MSC: 30C45

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CERTAIN CLASS OF HARMONIC MULTIVALENT FUNCTIONS

Making use of the generalized derivative operator, we introduce a new subclass of harmonic multivalent functions. We obtain the coefficient bounds, distortion inequalities and inclusion relationships involving the neighborhoods of subclasses of harmonic multivalent functions.

Keywords: harmonic multivalent functions, derivative operator, neighborhood.

§ 1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply-connected complex domain D is said to be harmonic in D if both functions u and v are real harmonic in D. Such functions can be expressed as

$$f = h + \overline{g}, \tag{1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|\overline{h}(z)| > |\overline{g}(z)|$ for all z in D (see [1]). Many researcher introduced and studied certain classes of harmonic univalent functions (see [2–8]). For $p \geqslant 1$, $n \in N$, denote by $SH(n,p)$ the class of functions of the form (1) that are harmonic multivalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$, where h and g are defined by

$$h(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, \quad g(z) = \sum_{k=n+p-1}^{\infty} b_k z^k, \quad z \in U, \quad |b_n + p - 1| < 1, \tag{2}$$

which are analytic and multivalent functions in U.

Now we introduce a generalized derivative operator

$$D^m_{p,\lambda} f^{(q)} = D^m_{p,\lambda} h^{(q)} + (-1)^m \overline{D^m_{p,\lambda} g^{(q)}}.$$

The derivative operator $D^m_{p,\lambda} f^{(q)}$ of p-valent functions was introduced and studied by Eljamal and Darus in [9], where

$$D^m_{p,\lambda} h^{(q)} = \frac{p!}{(p-q)!} z^{p-q} + \sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!} \left(\frac{k+\lambda-q}{p+\lambda-q}\right)^m a_k z^{k-q},$$

$$D^m_{p,\lambda} g^{(q)} = \sum_{k=n+p-1}^{\infty} \frac{k!}{(k-q)!} \left(\frac{k+\lambda-q}{p+\lambda-q}\right)^m b_k z^{k-q}, \quad m \in N_0, \quad z \in U.$$

For $q = \lambda = 0$, $p = k = 1$, the differential operator $D^m f^{(q)}$ was introduced by Salagean [10] for the class A of analytic functions and modified for the class $SH(1,1)$ by Jahangiri et. al [11].

Let

$$F(z) = (1-\ell) D^m_{p,\lambda} f^{(q)}(z) + \ell D^{m+1}_{p,\lambda} f^{(q)}(z) = H(z) + \overline{G(z)}, \quad f(z) \in SH(n,p), \quad 0 \leqslant \ell \leqslant 1,$$

where H and G are of the form

$$H(z) = (1 - \ell + \ell(p+\lambda-q)) \frac{(p+\lambda-q)^m}{(p-q)!} p! z^{p-q} +$$

$$+\sum_{k=n+p}^{\infty}(1-\ell+\ell(k+\lambda-q))\frac{(k+\lambda-q)^m}{(k-q)!}k!z^{k-q}a_k z^{k-q}, \quad (3)$$

$$G(z)=(-1)^m\sum_{k=n+p-1}^{\infty}(1-\ell+\ell(k+\lambda-q))\frac{(k+\lambda-q)^m}{(k-q)!}k!z^{k-q}b_k z^{k-q}. \quad (4)$$

Also, let $SH^{m,n}_{p,\lambda}(q,\ell,\alpha)$ denote the subclass of $SH(n,p)$ consisting of functions f defined in (1) that satisfy the following condition:

$$\operatorname{Re}\left\{\frac{zH'(z)-\overline{zG'(z)}}{H(z)+\overline{G(z)}}\right\}>\alpha(p+\lambda-q) \quad (0\leqslant\alpha<1,\ p>q,\ p\in N,\ q\in N_0,\ z\in U), \quad (5)$$

where $H(z)$ and $G(z)$ are given by (3) and (4) respectively.

Denote by $\overline{SH}(n,p)$ the subclass of $SH(n,p)$ consisting of harmonic functions $f_m=h+\overline{g_m}$, where h and g_m are of the form

$$h(z)=z^p-\sum_{k=n+p}^{\infty}a_k z^k, \quad g_m(z)=(-1)^m\sum_{k=n+p-1}^{\infty}b_k z^k, \quad b_k\geqslant 0. \quad (6)$$

Define $\overline{SH}^{m,n}_{p,\lambda}(q,\ell,\alpha)=SH^{m,n}_{p,\lambda}(q,\ell,\alpha)\cap\overline{SH}(n,p)$.

The classes $SH^{m,n}_{p,\lambda}(q,\ell,\alpha)$ and $\overline{SH}^{m,n}_{p,\lambda}(q,\ell,\alpha)$ include well-known subclasses of $SH(n,p)$. For example:

(i) $SH^{0,1}_{1,0}(0,0,0)\equiv SH^*$ is the class of sense-preserving, harmonic univalent functions f which are starlike in U (see [12,13]);

(ii) $\overline{SH}^{0,1}_{1,0}(0,0,\alpha)\equiv SH^*(\alpha)$ is the class of sense-preserving, harmonic univalent functions f which are starlike of order α in U (see [14]);

(iii) $\overline{SH}^{1,1}_{1,0}(0,0,\alpha)\equiv HK(\alpha)$ is the class of sense-preserving, harmonic univalent functions f which are convex of order α in U (see [12]);

(iv) $\overline{SH}^{0,1}_{1,p}(0,0,0)\equiv SH^*(p)$ is the class of sense-preserving, harmonic multivalent functions which are starlike in U (see [15]);

(v) $\overline{SH}^{m,n}_{p,0}(q,\ell,\alpha)\equiv\overline{SH}^{m,n}_{p}(q,\ell,\alpha)$ is the class of sense-preserving, harmonic multivalent functions in U (see [16]).

To prove our main results we need the following lemma.

Lemma 1. *Let $f_m=h+\overline{g}_m$ be of the form (6). Then $f_m\in\overline{SH}(n,p)$ if and only if*

$$\sum_{k=n+p}^{\infty}ka_k+\sum_{k=n+p-1}^{\infty}kb_k\leqslant p \quad (p\geqslant 1,\ n\in N). \quad (7)$$

§ 2. Main Result

We begin deriving a coefficient sufficient condition for the function f to belong to the class $SH^{m,n}_{p,\lambda}(q,\ell,\alpha)$. This result is contained in the following.

Theorem 1. *Let $f = h + \overline{g}$ be given by* (1). *Furthermore, let*

$$\sum_{k=n+p}^{\infty} \frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell+\ell(k+\lambda-q))}{[((1-\alpha)(p+\lambda-q)+1)-|(1-\alpha)(p+\lambda-q)-1|](1-\ell+\ell(p+\lambda-q))} \frac{q_p^m}{q_k^m}|a_k| + \tag{8}$$
$$+\sum_{k=n+p-1}^{\infty} \frac{(k+\lambda-q-\alpha(p+\lambda-q))|(1-\ell-\ell(k+\lambda-q))|}{[((1-\alpha)(p+\lambda-q)+1)-|(1-\alpha)(p+\lambda-q)-1|](1-\ell+\ell(p+\lambda-q))} \frac{q_p^m}{q_k^m}|b_k| \leqslant \frac{1}{2},$$

where $q_p^m = \frac{(p+\lambda-q)^m}{(p-q)!}p!$, $q_k^m = \frac{(k+\lambda-q)^m}{(k-q)!}k!$, *and* $(0 \leqslant \alpha < 1,\ p > q,\ p \in N,\ q \in N_0,\ z \in U)$. *Then f is sense-preserving, harmonic multivalent functions in U, and $f \in SH_{p,\lambda}^{m,n}(q,\ell,\alpha)$.*

P r o o f. We first show that if the inequality (8) holds for the coefficients of $f = h+\overline{g}$, then the required condition (7) is sense preserving and harmonic multivalent in U. In view of (5), we need to prove that $\operatorname{Re}\{w\} > 0$, where

$$w = \frac{zH'(z) - \overline{zG'(z)} - \alpha(p+\lambda-q)[H(z)+\overline{G(z)}]}{H(z)+\overline{G(z)}} = \frac{A(z)}{B(z)}.$$

By using the fact that $\operatorname{Re}\{w\} > 0 \Leftrightarrow |1+w| > |1-w|$, it suffices to show that $|A(z)+B(z)| - |A(z)-B(z)| > 0$, therefore we obtain $|A(z)+B(z)| - |A(z)-B(z)| \geqslant$

$$\geqslant [((1-\alpha)(p+\lambda-q)+1)-|(1-\alpha)(p+\lambda-q)-1|](1-\ell+\ell(p+\lambda-q))q_p^m|z|^{p-q} -$$
$$-\sum_{k=n+p}^{\infty} 2(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell+\ell(k+\lambda-q))q_k^m|a_k||z|^{k-q} -$$
$$-\sum_{k=n+p}^{\infty} 2(k+\lambda-q-\alpha(p+\lambda-q))|(1-\ell+\ell(k+\lambda-q))|q_k^m|b_k||z|^{k-q} >$$
$$> [((1-\alpha)(p+\lambda-q)+1)-|(1-\alpha)(p+\lambda-q)-1|](1-\ell+\ell(p+\lambda-q))q_p^m|z|^{p-q} \times$$
$$\times\Big\{1-\sum_{k=n+p}^{\infty} \frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell+\ell(k+\lambda-q))}{[((1-\alpha)(p+\lambda-q)+1)-|(1-\alpha)(p+\lambda-q)-1|](1-\ell+\ell(p+\lambda-q))} \frac{q_k^m}{q_p^m}|a_k| -$$
$$-\sum_{k=n+p}^{\infty} \frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell+\ell(k+\lambda-q))}{[((1-\alpha)(p+\lambda-q)+1)-|(1-\alpha)(p+\lambda-q)-1|](1-\ell+\ell(p+\lambda-q))} \frac{q_k^m}{q_p^m}|b_k|\Big\}. \quad \square$$

Theorem 2. *Let $f_m = h + \overline{g_m}$ be given by* (6). *Also, suppose that* $\lambda < \dfrac{1}{n+p+\lambda-q}$ *and* $\alpha \geqslant 1 - \dfrac{1}{p+\lambda-q}$. *Then $f_m \in \overline{SH}_{p,\lambda}^{m,n}(q,\ell,\alpha)$ if and only if*

$$\sum_{k=n+p}^{\infty} \frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))} \frac{q_k^m}{q_p^m}|a_k| +$$
$$+\sum_{k=n+p-1}^{\infty} \frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell-\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))} \frac{q_k^m}{q_p^m}|b_k| \leqslant 1. \tag{9}$$

P r o o f. Since $\overline{SH}^{m,n}_{p,\lambda}(q,\ell,\alpha) \subset SH^{m,n}_{p,\lambda}(q,\ell,\alpha)$, we only need to prove the necessary part of the theorem. Assume that $f_m \in \overline{SH}^{m,n}_{p,\lambda}(q,\ell,\alpha)$, then by virtue of (8) and (7), we obtain

$$\mathrm{Re}\left\{\frac{(1-\alpha)(p+\lambda-q)z^{p-q} - \sum_{k=n+p}^{\infty}(k+\lambda-q-\alpha(p+\lambda-q))\frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))}\frac{q_k^m}{q_p^m}a_k z^{k-q}}{z^{p-q} - \sum_{k=n+p}^{\infty}\frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))}\frac{q_k^m}{q_p^m}a_k z^{k-q} + (-1)^{2m}\sum_{k=n+p-1}^{\infty}\frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))}\frac{q_k^m}{q_p^m}b_k\overline{z^{k-q}}} \times\right.$$
$$\left.\times\frac{-(-1)^{2m}\sum_{k=n+p-1}^{\infty}(k+\lambda-q+\alpha(p+\lambda-q))\frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))}\frac{q_k^{m+1}}{q_p^m}b_k\overline{z^{k-q}}}{z^{p-q} - \sum_{k=n+p}^{\infty}\frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))}\frac{q_k^m}{q_p^m}a_k z^{k-q} + (-1)^{2m}\sum_{k=n+p}^{\infty}\frac{(1-\ell-\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))}\frac{q_k^m}{q_p^m}b_k\overline{z^{k-q}}}\right\} \geqslant 0. \quad (10)$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \leqslant z = r < 1$ we have

$$\frac{(1-\alpha)(p+\lambda-q) - \sum_{k=n+p}^{\infty}(k+\lambda-q-\alpha(p+\lambda-q))\frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))}\frac{q_k^m}{q_p^m}a_k r^{k-q}}{1 - \sum_{k=n+p}^{\infty}\frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))}\frac{q_k^m}{q_p^m}a_k r^{k-p} + \sum_{k=n+p-1}^{\infty}\frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))}\frac{q_k^m}{q_p^m}b_k r^{k-p}} \times$$
$$\times\frac{-\sum_{k=n+p-1}^{\infty}(k+\lambda-q-\alpha(p+\lambda-q))\frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))}\frac{q_k^{m+1}}{q_p^m}b_k r^{k-q}}{1 - \sum_{k=n+p}^{\infty}\frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))}\frac{q_k^m}{q_p^m}a_k r^{k-q} + \sum_{k=n+p-1}^{\infty}\frac{(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))}\frac{q_k^m}{q_p^m}b_k r^{k-q}} \geqslant 0. \quad (11)$$

If (9) does not hold, then the numerator in (11) is negative for r sufficiently close to 1. Therefore, there exists a point $z_0 = r_0$ in $(0,1)$ for which the quotient in (11) is negative. This contradicts our assumption that $f_m \in \overline{SH}^{m,n}_{p,\lambda}(q,\ell,\alpha)$. We thus conclude that it is both necessary and sufficient that the coefficient bound inequality (9) holds true when $f_m \in \overline{SH}^{m,n}_{p,\lambda}(q,\ell,\alpha)$. □

Theorem 3. *The class $\overline{SH}^{m,n}_{p,\lambda}(q,\ell,\alpha)$ is closed under convex combinations.*

P r o o f. Let $\ell < \dfrac{1}{p+\lambda-q}$, $\alpha \geqslant 1 - \dfrac{1}{p+\lambda-q}$ and $f_{mi} \in \overline{SH}^{m,n}_{p,\lambda}(q,\ell,\alpha)$ for $i = 1,2,\ldots$, where f_{mi} is given by

$$f_{mi}(z) = z^p - \sum_{k=n+p}^{\infty} a_{ki}z^k + (-1)^m \sum_{k=n+p-1}^{\infty} b_{ki}\overline{z^k}.$$

Then by (9),

$$\sum_{k=n+p}^{\infty}\frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))}\frac{q_k^m}{q_p^m}a_{ki} +$$
$$+\sum_{k=n+p-1}^{\infty}\frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell-\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))}\frac{q_k^m}{q_p^m}b_{ki} \leqslant (1-\alpha(p+\lambda-q)). \quad (12)$$

For $\sum_{i=1}^{\infty} t_i$, $0 \leqslant t_i \leqslant 1$, the convex combination of f_{mi} may be written as

$$\sum_{i=1}^{\infty} t_i f_{mi}(z) = z^p - \sum_{k=n+p}^{\infty}(\sum_{i=1}^{\infty} t_i a_{ki})z^k + (-1)^m \sum_{k=n+p-1}^{\infty}(\sum_{i=1}^{\infty} t_i b_{ki})\overline{z^k}.$$

Then by (12),

$$\sum_{k=n+p}^{\infty}\frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))}\frac{q_k^m}{q_p^m}(\sum_{i=1}^{\infty} t_i a_{ki}) +$$
$$+\sum_{k=n+p-1}^{\infty}\frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell-\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))}\frac{q_k^m}{q_p^m}(\sum_{i=1}^{\infty} t_i b_{ki}) =$$

$$= \sum_{i=1}^{\infty} t_i \Big\{ \sum_{k=n+p}^{\infty} \frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))} \frac{q_k^m}{q_p^m} a_{ki} +$$

$$+ \sum_{k=n+p-1}^{\infty} \frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell-\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))} \frac{q_k^m}{q_p^m} b_{ki} \Big\} \leqslant$$

$$\leqslant (1-\alpha)(p+\lambda-q) \sum_{i=1}^{\infty} = (1-\alpha)(p+\lambda-q).$$

This is the condition required by (10) so $\sum_{i=1}^{\infty} t_i f_{mi}(z) \in \overline{SH}_{p,\lambda}^{m,n}(q,\ell,\alpha)$. □

Theorem 4. *Let* $\ell < \dfrac{1}{p+\lambda-q}$ *and* $\alpha_1 \geqslant 1 - \dfrac{1}{p+\lambda-q}$. *For* $\alpha_1 < \alpha_2$,

$$\overline{SH}_{p,\lambda}^{m,n}(q,\ell,\alpha_2) \subset \overline{SH}_{p,\lambda}^{m,n}(q,\ell,\alpha_1).$$

P r o o f. Let $\ell < \dfrac{1}{p+\lambda-q}$, $\alpha_1 \geqslant 1 - \dfrac{1}{p+\lambda-q}$, and $f_m(z) \in \overline{SH}_{p,\lambda}^{m,n}(q,\ell,\alpha_2)$.

$$\sum_{k=n+p}^{\infty} \frac{(k+\lambda-q-\alpha_1(p+\lambda-q))(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))(1-\alpha_1)} \frac{q_k^m}{q_p^{m+1}} |a_k| +$$

$$+ \sum_{k=n+p-1}^{\infty} \frac{(k+\lambda-q-\alpha_1(p+\lambda-q))(1-\ell-\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))(1-\alpha_1)} \frac{q_k^m}{q_p^{m+1}} |b_k| \leqslant$$

$$\leqslant \sum_{k=n+p}^{\infty} \frac{(k+\lambda-q-\alpha_2(p+\lambda-q))(1-\ell+\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))(1-\alpha_2)} \frac{q_k^m}{q_p^{m+1}} |a_k| +$$

$$+ \sum_{k=n+p-1}^{\infty} \frac{(k+\lambda-q-\alpha_2(p+\lambda-q))(1-\ell-\ell(k+\lambda-q))}{(1-\ell+\ell(k+\lambda-q))(1-\alpha_2)} \frac{q_k^m}{q_p^{m+1}} |b_k| \leqslant 1,$$

then $f_m(z) \in \overline{SH}_{p,\lambda}^{m,n}(q,\ell,\alpha_1)$. □

Following Goodman [17] and Ruscheweyh [18] we refer to the neighborhood of $f = h + \overline{g} \in \overline{SH}(n,p)$,

$$N_{n,p,\lambda}^{\delta}(f_m^{(q)}, g_m^{(q)}) = \{g_m \in \overline{SH}(n,p):$$

$$g_m(z) = z^p - \sum_{k=n+p}^{\infty} A_k z^k + (-1)^m \sum_{k=n+p-1}^{\infty} B_k \overline{z}^k,\ A_k, B_k \geqslant 0,\ B_{n+p-1} < 1, \text{ and} \tag{13}$$

$$\sum_{k=n+p}^{\infty} \frac{k!}{(k+\lambda-q)!} k(|a_k - A_k| + |b_k - B_k|) + \frac{(n+p-1)!(n+p-1)}{(n+p-1+\lambda-q)!} |b_{n+p-1} - B_{n+p-1}| \leqslant \delta,\ \delta > 0\}.$$

In particular, for the function $h(z) = z^p$ we have

$$N_{n,p,\lambda}^{\delta}(h^{(q)}, g_m^{(q)}) = \{g_m \in \overline{SH}(n,p):$$

$$g_m(z) = z^p - \sum_{k=n+p}^{\infty} A_k z^k + (-1)^m \sum_{k=n+p-1}^{\infty} B_k \overline{z}^k,\ A_k, B_k \geqslant 0,\ B_{n+p-1} < 1, \text{ and} \tag{14}$$

$$\sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!} k(A_k + B_k) + \frac{(n+p-1)!(n+p-1)}{(n+p-1-q)!} |B_{n+p-1}| \leqslant \delta, \quad \delta > 0\}.$$

Theorem 5. *Let* $\ell < \dfrac{1}{p+\lambda-q}$ *and* $\alpha \geqslant 1 - \dfrac{1}{p+\lambda-q}$. *If* $g_m \in \overline{SH}^{m,n}_{p,\lambda}(q,\ell,\alpha)$, *then*

$$\overline{SH}^{m,n}_{p,\lambda}(q,\ell,\alpha) \subset N^{\delta}_{n,p,\lambda}(h^{(q)}, g^{(q)}_m),$$

where $h(z)$ *and* $g_m(z)$ *are given by* (14),

$$\delta = \frac{(n+p)(1-\alpha)(p+\lambda-q)}{(n+(1-\alpha)(p+\lambda-q))\psi} -$$
$$-\left(\frac{(n+p)(n-1+(1+\alpha)(p+\lambda-q))(1-\ell-\ell(n+p+\lambda-1-q))}{(n+(1-\alpha)(p+\lambda-q))(1-\ell-\ell(n+p+\lambda-q))(n+p+\lambda-q)^m} - \right.$$
$$\left. -\frac{(n+p-1)!(n+p-1)}{(n+p-1-q)!}\right)B_{n+p-1},$$

and

$$\psi = \left(\frac{(1-\ell-\ell(n+p+\lambda-q))(n+p+\lambda-q)^m}{((1-\ell)+\ell(p+\lambda-q))q^m_{n+p-1}}\right),$$

where $q^m_{n+p-1} = \frac{(p+\lambda-q)^m}{(n+p-1-q)!}(n+p-1)!$.

P r o o f. Let $f_m \in \overline{SH}^{m,n}_{p,\lambda}(q,\ell,\alpha)$. We need to show that $g_m(z) \in N^{\delta}_{n,p,\lambda}(h^{(q)}, g^{(q)}_m)$. It suffices to show that g_m satisfies the condition (14). In view of Theorem 2, we have

$$\psi\left[\sum_{k=n+p}^{\infty}(k+\lambda-q-\alpha(p+\lambda-q))\frac{k!}{(k-q)!}A_k + \sum_{k=n+p}^{\infty}(k+\lambda-q-\alpha(p+\lambda-q))\frac{k!}{(k-q)!}B_k\right] \leqslant$$
$$\leqslant (1-\alpha)(p+\lambda-q) - \frac{(n-1+(1+\alpha)(p+\lambda-q))(1-\ell-\ell(n+p+\lambda-1-q))q^m_{n+p-1}}{((1-\ell)+\ell(p+\lambda-q))q^m_p}B_{n+p-1}.$$

Then

$$\sum_{k=n+p}^{\infty}\left(\frac{k!}{(k-q)!}k\right)(A_k+B_k) \leqslant \frac{(1-\alpha)(p+\lambda-q)}{\psi} -$$
$$-\frac{(n-1+(1+\alpha)(p+\lambda-q))(1-\ell-\ell(n+p+\lambda-1-q))q^m_{n+p-1}}{\psi((1-\ell)+\ell(p+\lambda-q))q^m_p}B_{n+p-1} +$$
$$+(q+\alpha(p+\lambda-q))\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}k(A_k+B_k) \leqslant \frac{(1-\alpha)(p+\lambda-q)}{\psi} -$$
$$-\frac{(n-1+(1+\alpha)(p+\lambda-q))(1-\ell-\ell(n+p+\lambda-1-q))q^m_{n+p-1}}{\psi((1-\ell)-\ell(p+\lambda-q))(n+p+\lambda-q)^m}B_{n+p-1} +$$
$$+\frac{(q+\alpha(p+\lambda-q))}{n+p}\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}k(A_k+B_k),$$

so that,

$$\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}k(A_k+B_k) \leqslant \frac{(n+p)(1-\alpha)(p+\lambda-q)}{\psi(n+((1-\alpha)(p+\lambda-q)))} -$$
$$-\frac{(n+p)(n-1+(1+\alpha)(p+\lambda-q))(1-\ell-\ell(n+p+\lambda-1-q))q^m_{n+p-1}}{(n+(1-\alpha)(p+\lambda-q))(1-\ell-\ell(n+p+\lambda-q))(n+p+\lambda-q)^m}B_{n+p-1} =$$
$$= \delta - \frac{(n+p-1)!(n+p-1)}{(n+p-1-q)!}B_{n+p-1},$$

which, in view of definition (14), completes the proof. □

Theorem 6. *Let* $\ell < \dfrac{1}{p+\lambda-q}$ *and* $\alpha \geqslant 1 - \dfrac{1}{p+\lambda-q}$. *If* $g_m \in \overline{SH}^{m,n}_{p,\lambda}(q,\ell,\alpha)$, *then*

$$N^{\delta}_{n,p,\lambda}(h^{(q)}, g_m^{(q)}) \subset SH^*(n,p),$$

where $h(z)$ *and* $g_m(z)$ *are given by* (14),

$$\delta \leqslant \frac{(n+p-1)!}{(n+p-1-q)!}p - \chi(1-\alpha)(p-q) + \\ + \left(\frac{(n+p-q)!(n-1+(1+\alpha)(p+\lambda-q))(1-\ell-\ell)(n+p+\lambda-1-q)q^m_{n+p-1}}{(n+(1-\alpha)(p+\lambda-q))(1-\ell-\ell(n+p+\lambda-q))(n+p+\lambda-q)^m(n+p-1)!} - \\ - (n+p-1)\right) b_{n+p-1},$$

and

$$\chi = \frac{(1-\ell+\ell(n+p+\lambda-q))(n+p-q)!q^m_p}{((1-\ell)-\ell(p+\lambda-q))(n+p+\lambda-q)^m(n+p-1)!}.$$

P r o o f. Suppose that $f_m(z) \in \overline{SH}^{m,n}_{p,\lambda}(q,\ell,\alpha)$ and $g_m(z) \in N^{\delta}_{n,p,\lambda}(h^{(q)}, g_m^{(q)})$. We need to show that g_m satisfies the condition (7). We have

$$\sum_{k=n+p}^{\infty} k(A_k+B_k)+(n+p-1)B_{n+p-1} \leqslant \sum_{k=n+p}^{\infty} k[|a_k-A_k|+|b_k-B_k|]+(n+p-1)|b_{n+p-1}-B_{n+p-1}|+ \\ + \sum_{k=n+p}^{\infty} k(a_k+b_k) + (n+p-1)b_{n+p-1} \leqslant \\ \leqslant \frac{(n+p-1-q)!}{(n+p-1)!}\left[\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}k\Big[|a_k-A_k|+|b_k-B_k|\Big] + \\ + \frac{(n+p-1)!(n+p-1)}{(n+p-1-q)!|b_{n+p-1}-B_{n+p-1}|}\right] + (n+p-1)b_{n+p-1} \\ \chi\left(\sum_{k=n+p}^{\infty}\left(\frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell+\ell(k+\lambda-q))\frac{q^m_k}{q^m_p}a_k}{(1-\ell+\ell(p+\lambda-q))} + \\ + \frac{(k+\lambda-q-\alpha(p+\lambda-q))(1-\ell-\ell(k+\lambda-q))\frac{q^m_k}{q^m_p}b_k}{(1-\ell+\ell(p+\lambda-q))}\right)\right) \leqslant \\ \leqslant \frac{(n+p-1-q)!}{(n+p-1)!}\delta + (n+p-1)b_{n+p-1} + \\ + \chi\Big((1-\alpha)(p+\lambda-q) - \frac{(n-1+(1+\alpha)(p+\lambda-q))(1-\ell-\ell(n+p-1+\lambda-q))}{(1-\ell+\ell(p+\lambda-q))q^m_p}\frac{q^m_{n+p-1}}{q^m_p}b_{n+p-1}\Big).$$

This expression is never greater than p provided that

$$\delta \leqslant \frac{(n+p-1)!}{(n+p-1-q)!}\Big[p - \chi(1-\alpha)(p-q) + \\ + \frac{(n+p-q)!(n-1+(1+\alpha)(p+\lambda-q))(1-\ell-\ell(n+p-1+\lambda-q))q^m_{n+p-1}}{(n+(1-\alpha)(p+\lambda-q))(1-\ell-\ell(n+p+\lambda-q))(n+p+\lambda-q)^m(n+p-1)!} - \\ - (n+p-1))b_{n+p-1}\Big].$$

The proof of the other case is similar and so is omitted. □

Remarks. Different type of results involving the harmonic functions can be read in [2–8].

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Об одном классе гармонических многолистных функций

Ключевые слова: гармонические многолистные функции, оператор производной, окрестность.

УДК 517.53

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С помощью обобщенного оператора производной вводится новый подкласс в классе гармонических многолистных функций. Установлены оценки на коэффициенты, неравенства искажения и включения с окрестностями для различных подклассов в классе гармонических многолистных функций.

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