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FINITE SPECTRUM ASSIGNMENT PROBLEM FOR BILINEAR SYSTEMS WITH SEVERAL DELAYS

A bilinear control system defined by a linear stationary differential system with several non-commensurate delays in the state variable is considered. A problem of finite spectrum assignment by constant control is studied. One needs to construct constant control vectors such that the characteristic function of the closed-loop system is equal to a polynomial with arbitrary given coefficients. Conditions on coefficients of the system are obtained under which the criterion was found for solvability of the finite spectrum assignment problem. Interconnection of the criterion conditions with the property of consistency for the truncated system without delays is shown. Corollaries on stabilization of bilinear systems with delays are obtained. The similar results are obtained for discrete-time bilinear systems with several delays. An illustrative example is considered.

Keywords: linear systems with delays, spectrum assignment, stabilization, bilinear system.

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Introduction

To date, a fairly large number of studies have been devoted to the problems of spectrum control and stabilization of delayed systems, which have already become classic. These are works on stabilization of an object with delay [1–5], stabilization of a group of objects by a single controller [6], assignment of a given finite spectrum [7–9], spectral reducibility [10–12], i.e., reduction of systems to a finite (but not given) spectrum, modal controllability [13–17]. At present, for retarded and neutral type systems, as well as for completely regular differential-algebraic systems with several delays, spectral criteria of modal controllability have been obtained [13, 15, 16] that coincide in form with the criterion of complete controllability (see, for example, [18]). They are also solvability conditions for the problem of assigning a finite spectrum [7, 9].

One of the features that delay systems may possess is the presence of invariant eigenvalues that can be excluded from the spectrum only by using integral regulators [9]. Respectively, in general case, the closed-loop system becomes a system with lumped and distributed delays. In the case of practical realization, integrals containing distributed delay are replaced by finite sums, which, even when using quadrature formulas of high accuracy, can lead to undesirable effects [4, 19, 20]. As an alternative to the situation described, some works [13, 14, 16] (see also the introduction in [13]) offer sufficient conditions for modal controllability in the class of differential-difference controllers. Such works are based on the transformation of the object under study to a certain special form, which makes it easy to obtain the required controller, and formulate solvability conditions in terms of a controllability matrix or its analogue.

Stabilization problems for bilinear systems with delays were studied in many papers; see, e.g., [21–25]. In this article, we study the problem of assigning an arbitrary finite spectrum to a bilinear system with delays. In the paper [26] (see also [27]), the sufficient conditions have been obtained for solvability of arbitrary finite spectrum assignment problem for a bilinear system with one delay in the state variable. In the present paper, we extend the results of [26] to systems with several noncommensurate delays. Sufficient conditions for solvability of arbitrary finite spectrum assignment problem are obtained in the class of difference controllers. The possibility of extending the results to the case of discrete systems is shown. The results of Section 1 of this paper (Theorem 1 and Corollary 1) was announced, without a proof, in the conference proceedings [28].

§ 1. Continuous-time systems with delays

Suppose $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$; $\mathbb{K}^n = \{x = \operatorname{col}(x_1, \ldots, x_n) \colon x_i \in \mathbb{K}\}$ is the linear space of column vectors over $\mathbb{K}$; $M_{m,n}(\mathbb{K})$ is the space of $m \times n$-matrices over $\mathbb{K}$; $M_n(\mathbb{K}) := M_{n,n}(\mathbb{K})$; $I \in M_n(\mathbb{K})$ is the identity matrix; T is the transposition of a vector or a matrix; $*$ is the Hermitian conjugation, i.e., $A^* = \overline{A}^T$; $\chi(H; \lambda)$ is the characteristic polynomial of a matrix $H \in M_n(\mathbb{K})$; $\operatorname{Sp} H$ is the trace of a matrix $H \in M_n(\mathbb{K})$; for a matrix $H \in M_n(\mathbb{K})$, we use the denotation $H^0 := I$.

Consider a bilinear differential system with constant coefficients with several noncommensurate delays in the state variable of the following form:

$$\begin{aligned}\dot{x}(t) = {} & A_{00}x(t) + u_{01}A_{01}x(t) + \ldots + u_{0r_0}A_{0r_0}x(t) + \\ & + A_{10}x(t-h_1) + u_{11}A_{11}x(t-h_1) + \ldots + u_{1r_1}A_{1r_1}x(t-h_1) + \ldots + \\ & + A_{s0}x(t-h_s) + u_{s1}A_{s1}x(t-h_s) + \ldots + u_{sr_s}A_{sr_s}x(t-h_s), \quad t > 0,\end{aligned} \tag{1}$$

with initial conditions $x(\tau) = \mu(\tau)$, $\tau \in [-h_s, 0]$; here $h_j > 0$ are constant delays such that $0 = h_0 < h_1 < \ldots < h_s$, $\mu \colon [-h_s, 0] \to \mathbb{K}^n$ is a continuous function, $x \in \mathbb{K}^n$ is a state vector, $u_j = \operatorname{col}(u_{j1}, \ldots, u_{jr_j}) \in \mathbb{K}^{r_j}$ are control vectors, $A_{j\nu} \in M_n(\mathbb{K})$, $j = \overline{0, s}$, $\nu = \overline{0, r_j}$.

In [29] the following linear stationary differential control system with several noncommensurate delays was considered:

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{s} P_j x(t-h_j) + Bw(t), \quad t > 0, \tag{2}$$

$$y(t) = C^* x(t), \tag{3}$$

where $A, P_j \in M_n(\mathbb{K})$, $j = \overline{1, s}$, $B \in M_{n,m}(\mathbb{K})$, $C \in M_{n,k}(\mathbb{K})$, $h_j > 0$ are constant delays such that $0 = h_0 < h_1 < \ldots < h_s$, $x \in \mathbb{K}^n$ is a state vector, $w \in \mathbb{K}^m$ is an input vector, and $y \in \mathbb{K}^k$ is an output vector. For the system (2), (3) in [29] the controller is constructed as linear static output feedback with delays

$$w(t) = \sum_{j=0}^{s} Q_j y(t-h_j), \quad t > 0, \tag{4}$$

where $Q_j \in M_{m,k}(\mathbb{K})$ ($j = \overline{0, s}$) are constant. The closed-loop system (2), (3), (4) takes the form

$$\dot{x}(t) = (A + BQ_0C^*)x(t) + \sum_{j=1}^{s} (P_j + BQ_jC^*)x(t-h_j). \tag{5}$$

Sufficient conditions for assigning an arbitrary finite spectrum for the system (5) have been obtained in [29, § 2]. The system (5) can be considered as a particular case of the system (1). In fact, every system (5), where $B = [b_1, \ldots, b_m]$, $C = [c_1, \ldots, c_k]$, $b_i, c_\ell \in \mathbb{K}^n$, $Q_j = \{\alpha^j_{i\ell}\}$, $\alpha^j_{i\ell} \in \mathbb{K}$, $j = \overline{0, s}$, $i = \overline{1, m}$, $\ell = \overline{1, k}$, can be rewritten in the form (1), where $r_0 = r_1 = \ldots =$

$r_s = r := mk$,

$$A_{00} = A, \quad A_{j0} = P_j \ (j = \overline{1,s}), \tag{6}$$

$$\begin{aligned} &A_{01} = A_{11} = \ldots = A_{s1} = b_1 c_1^*, && \ldots, \quad A_{0k} = A_{1k} = \ldots = A_{sk} = b_1 c_k^*, \\ &A_{0,k+1} = A_{1,k+1} = \ldots = A_{s,k+1} = b_2 c_1^*, && \ldots, \quad A_{0,2k} = A_{1,2k} = \ldots = A_{s,2k} = b_2 c_k^*, \ldots, \\ &A_{0,r-k+1} = \ldots = A_{s,r-k+1} = b_m c_1^*, && \ldots, \quad A_{0r} = A_{1r} = \ldots = A_{sr} = b_m c_k^*, \end{aligned} \tag{7}$$

$$u_j = \operatorname{col}(\alpha_{11}^j, \alpha_{12}^j, \ldots, \alpha_{1k}^j, \alpha_{21}^j, \ldots, \alpha_{2k}^j, \ldots, \alpha_{m1}^j, \ldots, \alpha_{mk}^j), \quad j = \overline{0,s}. \tag{8}$$

In the present paper, we obtain sufficient conditions for assigning an arbitrary finite spectrum for the system (1). These results extend the results [29, § 2] from the system (5) to the system (1).

Let us denote by

$$\varphi(\lambda, e^{-\lambda}) = \det\Big[\lambda I - \big(A_{00} + \sum_{\nu=1}^{r_0} u_{0\nu} A_{0\nu}\big) - \sum_{j=1}^{s} e^{-\lambda h_j}\big(A_{j0} + \sum_{\nu=1}^{r_j} u_{j\nu} A_{j\nu}\big)\Big]$$

the characteristic function of the system (1). The characteristic equation $\varphi(\lambda, e^{-\lambda}) = 0$ of the system (1) has the form

$$\lambda^n + \sum_{i=0}^{n-1} \lambda^{n-1-i} \sum_{0 \leqslant \rho_0 \leqslant \ldots \leqslant \rho_i \leqslant s} \delta_{\rho_0 \ldots \rho_i} \prod_{\mu=0}^{i} \exp(-\lambda h_{\rho_\mu}) = 0. \tag{9}$$

Here numbers $\delta_{\rho_0 \ldots \rho_i}$ depend on $A_{j\nu}$, $u_{j\nu}$. The set $\sigma = \{\lambda \in \mathbb{C} : \varphi(\lambda, e^{-\lambda}) = 0\}$ of the roots of (9) is called the spectrum of the system (1). In general, the spectrum σ of a system with delays (1) is countable. If $\delta_{\rho_0 \ldots \rho_i} = 0$ for all $i = \overline{0, n-1}$, $0 \leqslant \rho_0 \leqslant \ldots \leqslant \rho_i \leqslant s$ (possibly with the exception of δ_0, δ_{00}, …, $\delta_{0 \ldots 0}$) in the equation (9), then the characteristic quasi-polynomial is polynomial and the spectrum σ is finite. Consider the problem of assigning an arbitrary finite spectrum σ for the system (1) by constant control.

Definition 1. The system (1) is called *arbitrary finite spectrum assignable by constant control* [26] if for any $\gamma_i \in \mathbb{K}$, $i = \overline{1,n}$, there exist $u_j \in \mathbb{K}^{r_j}$ $(j = \overline{0,s})$ such that

$$\varphi(\lambda, e^{-\lambda}) = \lambda^n + \gamma_1 \lambda^{n-1} + \ldots + \gamma_n.$$

Suppose that the matrices of the system (1) have the following special form: the matrix A_{00} has the lower Hessenberg form with nonzero superdiagonal entries; for some $p \in \{1, \ldots, n\}$, the first $p-1$ rows and the last $n-p$ columns of $A_{j\nu}$, $j = \overline{0,s}$, $\nu = \overline{0, r_j}$ $((j,\nu) \neq (0,0))$, are equal to zero, i.e.,

$$A_{00} = \begin{pmatrix} a_{11} & a_{12} & 0 & \ldots & 0 \\ a_{21} & a_{22} & a_{23} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n-1,1} & a_{n-1,2} & \ldots & \ldots & a_{n-1,n} \\ a_{n1} & a_{n2} & \ldots & \ldots & a_{nn} \end{pmatrix}, \qquad a_{i,i+1} \neq 0, \quad i = \overline{1, n-1}; \tag{10}$$

$$A_{j\nu} = \begin{pmatrix} 0 & 0 \\ \widehat{A}_{j\nu} & 0 \end{pmatrix}, \quad \widehat{A}_{j\nu} \in M_{n-p+1,p}(\mathbb{K}),\ j = \overline{0,s},\ \nu = \overline{0,r_j},\ (j,\nu) \neq (0,0). \tag{11}$$

For the system (1), (10), (11) without delays (i.e., for the case $A_{j\nu} = 0$, $j = \overline{1,s}$, $\nu = \overline{0,r_j}$), it was proved in [30] (see also [31]) that the system is arbitrary finite spectrum assignable by constant control iff the rank of the matrix $\mathit{\Gamma} = \{\mathrm{Sp}\,(A_{0j}A_{00}^{i-1})_{i,j=1}^{n,r_0}\}$ is equal to n. Here this result is extended to systems with several delays.

Let $\chi(A_{00};\lambda) = \lambda^n + \alpha_1\lambda^{n-1} + \ldots + \alpha_n$. Set $\alpha_0 := 1$. Let us construct the matrices

$$F_k = \alpha_0 A_{00}^k + \alpha_1 A_{00}^{k-1} + \ldots + \alpha_k I, \quad k = \overline{0, n-1}. \tag{12}$$

Further, we will use the following lemma (see [32, Lemma 1]).

Lemma 1. *Suppose a matrix A_{00} has the form (10) and a matrix $D \in M_n(\mathbb{K})$ has the following form for some $p \in \{1,\ldots,n\}$:*

$$D = \begin{pmatrix} 0 & 0 \\ D_1 & 0 \end{pmatrix}, \quad D_1 \in M_{n-p+1,p}(\mathbb{K}). \tag{13}$$

Let $\chi(A_{00}+D;\lambda) = \lambda^n + \gamma_1\lambda^{n-1} + \ldots + \gamma_n$. Then $\gamma_i = \alpha_i - \mathrm{Sp}\,(DF_{i-1})$ for all $i = 1,\ldots,n$.

From the system (1), we construct the matrices $\mathit{\Gamma}_j \in M_{n,r_j}(\mathbb{K})$ ($j = \overline{0,s}$), $\mathit{\Lambda}_j \in M_{n,1}(\mathbb{K})$ ($j = \overline{1,s}$):

$$\mathit{\Gamma}_0 = \begin{pmatrix} \mathrm{Sp}\,(A_{01}) & \mathrm{Sp}\,(A_{02}) & \ldots & \mathrm{Sp}\,(A_{0r_0}) \\ \mathrm{Sp}\,(A_{01}A_{00}) & \mathrm{Sp}\,(A_{02}A_{00}) & \ldots & \mathrm{Sp}\,(A_{0r_0}A_{00}) \\ \dots & \dots & \dots & \dots \\ \mathrm{Sp}\,(A_{01}A_{00}^{n-1}) & \mathrm{Sp}\,(A_{02}A_{00}^{n-1}) & \ldots & \mathrm{Sp}\,(A_{0r_0}A_{00}^{n-1}) \end{pmatrix}, \tag{14}$$

$$\mathit{\Gamma}_j = \begin{pmatrix} \mathrm{Sp}\,(A_{j1}) & \mathrm{Sp}\,(A_{j2}) & \ldots & \mathrm{Sp}\,(A_{jr_j}) \\ \mathrm{Sp}\,(A_{j1}A_{00}) & \mathrm{Sp}\,(A_{j2}A_{00}) & \ldots & \mathrm{Sp}\,(A_{jr_j}A_{00}) \\ \dots & \dots & \dots & \dots \\ \mathrm{Sp}\,(A_{j1}A_{00}^{n-1}) & \mathrm{Sp}\,(A_{j2}A_{00}^{n-1}) & \ldots & \mathrm{Sp}\,(A_{jr_j}A_{00}^{n-1}) \end{pmatrix}, \; \mathit{\Lambda}_j = \begin{pmatrix} \mathrm{Sp}\,(A_{j0}) \\ \mathrm{Sp}\,(A_{j0}A_{00}) \\ \dots \\ \mathrm{Sp}\,(A_{j0}A_{00}^{n-1}) \end{pmatrix}; \tag{15}$$

and construct the matrices $\mathit{\Delta}_j = [\mathit{\Gamma}_j, \mathit{\Lambda}_j] \in M_{n,r_j+1}(\mathbb{K})$, $j = \overline{1,s}$.

Theorem 1. *Suppose that the matrices of the system (1) have the special form (10), (11). Then the system (1) is arbitrary finite spectrum assignable by constant control iff the following conditions hold:*

$$\mathrm{rank}\,\mathit{\Gamma}_0 = n, \tag{16}$$

$$\mathrm{rank}\,\mathit{\Gamma}_j = \mathrm{rank}\,\mathit{\Delta}_j, \quad j = \overline{1,s}. \tag{17}$$

P r o o f. Suppose the matrices of the system (1) have the form (10), (11). Consider the problem of assigning an arbitrary finite spectrum. Let a polynomial

$$q(\lambda) = \lambda^n + \gamma_1\lambda^{n-1} + \ldots + \gamma_n \tag{18}$$

with numbers $\gamma_i \in \mathbb{K}$ be given. One needs to construct $u_j \in \mathbb{K}^{r_j}$, $j = \overline{0,s}$, such that the characteristic quasi-polynomial $\varphi(\lambda, e^{-\lambda})$ of the system (1) satisfies the equality

$$\varphi(\lambda, e^{-\lambda}) = q(\lambda). \tag{19}$$

Denote

$$D = \sum_{\nu=1}^{r_0} u_{0\nu}A_{0\nu} + \sum_{j=1}^{s} e^{-\lambda h_j}\big(A_{j0} + \sum_{\nu=1}^{r_j} u_{j\nu}A_{j\nu}\big). \tag{20}$$

We have

$$\varphi(\lambda, e^{-\lambda}) = \det\big(\lambda I - (A_{00} + D)\big) = \chi(A_0 + D; \lambda). \tag{21}$$

It follows from conditions (11) that the matrix (20) has the form (13). Taking into account (21), (19), (18), condition (10), and applying Lemma 1, we obtain that the system (1) is arbitrary finite spectrum assignable by constant control iff there exist $u_j \in \mathbb{K}^{r_j}$, $j = \overline{0, s}$, such that for all $i = \overline{1, n}$ the following equalities hold:

$$\gamma_i = \alpha_i - \operatorname{Sp}\Big(\big(\sum_{\nu=1}^{r_0} u_{0\nu} A_{0\nu}\big) F_{i-1}\Big) - \sum_{j=1}^{s} e^{-\lambda h_j} \operatorname{Sp}\Big(\big(A_{j0} + \sum_{\nu=1}^{r_j} u_{j\nu} A_{j\nu}\big) F_{i-1}\Big). \tag{22}$$

Equalities (22) hold iff for all $i = 1, \dots, n$

$$\gamma_i = \alpha_i - \operatorname{Sp}\Big(\big(\sum_{\nu=1}^{r_0} u_{0\nu} A_{0\nu}\big) F_{i-1}\Big), \tag{23}$$

$$\operatorname{Sp}\Big(\big(A_{j0} + \sum_{\nu=1}^{r_j} u_{j\nu} A_{j\nu}\big) F_{i-1}\Big) = 0 \quad \text{for all} \quad j = 1, \dots, s. \tag{24}$$

Taking into account the definition (12) of the matrices F_k, we have

$$\operatorname{Sp}\Big(\big(\sum_{\nu=1}^{r_0} u_{0\nu} A_{0\nu}\big) F_{i-1}\Big) = \sum_{\ell=0}^{i-1} \alpha_{i-1-\ell}\Big(\sum_{\nu=1}^{r_0} u_{0\nu} \operatorname{Sp}(A_{0\nu} A_{00}^{\ell})\Big),$$

$$\operatorname{Sp}\Big(\big(A_{j0} + \sum_{\nu=1}^{r_j} u_{j\nu} A_{j\nu}\big) F_{i-1}\Big) = \sum_{\ell=0}^{i-1} \alpha_{i-1-\ell}\Big(\operatorname{Sp}(A_{j0} A_{00}^{\ell}) + \sum_{\nu=1}^{r_j} u_{j\nu} \operatorname{Sp}(A_{j\nu} A_{00}^{\ell})\Big).$$

Therefore the equalities (23), (24) are equivalent to $(1 + s)$ systems of n linear equations, where every jth system has r_j unknown variables of the vector u_j, $j = \overline{0, s}$:

$$\sum_{\ell=0}^{i-1} \alpha_{i-1-\ell}\Big(\sum_{\nu=1}^{r_0} u_{0\nu} \operatorname{Sp}(A_{0\nu} A_{00}^{\ell})\Big) = \alpha_i - \gamma_i, \qquad i = \overline{1, n}, \tag{25}$$

$$\sum_{\ell=0}^{i-1} \alpha_{i-1-\ell}\Big(\sum_{\nu=1}^{r_j} u_{j\nu} \operatorname{Sp}(A_{j\nu} A_{00}^{\ell})\Big) = -\sum_{\ell=0}^{i-1} \alpha_{i-1-\ell} \operatorname{Sp}(A_{j0} A_{00}^{\ell})\Big), \quad i = \overline{1, n}, \quad j = \overline{1, s}. \tag{26}$$

Let us construct the matrices

$$G := \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \alpha_1 & 1 & 0 & \dots & 0 \\ \alpha_2 & \alpha_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{n-1} & \alpha_{n-2} & \alpha_{n-3} & \dots & 1 \end{pmatrix}, \tag{27}$$

and (14), (15). Denote $w_0 := \operatorname{col}(\alpha_1 - \gamma_1, \dots, \alpha_n - \gamma_n) \in \mathbb{K}^n$. Then one can rewrite $(1 + s)$ systems (25), (26) in the vector form

$$G\varGamma_0 u_0 = w_0, \tag{28}$$

$$G\varGamma_j u_j = -G\varLambda_j, \quad j = \overline{1, s}. \tag{29}$$

Taking into account that $\det G = 1 \neq 0$, we see that the system (28) is resolvable with respect to u_0 (over $\mathbb{K}$) for any pregiven $\gamma_i \in \mathbb{K}$, $i = \overline{1, n}$, iff condition (16) holds, and the systems (29) are resolvable with respect to u_j, $j = \overline{1, s}$, (over $\mathbb{K}$) iff conditions (17) hold. Finding u_0, u_j, $j = \overline{1, s}$, from (28), (29), we assign the polynomial (18) as the characteristic function for the system (1). □

Remark 1. Let us show that Theorem 1 is a generalization of [29, Theorem 2]. Suppose that the system (1) has the form (5), i.e., the equalities (6), (7), (8) are fulfilled. Suppose that the matrices of the system (5) (i.e., of the system (2.4) of [29]) have the form (1.6), (1.7), (2.5) of [29], that is the matrix A has the lower Hessenberg form with nonzero superdiagonal entries (i.e., the form (10)), the first $p-1$ rows of the matrix B and the last $n-p$ rows of the matrix C are equal to zero, and P_j ($j=\overline{1,s}$) have the form (11). Then the matrices (6), (7) will satisfy the form (10), (11). Next, the condition 1 of [29, Theorem 2], which states that, for the system (5), the matrices C^*B, C^*AB, ..., $C^*A^{n-1}B$ are linearly independent, is equivalent to the condition (16), where the matrices A_{0j}, $j=\overline{0,r_0}$, in (14) are defined by equalities (6), (7). It follows from Lemma 2 below. Next, by (7), we have $\Gamma_j=\Gamma_0$, $j=\overline{1,s}$. Hence, if $\operatorname{rank}\Gamma_0=n$, then $\operatorname{rank}\Gamma_j=n$, i.e., the matrices Γ_j ($j=\overline{1,s}$) have the full rank. Hence, $\operatorname{rank}\Delta_j=n$, $j=\overline{1,s}$, i.e., condition (17) holds. Thus, if the system (1) has the form (5), then the condition 1 of [29, Theorem 2] is equivalent to (16)&(17). Therefore, Theorem 1 is a generalization of [29, Theorem 2]. □

Lemma 2. *Let* $A\in M_n(\mathbb{K})$, $B=[b_1,\dots,b_m]\in M_{n,m}(\mathbb{K})$, $C=[c_1,\dots,c_k]\in M_{n,k}(\mathbb{K})$, $b_i,c_\ell\in\mathbb{K}^n$, $i=\overline{1,m}$, $\ell=\overline{1,k}$. *Set* $r:=mk$. *Let the following matrices be constructed:*

$$\begin{array}{llll} A_1:=b_1c_1^*, & A_2:=b_1c_2^*, & \dots, & A_k=b_1c_k^*,\\ A_{k+1}:=b_2c_1^*, & A_{k+2}:=b_2c_2^*, & \dots, & A_{2k}=b_2c_k^*, \quad \dots,\\ A_{r-k+1}:=b_mc_1^*, & A_{r-k+2}:=b_mc_2^*, & \dots, & A_r=b_mc_k^*; \end{array}$$

$$\Gamma=\begin{pmatrix} \operatorname{Sp}(A_1) & \operatorname{Sp}(A_2) & \dots & \operatorname{Sp}(A_r)\\ \operatorname{Sp}(A_1A) & \operatorname{Sp}(A_2A) & \dots & \operatorname{Sp}(A_rA)\\ \dots & \dots & \dots & \dots\\ \operatorname{Sp}(A_1A^{n-1}) & \operatorname{Sp}(A_2A^{n-1}) & \dots & \operatorname{Sp}(A_rA^{n-1}) \end{pmatrix}.$$

Then the following conditions are equivalent:

1. *The matrices* C^*AB, ..., $C^*A^{n-1}B$ *are linearly independent.*
2. $\operatorname{rank}\Gamma=n$.

P r o o f. (1 ⇒ 2). Suppose $\operatorname{rank}\Gamma<n$. Then there exists $\alpha=\operatorname{col}(\alpha_1,\dots,\alpha_n)\in\mathbb{K}^n$, $\alpha\neq 0$, such that $\alpha^T\Gamma=0\in(\mathbb{K}^r)^T$, i.e.,

$$\alpha_1\cdot\operatorname{Sp}(A_j)+\alpha_2\cdot\operatorname{Sp}(A_jA)+\dots+\alpha_n\cdot\operatorname{Sp}(A_jA^{n-1})=0,\quad \forall j=\overline{1,r}.$$

Hence, for all $i=\overline{1,m}$, $\ell=\overline{1,k}$

$$\begin{aligned} 0&=\operatorname{Sp}\big(b_ic_\ell^*(\alpha_1I+\alpha_2A+\dots+\alpha_nA^{n-1})\big)=\\ &=\operatorname{Sp}\big(c_\ell^*(\alpha_1I+\alpha_2A+\dots+\alpha_nA^{n-1})b_i\big)=\\ &=c_\ell^*(\alpha_1I+\alpha_2A+\dots+\alpha_nA^{n-1})b_i. \end{aligned}$$

Therefore,

$$C^*(\alpha_1I+\alpha_2A+\dots+\alpha_nA^{n-1})B=0\in M_{k,m}(\mathbb{K}),$$

i.e.,

$$\alpha_1(C^*B)+\alpha_2(C^*AB)+\dots+\alpha_n(C^*A^{n-1}B)=0\in M_{k,m}(\mathbb{K}).$$

This contradicts condition 1.

Arguments in the reverse order prove the implication (2 ⇒ 1). □

Remark 2. Theorem 1 extends results of [26, Theorem 1] from systems (1) with one delay ($s=1$) to systems (1) with several delays. Also, Theorem 1 extends the results of [30] from bilinear systems without delays to bilinear systems (1) with several delays (see [26, Remark 2]). □

The obvious corollary on stabilization follows from Theorem 1. Choosing the polynomial (18) in such a way that its roots belong to left half-plane $\omega_\eta = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < -\eta < 0\}$, one can obtain exponential stability for the system (1) with any pregiven decay rate $\eta > 0$.

Corollary 1. *Suppose that the matrices of the system* (1) *have the special form* (10)*,* (11)*. Suppose conditions* (16)*,* (17) *hold. Then the system* (1) *is exponentially stabilizable by constant control with an arbitrary pregiven decay rate.*

For the system (1), let us construct the "truncated system" (without delays) assuming $A_{j\nu} = 0$, $j = \overline{1, s}$, $\nu = \overline{0, r_j}$,

$$\dot{x}(t) = \Big(A_{00} + \sum_{\nu=1}^{r_0} u_{0\nu} A_{0\nu}\Big) x(t). \tag{30}$$

Let us denote by $X(t, s)$ the transition matrix of the free system $\dot{x}(t) = A_{00}x(t)$. Hence, $X(t, s) = e^{(t-s)A_{00}}$.

Definition 2. The system (30) is said to be *consistent on* $[t_0, t_1]$ if for any $H \in M_n(\mathbb{K})$ there exists a piecewise continuous control function $\widehat{u}_0 : [t_0, t_1] \to \mathbb{K}^{r_0}$ such that the solution of the $n \times n$-matrix initial value problem

$$\dot{Z}(t) = A_{00}Z(t) + \sum_{\nu=1}^{r_0} \big(\widehat{u}_{0\nu}(t) A_{0\nu}\big) X(t, t_0), \qquad Z(t_0) = 0,$$

satisfies condition $Z(t_1) = H$.

The property of consistency was introduced in [33] for continuous-time systems (30), which are not necessarily stationary. For systems (30) with a cyclic matrix A_{00} (in particular, with A_{00} of the form (10)), the property of consistency is sufficient for condition (16) to be fulfilled (see [34, Assertion 5]). Thus, the following theorem holds.

Theorem 2. *Suppose that the matrices of the system* (1) *have the special form* (10)*,* (11)*. Suppose that the truncated system* (30) *is consistent and conditions* (17) *hold. Then the system* (1) *is arbitrary finite spectrum assignable by constant control.*

Remark 3. Theorem 1 together with Theorem 2 extends Theorem 2 of [34] from bilinear systems without delay (30) to bilinear systems (1) with delays (see also [26, Remark 3]). □

§ 2. Discrete-time systems with delays

Consider a bilinear discrete-time system with constant coefficients with several delays in the state variable of the following form:

$$\begin{aligned} x(t+1) = {} & A_{00}x(t) + u_{01}A_{01}x(t) + \ldots + u_{0r_0}A_{0r_0}x(t) + \\ & + A_{10}x(t-h_1) + u_{11}A_{11}x(t-h_1) + \ldots + u_{1r_1}A_{1r_1}x(t-h_1) + \ldots + \\ & + A_{s0}x(t-h_s) + u_{s1}A_{s1}x(t-h_s) + \ldots + u_{sr_s}A_{sr_s}x(t-h_s), \end{aligned} \tag{31}$$

$t = 0, 1, 2, \ldots$, with initial conditions $x(\tau) = \mu(\tau)$, $\tau = -h_s, -h_s + 1, \ldots, 0$; here $h_j > 0$ are integer constant delays such that $0 = h_0 < h_1 < \ldots < h_s$, $\mu(\tau) \in \mathbb{K}^n$ $(\tau = -h_s, \ldots, 0)$, $x \in \mathbb{K}^n$ is a state vector, $u_j = \operatorname{col}(u_{j1}, \ldots, u_{jr_j}) \in \mathbb{K}^{r_j}$ are control vectors, $A_{j\nu} \in M_n(\mathbb{K})$, $j = \overline{0, s}$, $\nu = \overline{0, r_j}$.

Denote by

$$\psi(\lambda) = \det\Big[\lambda I - \big(A_{00} + \sum_{\nu=1}^{r_0} u_{0\nu} A_{0\nu}\big) - \sum_{j=1}^{s} \lambda^{-h_j}\big(A_{j0} + \sum_{\nu=1}^{r_j} u_{j\nu} A_{j\nu}\big)\Big]$$

the characteristic function of the system (31). This function is rational. The characteristic equation $\psi(\lambda) = 0$ of the system (31) has the form

$$\lambda^n + \sum_{i=0}^{n-1} \lambda^{n-1-i} \sum_{0 \leqslant \rho_0 \leqslant \ldots \leqslant \rho_i \leqslant s} \delta_{\rho_0 \ldots \rho_i} \prod_{\mu=0}^{i} \lambda^{-h_{\rho_\mu}} = 0. \tag{32}$$

The spectrum of the system (31) is the set $\theta = \{\lambda \in \mathbb{C} : \psi(\lambda) = 0\}$ of the roots of (32). In general, the spectrum θ of a discrete-time system with delay (31) consists of a finite amount $N \geqslant n$ of numbers $\lambda_m \in \mathbb{C}$, $m = \overline{1, N}$. The spectrum θ consists of exactly n points (with accounting the multiplicity) iff $\delta_{\rho_0 \ldots \rho_i} = 0$ for all $i = \overline{0, n-1}$, $0 \leqslant \rho_0 \leqslant \ldots \leqslant \rho_i \leqslant s$ (possibly with the exception of δ_0, δ_{00}, …, $\delta_{0\ldots0}$) in the equation (32); in that case $\psi(\lambda)$ is polynomial. Consider the problem of assigning an arbitrary n-point spectrum θ for the system (31) by constant control.

Definition 3. The system (31) is called *arbitrary n-point spectrum assignable by constant control* [26] if for any $\gamma_i \in \mathbb{K}$, $i = \overline{1, n}$, there exist $u_j \in \mathbb{K}^{r_j}$ $(j = \overline{0, s})$ such that

$$\psi(\lambda) = \lambda^n + \gamma_1 \lambda^{n-1} + \ldots + \gamma_n.$$

Theorem 3. *Suppose that the matrices of the system* (31) *have the special form* (10)*,* (11)*. Then the system* (31) *is arbitrary n-point spectrum assignable by constant control iff conditions* (16)*,* (17) *hold.*

The proof of Theorem 3 is the same as the proof of Theorem 1.

Remark 4. Theorem 3 extends results of [26, Theorem 3] from systems (31) with one delay $(s = 1)$ to systems (31) with several delays. □

Corollary 2. *Suppose that the matrices of the system* (31) *have the special form* (10)*,* (11)*. Suppose conditions* (16)*,* (17) *hold. Then the system* (31) *is exponentially stabilizable by constant control with an arbitrary pregiven decay rate.*

For the system (31), consider the truncated system

$$x(t+1) = \Big(A_{00} + \sum_{\nu=1}^{r_0} u_{0\nu} A_{0\nu}\Big) x(t). \tag{33}$$

Let us denote by $X(t, s)$ the transition matrix of the free system $x(t+1) = A_{00} x(t)$. Hence, $X(t, s) = A_{00}^{t-s}$, $t \geqslant s$.

Definition 4. The system (33) is said to be *consistent on* $[t_0, t_1) \subset \mathbb{Z}$ [35] if, for any matrix $H \in M_n(\mathbb{K})$, there exists a $\widehat{u}_0(t) = \operatorname{col}\big(\widehat{u}_{01}(t), \ldots, \widehat{u}_{0r_0}(t)\big)$, $t = t_0, \ldots, t_1 - 1$, such that the solution of the $n \times n$-matrix initial value problem

$$Z(t+1) = A_{00} Z(t) + \sum_{\nu=1}^{r_0} \big(\widehat{u}_{0\nu}(t) A_{0\nu}\big) X(t, t_0), \qquad Z(t_0) = 0,$$

satisfies condition $Z(t_1) = H$.

The property of consistency was introduced in [35] for discrete-time systems (33), which are not necessarily stationary. For time-invariant systems (33) with a cyclic matrix A_{00} (in particular, with A_{00} of the form (10)), the property of consistency is sufficient for condition (16) to be fulfilled for discrete-time systems (see [36, Assertion 3]). Thus, the following theorem holds.

Theorem 4. *Suppose that the matrices of the system (31) have the special form (10), (11). Suppose that the truncated system (33) is consistent, and condition (17) holds. Then the system (31) is arbitrary n-point spectrum assignable by constant control.*

Remark 5. Theorem 3 together with Theorem 4 extends Theorem 6 of [36] from bilinear systems without delays (33) to bilinear systems (31) with delays. □

Remark 6. The condition $r_0 \geqslant n$ is obviously necessary both for condition (16) and for the property of consistency of the truncated system (see [34, Corollary 5] for continuous-time systems and [36, Corollary 7] for discrete-time systems). Nevertheless, there is no necessary estimation to r_j $(j = \overline{1, s})$ for condition (17) to be fulfilled.

§ 3. Example

Consider an example illustrating Theorem 1. Suppose $\mathbb{K} = \mathbb{C}$, $n = 3$, $s = 2$, $r_0 = 3$, $r_1 = 3$, $r_2 = 2$, $0 = h_0 < h_1 < h_2$, $p = 2$, and the matrices of the system (1) have the following form:

$$A_{00} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & i & 1 \\ i & 1 & -i \end{pmatrix}, \quad A_{01} = \begin{pmatrix} 0 & 0 & 0 \\ i & i & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_{02} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_{03} = \begin{pmatrix} 0 & 0 & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$A_{10} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_{11} = \begin{pmatrix} 0 & 0 & 0 \\ -i & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -i & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_{13} = \begin{pmatrix} 0 & 0 & 0 \\ -i & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix},$$
$$A_{20} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} 0 & 0 & 0 \\ i & 1 & 0 \\ 1 & -i & 0 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -i & -1 & 0 \end{pmatrix}. \tag{34}$$

The matrices (34) of the system (1) have the special form (10), (11). We have

$$\chi(A_{00}; \lambda) = \lambda^3 - \lambda^2 + \lambda,$$

hence, $\alpha_1 = -1$, $\alpha_2 = 1$, $\alpha_3 = 0$. Let's calculate the matrices (27), (14), (15):

$$G = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \quad \varGamma_0 = \begin{pmatrix} i & 0 & 1 \\ i & -1 & 0 \\ -1 & -i & -i \end{pmatrix}, \tag{35}$$

$$\varGamma_1 = \begin{pmatrix} -1 & -i & 0 \\ -1-2i & 2 & -1-i \\ 2-i & 2i & -i \end{pmatrix}, \quad \varLambda_1 = \begin{pmatrix} i \\ -1 \\ 1-i \end{pmatrix}, \quad \varGamma_2 = \begin{pmatrix} 1 & -1 \\ i & -i \\ -1+i & 2 \end{pmatrix}, \quad \varLambda_2 = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}. \tag{36}$$

One can see that conditions (16), (17) hold. Hence, by Theorem 1, the system (1) with the matrices (34) is arbitrary finite spectrum assignable by constant control. Let us construct that control $u_0 \in \mathbb{K}^3$, $u_1 \in \mathbb{K}^3$, $u_2 \in \mathbb{K}^2$. Suppose, for example, that

$$q(\lambda) = (\lambda + 1)^3.$$

We have $\gamma_1 = 3$, $\gamma_2 = 3$, $\gamma_3 = 1$. Hence,

$$w_0 = \operatorname{col}(\alpha_1 - \gamma_1, \alpha_2 - \gamma_2, \alpha_3 - \gamma_3) = \operatorname{col}(-4, -2, -1). \tag{37}$$

Resolving the systems (28), (29) with coefficients (35), (36), (37), we obtain

$$u_0 = \operatorname{col}(3 - 2i, 8 + 3i, -6 - 3i), \quad u_1 = \operatorname{col}(-i, 2, 1), \quad u_2 = \operatorname{col}(-1 + i, i). \tag{38}$$

The system (1) with the matrices (34) and with the control (38) takes the form

$$\dot{x}(t) = \begin{pmatrix} 1 & 1 & 0 \\ -10+6i & -4+i & 1 \\ 8+4i & 4-2i & -i \end{pmatrix} x(t) + \\ + \begin{pmatrix} 0 & 0 & 0 \\ 1-i & 0 & 0 \\ -2 & -1+i & 0 \end{pmatrix} x(t-h_1) + \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1+i & 1 & 0 \end{pmatrix} x(t-h_2). \tag{39}$$

Calculating the characteristic function for the system (39), we obtain that

$$\varphi(\lambda, e^{\lambda}) = \lambda^3 + 3\lambda^2 + 3\lambda + 1.$$

In particular the system (39) is exponentially stable. □

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Задача назначения конечного спектра для билинейных систем с несколькими запаздываниями

Ключевые слова: линейные системы с запаздываниями, назначение спектра, стабилизация, билинейная система.

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Рассматривается билинейная управляемая система, заданная линейной стационарной дифференциальной системой с несколькими несоизмеримыми запаздываниями в состоянии. Исследуется задача назначения произвольного конечного спектра посредством стационарного управления. Требуется построить постоянные векторы управления таким образом, чтобы характеристическая функция замкнутой системы равнялась многочлену с произвольными наперед заданными коэффициентами. Получены условия на коэффициенты системы, при которых найден критерий разрешимости данной задачи назначения конечного спектра. Показана взаимосвязь условий критерия со свойством согласованности усеченной системы без запаздываний. Получены следствия о стабилизации билинейных систем с запаздываниями. Аналогичные результаты получены для билинейных системы с несколькими запаздываниями с дискретным временем. Рассмотрен иллюстрирующий пример.

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