphi^+(w_j) + \frac{s+2}{2} + \frac{ns(s+2d)}{4} - 2^{d-1} + 2\sum_{i=0}^{(d-4)/2} 4^i \quad \text{for } k \in \{1,3,\dots,d-1\}.$$

From the equations (1.2) and (1.4), we have

$$\vartheta^+(w_j) = \frac{d(nd+2)}{4} + \frac{2^d-1}{3} + \frac{s+n+2}{2} + \frac{ns(s+2d)}{4} - 2^{d-1} + 2\sum_{i=0}^{(d-4)/2} 4^i \quad \text{for } 0 \le k \le d-1.$$

From the equations (1.1) and (1.3),

$$\vartheta^+(v_i) = \frac{d(nd+2)}{4} + \frac{ns(2d+3s)+2s}{4} \quad \text{for } 0 \le i \le (n-2)/2.$$

Then the leftover colors of $\mathcal{G} \vee O_s$ are as follows:

$$\vartheta^+(z_t) = n(nd+ns+1)/2 \quad \text{for } 0 \le t \le s.$$

Thus, ϑ induces a proper vertex coloring ϑ^+ of $\mathcal{G} \vee O_s$ with 3 colors. Hence, $\chi_{v\ell a}(\mathcal{G} \vee O_s) \le 3$ and $\chi_{v\ell a}(\mathcal{G} \vee O_s) \ge \chi(\mathcal{G} \vee O_s) = 3$ (Table 2 shows the example of vertex local anti-magic chromatic number of $W_{4,16} \vee O_8$.) □

	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	$\varphi^+(v_i)$	$\Re^+(v_i)$
v_0	160	145	144	129	128	113	112	97	66	1094
v_1	159	146	143	130	127	114	111	98	66	1094
v_2	158	147	142	131	126	115	110	99	66	1094
v_3	157	148	141	132	125	116	109	100	66	1094
v_4	156	149	140	133	124	117	108	99	66	1094
v_5	155	150	139	134	123	118	107	102	66	1094
v_6	154	151	138	135	122	119	106	103	66	1094
v_7	153	152	137	136	121	120	105	104	66	1094
w_0	**36**	48	49	64	65	80	81	96	63	582
w_3	**37**	47	50	63	66	79	82	95	63	582
w_4	**38**	46	51	62	67	78	83	94	63	582
w_5	**39**	45	52	61	68	77	84	93	63	582
w_6	**40**	44	53	60	69	76	85	92	63	582
w_1	**33**	43	54	59	70	75	86	91	71	582
w_2	**34**	42	55	58	71	74	87	90	71	582
w_7	**35**	41	56	57	72	73	88	89	71	582
$g^+(z_t)$	1544	1544	1544	1544	1544	1544	1544	1544		

Table 2. $\chi_{v\ell a}(W_{4,16} \vee O_8) = 3$

Lemma 1.2. *For $s \geq 3$ odd, $\chi_{v\ell a}(\mathcal{G} \vee O_s) = 3$.*

P r o o f. Label the edges of $\mathcal{G} \vee O_{s-2}$ by

$$\left[\frac{nd}{2}+1, \frac{n(d+s-2)}{2}\right] \cup \left[\frac{nd}{2}+ns-\frac{n(s-2)}{2}+1, \frac{nd}{2}+ns\right]$$

as in the Lemma 1.1. Next, we label the remaining edges $w_j z_{t-1}$ and $w_j z_t$ as follows.

Let $2^k - 1 \leq j \leq 2^{k+1} - 2 \leq (n-2)/2$ and $0 \leq i \leq (n-2)/2$. For $k \in \{0, 2, \ldots, d-2\}$,

$$\vartheta(w_j z_{s-1}) = \frac{n(d+s-2)+2}{2} + (j - 2^k + 1) + \sum_{i=0}^{(k-2)/2} 2^{2i},$$

$$\vartheta(w_j z_s) = \frac{n(d+s+1)-2}{2} - 2(j - 2^k + 1) - 2\sum_{i=0}^{(k-2)/2} 2^{2i}.$$

For $k \in \{1, 3, \ldots, d-1\}$,

$$\vartheta(w_j z_{s-1}) = \frac{n(d+s-2)+2}{2} + (j - 2^k + 1) + \sum_{i=0}^{(d-2)/2} 4^i + \sum_{i=1}^{(k-1)/2} 2^{2i-1},$$

$$\vartheta(w_j z_s) = \frac{n(d+s+1)-2}{2} - 2(j - 2^k + 1) - 2\sum_{i=0}^{(d-2)/2} 4^i - 2\sum_{i=1}^{(k-1)/2} 2^{2i-1}.$$

Further, label the edges of $v_i z_{s-1}$ and $v_i z_s$ as shown in the table below:

Hence, the induced vertex colors of ϑ are as follows:

$$\vartheta^+(v_i) = \varphi^+(v_i) + \frac{ns(2d+3s)+2(s+1)-n}{4} \quad \text{for } 0 \leq i \leq (n-2)/2, \tag{1.5}$$

	v_0	v_1	v_2	v_3	$\ldots$	$v_{\frac{n-2}{2}}$
z_{s-1}	$\dfrac{n(2+d+s)}{2}$	$\dfrac{n(2+d+s)}{2}-1$	$\dfrac{n(2+d+s)}{2}-2$	$\dfrac{n(2+d+s)}{2}-3$	$\ldots$	$\dfrac{n(1+d+s)}{2}+1$
z_s	$\dfrac{n(s+d-1)+4}{2}$	$\dfrac{n(s+d-1)+4}{2}+2$	$\dfrac{n(s+d-1)+4}{2}+4$	$\dfrac{n(s+d-1)+4}{2}+6$	$\ldots$	$\dfrac{n(s+d+1)}{2}$

Table 3. The edge labeling of v_iz_{s-1} and v_iz_s

For $2^k-1 \leq j \leq 2^{k+1}-2 \leq (n-2)/2$,

$$\vartheta^+(w_j) = \varphi^+(w_j) + \frac{ns(s+2d)+2(s+1)+3n}{4} - 2^{d-1} + 2\sum_{i=0}^{(d-4)/2} 4^i, \quad \text{if } k \in \{0,2,\ldots,d-2\};$$
$$\vartheta^+(w_j) = \varphi^+(w_j) + \frac{ns(s+2d)+2(s+1)+n}{4} - 2^{d-1} + 2\sum_{i=0}^{(d-4)/2} 4^i, \quad \text{if } k \in \{1,3,\ldots,d-1\}. \tag{1.6}$$

From (1.2) and (1.6),

$$\vartheta^+(w_j) = \frac{d(nd+2)}{4} + \frac{2^d-1}{3} + \frac{ns(s+2d)+2(s+1)+3n}{4} - 2^{d-1} + 2\sum_{i=0}^{(d-4)/2} 4^i, \quad \text{if } k \in \{0,1,\ldots,d-1\},$$

From (1.1) and (1.5),

$$\vartheta^+(v_i) = \frac{d(nd+2)}{4} + \frac{ns(2d+3s)+2(s+1)-n}{4}, \quad \text{if } 0 \leq i \leq (n-2)/2.$$

Then, the leftover colors of $\mathcal{G} \vee O_s$ are as follows:

$$\vartheta^+(z_t) = \frac{n(nd+ns+1)}{2} \quad \text{for } 0 \leq t \leq s.$$

Thus, ϑ induces a proper vertex coloring ϑ^+ of $\mathcal{G} \vee O_s$ with 3 colors. Hence, $\chi_{v\ell a}(\mathcal{G} \vee O_s) \leq 3$ and $\chi_{v\ell a}(\mathcal{G} \vee O_s) \geq \chi(\mathcal{G} \vee O_s) = 3$ (Table 4 shown the example of vertex local anti-magic chromatic number of $W_{4,16} \vee O_7$.) $\square$

Due to the proof of Lemma 1.1 and 1.2, we have the following

Theorem 1.3. *For $s \geq 2$, $\chi_{v\ell a}(\mathcal{G} \vee O_s) = 3$.*

Theorem 1.4. *For $s \geq 3$, $\chi_{v\ell a}(\mathcal{G} \vee C_s) = 5$.*

P r o o f. Let ϑ is a VLAC as in the proof of Lemma 1.2. Since $\theta\colon E(C_s) \to [1,s]$ is a VLAC of C_s by

$$\theta(z_tz_{t+1}) = \begin{cases} t/2, & \text{if } t \text{ is even,} \\ s-(t-1)/2, & \text{if } t \text{ is odd.} \end{cases}$$

Next, define edge labeling function $\tau\colon E(\mathcal{G} \vee C_s) \to [1, q+s]$ as follows:

$$\tau(e) = \begin{cases} \vartheta(e), & \text{if } e \in E(\mathcal{G} \vee O_s), \\ q+\theta(e), & \text{if } e \in E(C_s), \end{cases}$$

	z_1	z_2	z_3	z_4	z_5	z_6	z_7	$f^+(v_i)$	$g^+(v_i)$
v_0	144	129	128	113	112	104	82	66	878
v_1	143	130	127	114	111	103	84	66	878
v_2	142	131	126	115	110	102	86	66	878
v_3	141	132	125	116	109	101	88	66	878
v_4	140	133	124	117	108	100	90	66	878
v_5	139	134	123	118	107	99	92	66	878
v_6	138	135	122	119	106	98	94	66	878
v_7	137	136	121	120	105	97	96	66	878
w_0	**36**	48	49	64	65	73	95	63	493
w_3	**37**	47	50	63	66	74	93	63	493
w_4	**38**	46	51	62	67	75	91	63	493
w_5	**39**	45	52	61	68	76	89	63	493
w_6	**40**	44	53	60	69	77	87	63	493
w_1	**33**	43	54	59	70	78	85	71	493
w_2	**34**	42	55	58	71	79	83	71	493
w_7	**35**	41	56	57	72	80	81	71	493
$g^+(z_t)$	1416	1416	1416	1416	1416	1416	1416		

Table 4. $\chi_{v\ell a}(W_{4,16} \vee O_7) = 3$

and

$$\tau(z_t z_{t+1}) = \begin{cases} q + t/2, & \text{if } t \text{ is even,} \\ q + s + (1-t)/2, & \text{if } t \text{ is odd.} \end{cases}$$

If $1 \leq i \leq j \leq (n-2)/2$,

$$\tau^+(z_t) = \vartheta^+(z_t) + s, \text{ for odd } t, \tag{1.7}$$

$$\tau^+(z_t) = \vartheta^+(z_t) + s + 1, \text{ for even } t, \tag{1.8}$$

$$\tau^+(z_1) = \vartheta^+(z_1) + (3s+1)/2, \text{ for even } t. \tag{1.9}$$

Furthermore,

$$\tau^+(w_j) = \vartheta^+(w_j), \tag{1.10}$$

$$\tau^+(v_i) = \vartheta^+(v_i). \tag{1.11}$$

Clearly, (1.7) < (1.8) < (1.9) and (1.10) < (1.11).

Next, we show that

$$\begin{aligned}
(1.9) - (1.11) &= \tau^+(z_1) - \tau^+(v_i) \\
&= \frac{n(nd+ns+1)+3s+1}{2} - \left(\frac{d(nd+2)}{4} + \frac{ns(3s+2d)+2(s+1)-n}{4}\right) \\
&= \frac{2n^2d + 2n^2s + 2n + 6s + 2 - nd^2 - 2d - 3ns^2 - 2nsd - 2s - 2 + n}{4} \\
&= \frac{nd(2n-d-2s) + ns(2n-3s) + 3n + 4s - 2d}{4} \neq 0.
\end{aligned}$$

Therefore, τ is a VLAC that induces 5 distinct vertex colors. Hence, $\chi_{v\ell a}(\mathcal{G} \vee C_s) \leq 5$ and $\chi_{v\ell a}(\mathcal{G} \vee C_s) \geq \chi(\mathcal{G} \vee C_s) = 5$. □

Theorem 1.5. *For $s \geq 3$, $\chi_{v\ell a}(\mathcal{G} \vee K_s) = s + 2$.*

P r o o f. Recall that: ϑ is a VLAC of $\mathcal{G} \vee O_s$ (by Theorem 1.3).

Let $\tau\colon E(K_s) \to [1, s(s-1)/2]$ be the VLAC of K_s that induces the vertex coloring of τ as

$$\tau^+(z_t) = \frac{2t^3 + 6t^2(1-s) + 2t(3s^2 - 3s - 4) + 3s(3-s)}{6} \quad \text{for } 1 \le t \le s.$$

Observe that τ induces t distinct vertex colors. Next we define edge labeling function $\zeta\colon E(\mathcal{G} \vee K_s) \to [1, q + s(s-1)/2]$ as

$$\zeta(e) = \begin{cases} \vartheta(e), & \text{if } e \in E(\mathcal{G} \vee K_s), \\ \tau(e) + q, & \text{if } e \in E(K_s). \end{cases}$$

Note that $\zeta^+(z_t) = \vartheta^+(z_t) + \tau^+(z_t) + q(s-1)$. Easily, we see that ϑ induces same vertex labels and τ, ζ induce distinct vertex labels. Furthermore, we consider two cases.

Case 1. s is even. We know that ϑ is the VLAC of $\mathcal{G} \vee O_s$ (by Lemma 1.1).

(a) $\zeta^+(v_i) = \vartheta^+(v_i) = \dfrac{d(nd+2)}{4} + \dfrac{ns(2d+3s)+2s}{4}$ for $0 \le i \le (n-2)/2$;

(b) $\zeta^+(w_j) = \vartheta^+(w_j) = \dfrac{d(nd+2)}{4} + \dfrac{2^d-1}{3} + \dfrac{p+2}{2} + \dfrac{ns(s+2d)}{4} - 2^{d-1} + 2\displaystyle\sum_{i=0}^{(d-4)/2} 4^i$, for $0 \le k \le d-1$;

(c)
$$\begin{aligned}\zeta^+(z_t) &= \vartheta^+(z_t) + \tau^+(z_t) + q(s-1) \\ &= \frac{n(nd+ns+1)}{2} + \frac{2t^3 + 6t^2(1-s) + 2t(3s^2-3s-4) + 3s(3-s)}{6} + q(s-1).\end{aligned}$$

Finally, we see that (a)>(b) for $1 \le i \le j \le n/2 - 1$.

Thus,

$$\begin{aligned}\zeta^+(z_t) - \zeta^+(v_i) = \frac{n(nd+ns+1)}{2} &+ \frac{2t^3 + 6t^2(1-s) + 2t(3s^2-3s-4) + 3s(3-s)}{6} \\ &+ q(s-1) - \frac{d(nd+2)}{4} + \frac{ns(2d+3s)+2s}{4} > 0.\end{aligned}$$

Case 2. s is odd. We know that ϑ is the VLAC of $\mathcal{G} \vee O_s$ (by Lemma 1.2).

(a) $\zeta^+(v_i) = \vartheta^+(v_i) = \dfrac{d(nd+2)}{4} + \dfrac{ns(2d+3s)+2(s+1)-n}{4}$ for $0 \le i \le (n-2)/2$;

(b) $\zeta^+(w_j) = \vartheta^+(w_j) = \dfrac{d(nd+2)}{4} + \dfrac{2^d-1}{3} + \dfrac{ns(s+2d)+2(s+1)+3n}{4} - 2^{d-1} + 2\displaystyle\sum_{i=0}^{(d-4)/2} 4^i$, for $0 \le k \le d-1$;

(c)
$$\begin{aligned}\zeta^+(z_t) &= \vartheta^+(z_t) + \tau^+(z_t) + q(s-1) \\ &= \frac{n(nd+ns+1)}{2} + \frac{2t^3 + 6t^2(1-s) + 2t(3s^2-3s-4) + 3s(3-s)}{6} + q(s-1).\end{aligned}$$

Finally, see that (a)>(b) for $1 \le i \le j \le n/2 - 1$.

Thus,

$$\begin{aligned}\zeta^+(z_t) - \zeta^+(v_i) = \frac{n(nd+ns+1)}{2} &+ \frac{2t^3 + 6t^2(1-s) + 2t(3s^2-3s-4) + 3s(3-s)}{6} \\ &+ q(s-1) - \frac{d(nd+2)}{4} + \frac{ns(2d+3s)+2(s+1)-n}{4} > 0.\end{aligned}$$

Clearly, in both cases, $\zeta^+(z_t) \neq \zeta^+(v_i)$. Thus, $\zeta^+(w_j) < \zeta^+(v_i) < \zeta^+(z_t)$. Therefore, ζ is a VLAC of $\mathcal{G} \vee K_s$ that induces $s+2$ distinct vertex colors. It concludes that $\chi_{v\ell a}(\mathcal{G} \vee K_s) \leq s+2$. Since $\chi_{v\ell a}(\mathcal{G} \vee K_s) \geq \chi(\mathcal{G} \vee K_s) = s+2$, we obtain $\chi_{v\ell a}(\mathcal{G} \vee K_s) = s+2$. □

Theorem 1.6. *Let $s, \ell \geq 2$ and $s \neq \ell$, where s and ℓ are even (s and ℓ are odd). If (a) $p+s > \ell$ and (b) $\ell < s$, then $\chi_{v\ell a}\big([\mathcal{G} \vee O_s] \vee O_\ell\big) = \chi_{v\ell a}(\mathcal{G} \vee K_{s,\ell}) = 4$.*

P r o o f. According to Theorem 1.3, $p = n+s$, $q = ns + nd/2$, and $r = \ell$. Recall that by the proof of Theorem 1.3, $\vartheta^+(w_j) < \vartheta^+(v_i) < \vartheta^+(z_t)$.
(a) Using Theorem 0.5, we get $\chi_{v\ell a}(\mathcal{G} \vee K_{s,\ell}) \leq 4$.
(b) Consider

$$\begin{aligned}
p - \ell = n \leq n + s - \ell &= (1 + \ell p + 2q)(p - \ell) - 2g^+(z_t) \\
&= (n+s-\ell)\big(\ell(n+s) + 2ns + nd + 1\big) - 2\left(\frac{n(1+nd+ns)}{2}\right) \\
&= (s+n-\ell)\big(\ell(s+n) + 2ns + nd + 1\big) - n(nd+ns+1) \\
&= \ell n^2 - \ell^2 n + n^2 s + \ell s^2 - \ell^2 s + 2ns^2 + nds - \ell nd - \ell + s \\
&= \ell n(n-\ell) + ns(n+2s) + \ell s(s-\ell) + nd(s-\ell) + s - \ell \\
&= \ell n(n-\ell) + ns(n+2s) + (s-\ell)(\ell s + nd + 1) > 0.
\end{aligned}$$

Thus, $2\vartheta^+(z_t) < (p-\ell)(\ell p + 2q + 1)$. By Theorem 0.5, we have $\chi_{v\ell a}(\mathcal{G} \vee K_{s,\ell}) \leq 4$ and $\chi_{v\ell a}(\mathcal{G} \vee K_{s,\ell}) \geq \chi(\mathcal{G} \vee K_{s,\ell}) = 4$. □

For simplicity, we relabel the vertices of a Fibonacci graph as follows. The vertices $\big\{(1,0),(1,1),\ldots,(1,(n-2)/2)\big\}$ are labelled as $V = \big\{v_0, v_1, \ldots, v_{\frac{n-2}{2}}\big\}$, while the other vertices $\big\{(2,0),(2,1),\ldots,(2,(n-2)/2)\big\}$ are labelled as $W = \big\{w_0, w_1, \ldots, w_{\frac{n-2}{2}}\big\}$. Then, two vertices of v_i and w_j are adjacent $\Leftrightarrow j \in \big\{i + F(1) - 1, i + F(2) - 1, \ldots, i + F(d+1) - 1\big\}$ $\big(j - i \in \big\{F(1)-1, F(2)-1, \ldots, F(d+1)-1\big\}\big)$. Here $V \cup W$ are the vertices of $F_{d,n}$ and we utilise the subscript addition modulo $n/2$. Clearly, $F_{2,4} \cong C_4$ and $F_{3,6} \cong K_{3,3}$. Since $\chi_{v\ell a}(K_{3,3}) = 3$ and $\chi_{v\ell a}(C_s) = 3$ [5,6], we get $\chi_{v\ell a}(F_{2,4}) = \chi_{v\ell a}(F_{3,6}) = 3$. Also, $F_{3,8} \cong C_4 \times P_2$ and $\chi_{v\ell a}(C_4 \times P_2) = 4$ [28]. Hence, $\chi_{v\ell a}(F_{3,8}) = 4$.

For $i \in \{0, 1, \ldots, d-1\}$, *if $d > 2$ is even* and $n = 2(F(d)+i)$, then the Fibonacci graph $F_{d,n}$ *is even regular*; if $d \geq 3$ *is odd* and $n = 2(F(d)+i)$, then the Fibonacci graph $F_{d,n}$ *is odd regular*.

§ 2. Fibonacci graphs

In this Section, we find the vertex local anti-magic chromatic number of Fibonacci graphs.

Theorem 2.1. *For both $n \geq 10$ and $d > 2$ $(1 \leq d \leq F^{-1}(n) - 1)$ even, $\chi_{v\ell a}(F_{d,n}) = 3$.*

P r o o f. Let $|V(F_{d,n})| = n$ and $|E(F_{d,n})| = nd/2$. Clearly,

$$V(F_{d,n}) = \{v_i, w_j \colon 0 \leq i, j \leq (n-2)/2\}, \quad \text{and}$$
$$E(F_{d,n}) = \big\{v_i w_j \colon 0 \leq i \leq (n-2)/2, j \in \{i + F(1) - 1, i + F(2) - 1, \ldots, i + F(d+1) - 1\}\big\}.$$

Define $\varphi \colon E(F_{d,n}) \to \{1, 2, \ldots, nd/2\}$ as follows.
For $i \in \{0, 1, \ldots, (n-2)/2\}$, $j \in \{i + F(k+1) - 1 \colon 0 \leq i \leq (n-2)/2 \text{ and } 0 \leq k \leq d\}$,

$$\varphi(v_i w_j) = \begin{cases} \dfrac{n(k+1) - 2i}{2}, & \text{if } k \in \{1, 3, \ldots, d-1\}, \\[2ex] \dfrac{2(i+1) + nk}{2}, & \text{otherwise.} \end{cases}$$

Thus, φ is a VLAC of $F_{d,n}$ with vertex colors as follows:

$$\varphi^+(v_i) = \frac{d(nd+2)}{4}, \quad \text{if } \ i \in \{0, 1, \dots, n/2 - 1\}.$$

For $j \in \{0, 1, \dots, n/2 - 1\}$, we have

$$\varphi^+(w_j) = \begin{cases} \dfrac{d(nd+2)}{4} + F(d-1), & \text{if } \ F(k+1) - 1 \le j \le F(k+1) + k - 1, \\ & k \in \{1, 3, \dots, d-1\}, \\ \dfrac{d(nd+2)}{4} - \dfrac{n}{2} + F(d-1), & \text{otherwise.} \end{cases}$$

Thus, φ induces a proper vertex coloring φ^+ of $F_{d,n}$ with 3 colors and $\chi_{v\ell a}(F_{d,n}) \le 3$. By Theorem 0.4, $\chi_{v\ell a}(F_{d,n}) \ge 3$. Hence, $\chi_{v\ell a}(F_{d,n}) = 3$ (for example, see $\chi_{v\ell a}(F_{4,14})$ in Fig. 2). $\square$

Conclusion

In this article, we discussed the vertex local anti-magic chromatic number of some Knödel graphs and Fibonacci graphs. Also we determined the vertex local anti-magic chromatic number of disjoint union of Knödel graphs and obtained the vertex local anti-magic chromatic number of some join graphs. Further, the following problems naturally arise.

Problem 2.1. Determine $\chi_{v\ell a}(W_{d,n})$ for even $n \ge 8$ and odd $d > 1$ $(1 \le d \le \lfloor \log_2(n) \rfloor)$.

Problem 2.2. Determine $\chi_{v\ell a}(F_{d,n})$ for even $n \ge 10$ and odd $d > 3$ $(1 \le d \le F^{-1}(n) - 1)$.

In addition, our aim is to find the vertex local anti-magic chromatic number of generalized Knödel graphs.

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Локальная антимагичность вершин для графов Кнёделя и Фибоначчи

Ключевые слова: локальная антимагическая маркировка вершин, полные графы, тривиальные графы, полные двудольные графы, графы Кнёделя, графы Фибоначчи, соединенные графы.

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В этой статье мы представляем вершинное локальное антимагическое хроматическое число для некоторых графов Кнёделя $\mathcal{G}$ и графов Фибоначчи, дизъюнктного объединения графов Кнёделя и соединенных графов $\mathcal{G} \vee \mathcal{H}$, где $\mathcal{H} \in \{O_s = K_s^C, K_s, C_s, K_{s,\ell}\}$.

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