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STEADY SOLITARY WAVE SOLUTIONS OF THE GENERALIZED SIXTH-ORDER BOUSSINESQ–OSTROVSKY EQUATION

An overview of models that lead to the nonintegrable Ostrovsky equation and its generalizations having no exact solitary-wave solutions is given. A brief derivation of the Ostrovsky equation for longitudinal waves in a geometrically nonlinear rod lying on an elastic foundation is performed. It is shown that in the case of axially symmetric propagation of longitudinal waves in a physically nonlinear cylindrical shell interacting with a nonlinear elastic medium the displacement component obeys the generalized sixth-order Boussinesq–Ostrovsky equation. We construct an exact kink-like solution of this equation, establish a connection with the generalized nonlinear Schrödinger (GNLS) equation and find the steady travelling wave solution of the GNLS in the form of simple soliton with monotonic or oscillating tails.

Keywords: nonlinear evolution equations, solitary-wave solutions, generalized nonlinear Schrödinger equation.

Introduction

In 1978 [1] the nonlinear evolution equation for internal waves in rotating ocean was derived:

$$\left(\eta_t + \frac{3c_0}{2h}\eta\eta_\xi + \frac{c_0\beta}{6}\eta_{\xi\xi\xi}\right)_\xi = \frac{\Omega^2}{2c_0}\eta, \tag{1}$$

where η is the perturbation of the free surface of liquid layer with depth h, c_0 represents the speed of propagation of perturbations, β is a high frequency dispersion parameter and Ω is the Coriolis parameter characterizing the rotation of the liquid. Equation (1) called the Ostrovsky equation is not integrated by the inverse scattering procedure and does not have exact soliton-like solutions.

In [2] it is noted that (1) actually refers to a great range of nonlinear systems characterized by the presence of wide nondispersive band in the frequency spectrum that separates the regions with low- and high-frequency dispersion. Examples of this can be the unusual electromagnetic and oblique magnetosonic waves in a magnetized plasma, excitation of atoms in the chain described by the Frenkel–Kontorova (FK) model, the waves in transmission lines of band-pass filter type, acoustic waves in a curved rod and waves in randomly inhomogeneous media.

A large number of works are devoted to the truncated dispersionless version of equation (1):

$$\left(\eta_t + \frac{3c_0}{2h}\eta\eta_\xi\right)_\xi = \frac{\Omega^2}{2c_0}\eta. \tag{2}$$

It is well known [3] that by introducing new independent and dependent variables, equation (2) becomes integrable in a class of exact solutions [4].

Equations that can be considered in some way as generalization of (1) and have in contrast to (1) exact solitary-wave solutions were obtained in [5,6] a few years before the publication of [1]. In particular, in [5] the following evolution equations were proposed using FK model supplemented by anharmonicity of interaction of neighbor atoms:

$$U_{tt} = \alpha\left(U_{xx} + \alpha_1 U_{xxxx} - \beta U_x U_{xx}\right) - \gamma U\left(a - U\right)\left(a - 2U\right), \tag{3}$$

$$U_{tt} = \alpha\left(U_{xx} + \alpha_1 U_{xxxx} - \beta U_x^2 U_{xx}\right) - \gamma\frac{2\pi}{a}\sin\left(\frac{2\pi}{a}U\right). \tag{4}$$

A reduction of (4) called the Konno–Kameyama–Sanuki (KKS) equation

$$U_{zt} = -\sigma \sin U + bU_t^2 U_{tt} + \frac{k_3}{6} U_{tttt} \tag{5}$$

was considered in [6]. It was shown that at a certain ratio between the parameters of nonlinearity and dispersion the KKS equation (5) becomes integrable and has the N-soliton solutions [7].

In 1997, the exact soliton solutions in generalized dynamic models of the quasi-one-dimensional crystal were obtained in [8]. After analyzing the generalized Boussinesq equations

$$U_{tt} - c^2 U_{xx} - 6GU_x U_{xx} - FU_{xxxx} = -\frac{\partial \Psi}{\partial U}, \tag{6}$$

$$U_{tt} - c^2 U_{xx} - HU_x^2 U_{xx} - PU_{xxxx} = -\frac{\partial \Phi}{\partial U}, \tag{7}$$

where $\Psi = A\left(U^2-1\right)^2$, $\Phi = B\left(1+\cos \pi U\right)$, the authors noted that [5, 6] with equations (3)–(5) are the only known papers related to the generalized models. In addition, they showed that simple generalizations of the potentials Ψ and Φ —

$$\Psi = A_1\left(U^2-1\right)^2 + A_2\left(U^2-1\right)^3,$$
$$\Phi = B_1\left(1+\cos \pi U\right) + B_2\left(1+\cos 2\pi U\right)$$

— lead to exact solutions without any restrictions on the relationships between the parameters in the left sides of equations (6) and (7).

In the recent years, the interest in generalization of the Boussinesq and Ostrovsky equations and corresponding mathematical models is growing. In [9, 10], on the basis of the system of two coupled Boussinesq-type equations

$$\begin{aligned} f_{tt} - f_{xx} &= \left(\frac{1}{2} f^2 + f_{tt}\right)_{xx} - \delta\left(f-g\right), \\ g_{tt} - c^2 g_{xx} &= \left(\frac{\alpha}{2} g^2 + \beta g_{tt}\right)_{xx} + \gamma\left(f-g\right), \end{aligned} \tag{8}$$

it was shown that in a two-layer elastic waveguide with non-ideal glued layers a soliton of elastic deformation may be followed by a zone of resonance radiation. The initial value problem for the system (8) is examined in [11]. Using the method of multiscale expansions it was found that various asymptotic regimes correspond to the coupled or uncoupled system of the Ostrovsky equations. A system of coupled Ostrovsky equations for internal waves in the shear flows was studied in [12]. Shoaling oceanic internal solitary waves based on the Ostrovsky equation with variable coefficients were asymptotically and numerically investigated in [13]. It was demonstrated that a combined effect of shoaling and rotation is to induce a secondary trailing wave packet, induced by enhanced radiation from the leading wave.

In the problems of nonlinear wave dynamics of deformable systems, the Ostrovsky equation arises naturally [14]. Consider the generalized Boussinesq equation

$$U_{tt} - c_0^2 U_{xx} + \alpha U_x U_{xx} + \beta U_{xxxx} = \gamma U, \tag{9}$$

which models the propagation of longitudinal waves in the geometrically nonlinear rod lying on the linear elastic foundation [15]. Here U is the displacement, c_0 is the speed of sound in the rod, α, β, γ are the parameters characterizing nonlinearity, dispersion and effect of the elastic foundation.

Assuming that α, β, γ are of the same order of smallness ε, introduce the new independent variables $\xi = x - ct$, $\tau = \varepsilon t$ and expand the displacement in powers of the small parameter: $U = U_0 + \varepsilon U_1 + \ldots$. Then, in the linear approximation, we obtain the expression for the velocity of propagation of the disturbance $c = c_0$, and the first non-linear order in the parameter ε gives us the evolution equation

$$U_{0\tau\xi} + \alpha_1 U_{0\xi} U_{0\xi\xi} + \beta_1 U_{0\xi\xi\xi\xi} = \gamma_1 U_0. \tag{10}$$

Differentiating both sides of (10) with respect to ξ and denoting $U_{0\xi} = W$ for the component of longitudinal deformation, we obtain the Ostrovsky equation

$$(W_\tau + \alpha_1 W W_\xi + \beta_1 W_{\xi\xi\xi})_\xi = \gamma_1 W.$$

In the case of nonlinear elastic foundation the equation (9) takes the form

$$U_{tt} - c_0^2 U_{xx} + \alpha U_x U_{xx} + \beta U_{xxxx} = \gamma_1 U \pm \gamma_2 U^3, \tag{11}$$

where the sign before the last term is selected depending on the type of nonlinearity: "hard" corresponds to plus, and "soft" to minus. In the latter case, equation (11) substantially coincides with the equation (6) and has the exact soliton-like solution.

The models of nonlinear wave dynamics of deformable media with microstructure can lead to equations with spatial derivatives of the highest order. The work [16] contains a non-integrable fifth-order evolution equation with cubic nonlinearity

$$\varphi_t - \alpha\varphi^2\varphi_x - \beta\varphi_{xxx} + \beta_1\varphi_{xxxxx} = 0, \tag{12}$$

to model the axisymmetric propagation of longitudinal deformation waves in a physically nonlinear cylindrical shell reinforced by ribs, the exact soliton-like solution to (12) founded on the method of the singular manifold and nonlinear Schrödinger (NLS) equation derived from (12). Considering the case of physical and geometric nonlinearity leads us to the fifth-order Gardner equation

$$\varphi_t - \alpha\varphi^2\varphi_x + \alpha_1\varphi\varphi_x - \beta\varphi_{xxx} + \beta_1\varphi_{xxxxx} = 0 \tag{13}$$

that has the exact solution [17]. When the shell interacts with external nonlinear-elastic medium, the equation (13) written for the component of longitudinal displacement U will contain additional terms similar to those in (11):

$$U_{tx} - \alpha U_x^2 U_{xx} + \alpha_1 U_x U_{xx} - \beta U_{xxxx} + \beta_1 U_{xxxxxx} = \gamma_1 U \pm \gamma_2 U^3. \tag{14}$$

The equation (14) which summarizes the KdV, mKdV, Gardner, Kawahara, Boussinesq and Ostrovsky equations, we shall call the generalized Boussinesq–Ostrovsky equation.

The main purpose of this paper is to find the solitary wave solution for (14). First, we show that (14) has the exact kink-like solution for a certain ratio between the parameters of the equation. Secondly, we establish a connection between (14) and the generalized nonlinear Schrödinger (GNLS) equation. Finally, we find steady travelling wave solutions for the GNLS in the form of a simple soliton with monotonic or oscillating tails and identify the modes that allow for existence and distribution of such wave solutions.

§ 1. Exact soliton-like solutions

Consider the equation

$$U_{qz} - \alpha_1 U_z^2 U_{zz} + \alpha_2 U_z U_{zz} - \alpha_3 U_{zzzz} + \alpha_4 U_{zzzzzz} = \alpha_5 U - \alpha_6 U^3 \tag{15}$$

where all parameters α_1–α_6 are positive numbers. Using the transformation

$$U(q,z) = \frac{\sqrt[4]{\alpha_3^3\alpha_5}}{\alpha_2} u(t,x), \quad x = \sqrt[4]{\frac{\alpha_5}{\alpha_3}} z, \quad t = \sqrt[4]{\alpha_3\alpha_5^3} q,$$

we obtain a simplified equation

$$u_{tx} - \alpha u_x^2 u_{xx} + u_x u_{xx} - u_{xxxx} + \beta u_{xxxxxx} = u - \gamma u^3 \tag{16}$$

with

$$\alpha = \frac{\alpha_1 \sqrt[4]{\alpha_3^3\alpha_5}}{\alpha_2^2}, \quad \beta = \frac{\alpha_4\sqrt{\alpha_5}}{\sqrt{\alpha_3^3}}, \quad \gamma = \frac{\alpha_6\sqrt{\alpha_3^3}}{\alpha_2^2\sqrt{\alpha_5}}.$$

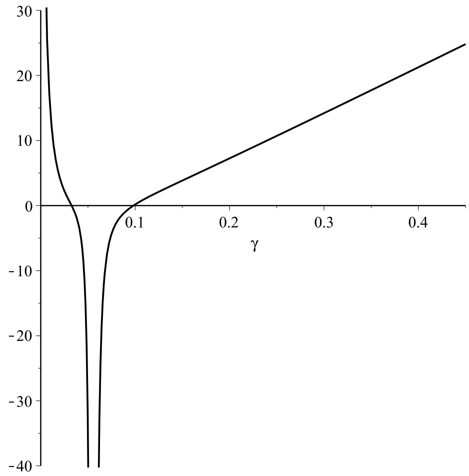

Fig 1. The velocity C of the soliton (17) at $\alpha = 1$

We will seek for the exact solution of (16) given by

$$u = A \tanh\left[B\,(x - Ct)\right] \tag{17}$$

with three constants A, B, C to be determined. Substitute the solution (17) into (16) and collect the terms proportional to tanh powers. Equating the coefficients in front of these collections to zero, we obtain a system of four equations, where the solution can be as follows:

$$A = \frac{1}{\sqrt{\gamma}}, \quad B = \frac{3\sqrt{\gamma}}{2(\alpha - 18\gamma)}, \quad C = \frac{2(\alpha - 18\gamma)^2}{9\gamma} + \frac{\alpha - 90\gamma}{10(\alpha - 18\gamma)^2},$$

with the relationship between the parameters of the equation (16) given by

$$\beta = \frac{\alpha(\alpha - 18\gamma)^2}{810\gamma^2}.$$

We see that exact solitary wave solution (17) exists only in the case of "soft nonlinearity" of the elastic foundation, when $\gamma > 0$ and the summands in the right-hand side of (16) have opposite signs. The velocity C of the solitary wave tends to infinity as $\gamma \to 0$ or $\gamma \to \frac{\alpha}{18}$. There are two values of γ at which the velocity C becomes zero (Fig. 1) and solution (17) is the standing wave. The value of C is almost a linear function of γ, if $\gamma \gg \frac{\alpha}{18}$; the larger the value of γ, the smaller the amplitude A of the wave.

Thus, equation (16) has the exact kink-like solution (17). This solution does not belong to localized solutions whose wave field vanishes at infinity: $u \to 0$ as $|x| \to \infty$. To find such a localized soliton-like solution we apply the approach based on the GNLS equation.

§ 2. Derivation of the GNLS equation

It is well known that the solitary wave solution of the ordinary nonlinear Schrödinger (NLS) equation describes the solution of the original equation ((16) for the given case) in which the carrier quasi-sinusoidal wave propagates with the linear phase velocity $c\,(k)$, while the envelope propagates with the group velocity $c_g\,(k)$, where k is wavenumber [18]. In general, these velocities do not coincide, therefore this is not a steady travelling wave solution. However, for the values k such that $c\,(k) = c_g\,(k)$ this becomes, at least to leading order, a steady solitary wave solution to the original equation. It was shown in [18] that the condition of equality the phase and group velocities requires consideration of higher-order corrections to the NLS equation leading to the GNLS equation.

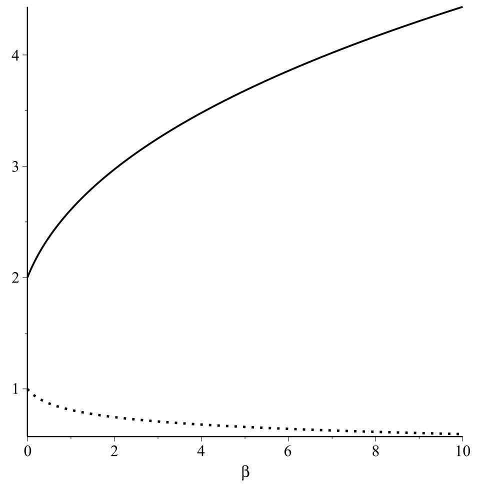

Fig 2. Linear phase velocity's minimum c_m (solid line) and the corresponding wavenumber k_m (dotted line)

In the case of small-amplitude perturbations we can neglect the nonlinear terms in equation (16) and then seek for the solution of the reduced linear equation in the form of a travelling wave $u = A\mathrm{e}^{i(kx-\omega t)}$. This leads to the following dispersion relation

$$\omega = \beta k^5 + k^3 + \frac{1}{k}. \tag{18}$$

The linear phase velocity $c = \frac{\omega}{k}$ is positive and monotonically increases as $k \to 0$ or $k \to \infty$, hence it has the minimum c_m whose dependence on the parameter β is shown in Fig. 2.

This means that for any possible solitary wave propagating with the velocity $v \geqslant c_m$, there is a synchronous linear wave which is radiated by the solitary wave and gradually destroys it due to radiative energy losses [19]. Thus, localized solitary wave can exist only in the case $v < c_m$. Further analysis using the GNLS equation confirmed this hypothesis.

At the first step of the GNLS derivation [20], we allow for function u to depend on the phase variable $\theta = kx - \omega t$, slow coordinate $X = \varepsilon x$ and slow time $T = \varepsilon t$, where ε is a small parameter, so that equation (16) takes the following form

$$\begin{aligned}
&\beta\varepsilon^6 u_{XXXXXX} + 6\beta\varepsilon^5 k u_{\theta XXXXX} + 15\beta\varepsilon^4 k^2 u_{\theta\theta XXXX} + 20\beta\varepsilon^3 k^3 u_{\theta\theta\theta XXX} + \\
&+ 15\beta\varepsilon^2 k^4 u_{\theta\theta\theta\theta XX} + 6\beta\varepsilon k^5 u_{\theta\theta\theta\theta\theta X} + \beta k^6 u_{\theta\theta\theta\theta\theta\theta} - \varepsilon^4 u_{XXXX} - \\
&- 4\varepsilon^3 k u_{\theta XXX} - 6\varepsilon^2 k^2 u_{\theta\theta XX} - 4\varepsilon k^3 u_{\theta\theta\theta X} - k^4 u_{\theta\theta\theta\theta} + \varepsilon^2 u_{XT} + \varepsilon k u_{\theta T} - \\
&- \left[\alpha\varepsilon^4 u_X^2 + \left(2\alpha\varepsilon^3 k u_\theta + \varepsilon^3\right) u_X + \alpha\varepsilon^2 k^2 u_\theta^2 - \varepsilon^2 k u_\theta\right] u_{XX} - \\
&- \left[2\alpha\varepsilon^3 k u_X^2 + \left(4\alpha\varepsilon^2 k^2 u_\theta - 2\varepsilon^2 k\right) u_X + 2\alpha\varepsilon k^3 u_\theta^2 - 2\varepsilon k^2 u_\theta + \varepsilon\omega\right] u_{\theta X} - \\
&- \left[\alpha\varepsilon^2 k^2 u_X^2 + \left(2\alpha\varepsilon k^3 u_\theta - \varepsilon k^2\right) u_X + \alpha k^4 u_\theta^2 - k^3 u_\theta + k\omega\right] u_{\theta\theta} + \\
&+ \gamma u^3 - u = 0.
\end{aligned} \tag{19}$$

Next we substitute the solution into equation (19) (the bar designates complex conjugation)

$$\begin{aligned}
u = {}&\varepsilon\left(A(X,T)\,\mathrm{e}^{i\theta} + \bar{A}(X,T)\,\mathrm{e}^{-i\theta}\right) + \\
&+ \varepsilon^2\left(B(X,T)\,\mathrm{e}^{2i\theta} + \bar{B}(X,T)\,\mathrm{e}^{-2i\theta} + C(X,T)\right) + O\left(\varepsilon^3\right)
\end{aligned} \tag{20}$$

and collect the terms proportional to $\mathrm{e}^{i\theta}$. At the leading order ε-term, we obtain the dispersion relation (18) as well as expressions for the phase and group velocities

$$c = \beta k^4 + k^2 + \frac{1}{k^2}, \quad c_g \equiv \frac{\partial\omega}{\partial k} = 5\beta k^4 + 3k^2 - \frac{1}{k^2}, \tag{21}$$

while the remaining terms give

$$\begin{aligned}
&\varepsilon^2 \left[i \left(6\beta k^5 + 4k^3 - \omega\right) A_X + ikA_T\right] + \\
&+ \varepsilon^3 \left[\left(15\beta k^4 + 6k^2\right) A_{XX} + A_{TX} + \left(\alpha k^4 + 3\gamma\right) A^2 \bar{A} + 2ik^3 \bar{A}B\right] + \\
&+ \varepsilon^4 \left[-i \left(20\beta k^3 + 4k\right) A_{XXX} - k^2 AC_X + 3k^2 \bar{A}B_X - 4i\alpha k^3 A\bar{A}A_X\right] + \\
&+ O\left(\varepsilon^5\right) = 0.
\end{aligned} \tag{22}$$

At the leading ε^2-term in (22), we have obtain

$$A_T + c_g A_X = O\left(\varepsilon\right). \tag{23}$$

Turning to the superslow time $\tau = \varepsilon T$ and running variable $\xi = X - c_g T$ we can obtain the solution of the last equation in the form of a travelling wave with a slowly varying amplitude $A\left(X, T\right) = A\left(\xi, \tau\right)$. After convertion A_{TX} to $A_{\xi\xi}$ by (23), we present the value of $\bar{A}B$ from ε^3-term and substitute it to ε^4-term to get the factor $\frac{1}{6}\frac{\partial^3\omega}{\partial k^3}$ in the front of the third derivative A_{XXX}.

At the next step in the transformation of the equation (22) to the GNLS equation, we need to express B and C as the functions of A. To do that, we ought to collect the terms proportional to $e^{2i\theta}$ and get

$$\begin{aligned}
&\varepsilon^2 \left(-ik^3 A^2 - \left(60\beta k^6 + 12k^4 - 3\right) B\right) + \\
&+ \varepsilon^3 \left[i \left(190\beta k^5 + 30k^3 - \frac{2}{k}\right) B_X + 2ikB_T - 3k^2 AA_X\right] + \\
&+ O\left(\varepsilon^4\right) = 0.
\end{aligned} \tag{24}$$

Note that B, a similar way as A, can be regarded as a function of ξ and τ; we can convert B_T to B_ξ and then seek the solution of (24) in the form $B = iB_0\left(\xi\right) + \varepsilon B_1\left(\xi\right)$. Equating to zero the real and imaginary parts of the equation (24) will allow us to find B_0 and B_1 separately, so that function B can be written as

$$B = -i\frac{k^3}{3\left(20\beta k^6 + 4k^4 - 1\right)}A^2 + \varepsilon\frac{k^2\left(60\beta k^6 + 4k^4 + 3\right)}{3\left(20\beta k^6 + 4k^4 - 1\right)}AA_\xi + O\left(\varepsilon^2\right). \tag{25}$$

Finally, we collect the "mean flow" terms which are independent on θ, and obtain

$$-\varepsilon^2 C + \varepsilon^3 k^2 \left(A\bar{A}\right)_X + O\left(\varepsilon^4\right) = 0. \tag{26}$$

Similarly introducing the variables ξ and τ, one can see that the equation (26) has the solution

$$C = \varepsilon k^2 \left(A\bar{A}\right)_\xi + O\left(\varepsilon^2\right). \tag{27}$$

As a final step in the derivation we substitute (25) and (27) into (22) to obtain the GNLS equation

$$iA_\tau + \beta_1 A_{\xi\xi} + \beta_2 \left|A\right|^2 A + i\varepsilon\left(\beta_3 A_{\xi\xi\xi} + \frac{\beta_2}{k}\left(\left|A\right|^2 A\right)_\xi + \beta_4 \left|A\right|^2 A_\xi\right) + O\left(\varepsilon^2\right) = 0, \tag{28}$$

where

$$\beta_1 = \frac{1}{2}\frac{\partial^2\omega}{\partial k^2} = 10\beta k^3 + 3k + \frac{1}{k^3}, \qquad \beta_2 = \alpha k^3 + \frac{2k^5}{3\left(20\beta k^6 + 4k^4 - 1\right)} + \frac{3\gamma}{k},$$

$$\beta_3 = -\frac{1}{6}\frac{\partial^3\omega}{\partial k^3} = -10\beta k^2 - 1 + \frac{1}{k^4}, \qquad \beta_4 = -\left(4\alpha k^2 + \frac{4k^4\left(4k^4 - 3\right)}{3\left(20\beta k^6 + 4k^4 - 1\right)^2}\right).$$

Note that as $20\beta k^6 + 4k^4 - 1 \to 0$ coefficients β_2 and β_4 tend to infinity as a result of the resonance of the fundamental and second harmonics of the solution (20) [19], but investigation of the case is beyond the scope of this paper.

§ 3. Solution of the GNLS equation

Consider a simplified case of the equation (28) when the value of the wavenumber k satisfies the equation $4k^4 - 3 = 0$, i. e. $k = \sqrt[4]{\frac{3}{4}}$ if we assume that $k > 0$. The condition of equality of the phase and group velocities (21) leads to the values $\beta = \frac{\sqrt{3}}{9}$ and $c = c_g = \frac{5\sqrt{3}}{4}$. For any differentiable function $\omega(k)$ the necessary condition for the extremum of $\frac{\omega}{k}$ can be written in the form $\omega' = \frac{\omega}{k}$, hence the phase velocity $c = \frac{\omega}{k}$ can coincide with the group velocity $c_g = \omega'$ only at the extreme point $(k_m,\, c_m)$ of function c. Thus, going in the GNLS equation (28) back to the variables T and X and putting $k = k_m = \sqrt[4]{\frac{3}{4}}$, $c_g = c_m = \frac{5\sqrt{3}}{4}$, $\beta = \frac{\sqrt{3}}{9}$ we can write the GNLS equation as

$$\begin{gathered} i\left(A_T + \frac{5\sqrt{3}}{4}A_X\right) + \sqrt[4]{12}\varepsilon\left(3A_{XX} + \gamma_1 |A|^2 A\right) + \\ + 2\varepsilon^2 i\left(-\frac{2}{3}A_{XXX} + \gamma_1\left(|A|^2 A\right)_X - \sqrt{3}\alpha |A|^2 A_X\right) = 0, \end{gathered} \tag{29}$$

where $\gamma_1 = \frac{1}{18} + \frac{\sqrt{3}}{4}\alpha + \sqrt{3}\gamma$. At leading order (29) reduces to the ordinary NLS equation with a well-known solitary wave solution

$$A = a\,\mathrm{sech}\,[\delta(\xi - \varepsilon v T)]\mathrm{e}^{i(l\xi - \varepsilon\sigma T)}, \tag{30}$$

where $\xi = X - \frac{5\sqrt{3}}{4}T$, $v = 6\sqrt[4]{12}l$, $\sigma = -3\sqrt[4]{12}\left(\delta^2 - l^2\right)$, $a^2 = \frac{6\delta^2}{\gamma_1}$ and l, δ are free parameters. Note that in general the solution (30) is not a steady solitary wave solution of the original equation (16). In [18] it was shown that the requirement $c = c_g$ needed for steady solitary wave solution leads to the parameters v and l of (30) to be $O(\varepsilon)$. Therefore, the requirement $c = c_g$ can be correctly taken into account only when considering the full GNLS equation (29) with the last ε^2-term. Following the paper [18], we will seek for the solution of (29) in the form

$$A = R(\eta)\,\mathrm{e}^{i\varepsilon[\phi(\eta) + l\eta - \sigma T]}, \tag{31}$$

where $\eta = X - c_m T - \varepsilon^2 V T$ and R, ϕ are functions to be determined. The total phase of the first harmonic of the solution (20) is

$$\begin{aligned} &\theta + \varepsilon[\phi(\eta) + l\eta - \sigma T] = \\ &= k_m x - c_m k_m t + \varepsilon\phi\left[\varepsilon\left(x - c_m t - \varepsilon^2 V t\right)\right] + \varepsilon^2\left[l\left(x - c_m t - \varepsilon^2 V t\right) - \sigma t\right]. \end{aligned}$$

For a steady wave the total phase must be a function on η alone, hence we have to put $\sigma = k_m V$. After substituting (31) into (29), equating the real and imaginary parts of the resulting equation to zero, and omitting terms of $O\left(\varepsilon^2\right)$, we obtain:

$$\begin{aligned} &3R_{\eta\eta} + \frac{1}{2}VR + \gamma_1 R^3 = 0, \\ &18k_m\left(R\phi_{\eta\eta} + 2R_\eta\phi_\eta\right) + 3\left[\left(6\gamma_1 - 2\sqrt{3}\alpha\right)R^2 + 12k_m l - V\right]R_\eta - 4R_{\eta\eta\eta} = 0. \end{aligned} \tag{32}$$

The first equation in the set (32) has a solitary wave solution

$$R = a\,\mathrm{sech}\left(\sqrt{\frac{\gamma_1}{6}}a\left(X - c_m T - \varepsilon^2 V T\right)\right), \tag{33}$$

where $V = -a^2\gamma_1$. From physical considerations it follows that $\alpha > 0$, $\gamma > 0$, hence $V < 0$ and velocity of the steady solitary wave is less then c_m, in accordance with the previously formulated hypothesis.

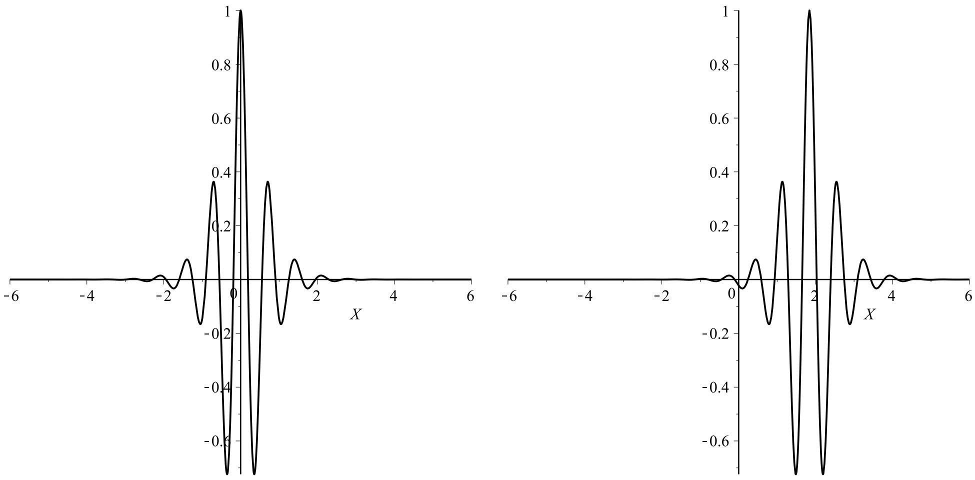

Fig 3. Plots of the steady solitary wave defined by (31) with $a = 5$, $\varepsilon = 0.1$, $\alpha = \gamma = \frac{1}{\sqrt{3}}$ at $T = 0$ and $T = 1$

With R given by (33), the second equation of the set (32) becomes the equation for the function ϕ alone with the solution of the form

$$\phi = -\frac{3a\left(11\sqrt{3}\gamma_1 - 9\alpha\right)}{54\sqrt[4]{3}\sqrt{\gamma_1}}\tanh\left(\sqrt{\frac{\gamma_1}{6}}a\left(X - c_m T - \varepsilon^2 VT\right)\right), \tag{34}$$

where $l = -\frac{7\sqrt{2}}{108\sqrt[4]{3}}\gamma_1 a^2$. It follows that the wavenumber correction for the first harmonic is negative.

The plots of the steady solitary wave (20) for leading order at $T = 0$ and $T = 1$ show that the wave form really does not change over time (Fig. 3).

Therefore, the GNLS equation (28) in its simplified form (29) has a one-parametric solution in the form of a steady solitary wave defined by (31), (33), and (34). The larger the amplitude of the wave, the smaller its scale and lower its velocity. With the increase of the small parameter ε, the oscillating tails of the soliton (Fig. 3) gradually transformed into monotonically decreasing tails.

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Устойчивые уединенно-волновые решения обобщенного уравнения Буссинеска–Островского шестого порядка

Ключевые слова: нелинейные эволюционные уравнения, уединенно-волновые решения, обобщенное нелинейное уравнение Шрёдингера.

УДК 517.95

Проведен обзор моделей, приводящих к неинтегрируемому уравнению Островского и его обобщениям, не имеющим точных уединенно-волновых решений. Приведен краткий вывод уравнения Островского для продольных волн в геометрически нелинейном стержне, лежащем на упругом основании. Показано, что в случае осесимметричного распространения пучка продольных волн в физически нелинейной цилиндрической оболочке, взаимодействующей с нелинейно-упругой средой, для компоненты перемещения возникает обобщенное уравнение Буссинеска–Островского шестого порядка. Построено точное кинкоподобное решение этого уравнения, установлена связь с обобщенным нелинейным уравнением Шрёдингера и найдено решение последнего уравнения в форме устойчивой солитоноподобной бегущей волны с монотонно затухающими или колебательными хвостами.

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