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PROBLEM WITH DATA ON THE CHARACTERISTICS FOR A LOADED SYSTEM OF HYPERBOLIC EQUATIONS

We consider a problem with data on the characteristics for a loaded system of hyperbolic equations of the second order on a rectangular domain. The questions of the existence and uniqueness of the classical solution of the considered problem, as well as the continuity dependence of the solution on the initial data, are investigated. We propose a new approach to solving the problem with data on the characteristics for the loaded system of hyperbolic equations second order based on the introduction new functions. By introducing new unknown functions the problem is reduced to an equivalent family of Cauchy problems for a loaded system of differential with a parameters and integral relations. An algorithm for finding an approximate solution to the equivalent problem is proposed and its convergence is proved. Conditions for the unique solvability of the problem with data on the characteristics for the loaded system of hyperbolic equations of the second order are established in the terms of coefficient's system.

Keywords: loaded systems of hyperbolic equations, problem with data on the characteristics, family of Cauchy problems, algorithm, solvability criteria.

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Introduction

Mathematical modeling of phenomena in the theory of predicting ground and groundwater, the theory of wave propagation in dispersive media, the theory of shells lead to loaded differential equations various type [1–15].

To date, various methods and approaches have been developed for solving boundary value problems for such equations (see [2, 4, 5] and the bibliography therein). However, despite this, there are still many problems and questions concerning problems for loaded differential equations, for example, generalization of results and solution methods to systems of equations, development of methods for finding approximate solutions and their convergence to an exact solution, construction of numerical solutions, etc. [1, 3, 6–15].

In this paper, we study a class of loaded differential equations of hyperbolic type. The existence of a unique solution to a problem with data on the characteristics for the loaded system of hyperbolic equations is discussed.

We consider the problem with data on the characteristics for the loaded system of hyperbolic equations in the following form

$$\frac{\partial^2 u}{\partial t \partial x} = A(t,x)\frac{\partial u(t,x)}{\partial x} + B(t,x)\frac{\partial u(t,x)}{\partial t} + C(t,x)u(t,x) + D(t,x)\frac{\partial u(t,x_0)}{\partial t} + f(t,x), \quad (0.1)$$

$$u(0,x) = \varphi(x), \qquad x \in [0,\omega], \quad (0.2)$$

$$u(t,0) = \psi(t), \qquad t \in [0,T]. \quad (0.3)$$

where $\Omega = [0,T] \times [0,\omega]$, $u(t,x) = \operatorname{col}(u_1(t,x), u_2(t,x), \ldots, u_n(t,x))$ is an unknown vector function, the $(n \times n)$-matrices $A(t,x)$, $B(t,x)$, $C(t,x)$, $D(t,x)$, and the n-vector function $f(t,x)$ are continuous on Ω, $0 < x_0 \leqslant \omega$, the n-vector function $\varphi(x)$ is continuously differentiable on $[0, \omega]$, the n-vector function $\psi(t)$ is continuously differentiable on $[0, T]$, and the compatibility condition is valid: $\varphi(0) = \psi(0)$.

Let $C(\Omega, \mathbb{R}^n)$ be the space of continuous on Ω vector functions $u(t, x)$ with the norm

$$||u||_0 = \max_{(t,x)\in\Omega} ||u(t, x)||, \qquad ||u(t, x)|| = \max_{i=\overline{1,n}} |u_i(t, x)|.$$

The function $u(t, x) \in C(\Omega, \mathbb{R}^n)$ that has partial derivatives $\dfrac{\partial u(t, x)}{\partial x} \in C(\Omega, \mathbb{R}^n)$, $\dfrac{\partial u(t, x)}{\partial t} \in$ $\in C(\Omega, \mathbb{R}^n)$, $\dfrac{\partial^2 u(t, x)}{\partial t \partial x} \in C(\Omega, \mathbb{R}^n)$ is called a *classical solution* to problem (0.1)–(0.3) if it satisfies the loaded system of hyperbolic equations (0.1) for all $(t, x) \in \Omega$ and the conditions on the characteristics (0.2) and (0.3) for all $x \in [0, \omega]$ and $t \in [0, T]$, respectively.

It is well known that the problem with data on the characteristics for the system of hyperbolic equations with continuous initial data (0.1)–(0.3) always has a unique classical solution for $D(t, x) = 0$. The loaded term $D(t, x)\dfrac{\partial u(t, x_0)}{\partial t}$ in the system of equations (0.1) plays an essential role for the unique solvability of the problem (0.1)–(0.3). Additional requirements for the coefficients of the system (0.1) allow us to distinguish a class of solvable problems (0.1)–(0.3). At the same time, the imposed conditions must be verifiable and consistent with the theory of boundary value problems for loaded differential equations [1–15]. The main method for solving problems with data on the characteristics for the system of loaded hyperbolic equations is the Riemann method [2]. However, the application of the Riemann method requires continuous differentiability of the coefficients $A(t, x)$ and $B(t, x)$ of the system of equations (0.1). Earlier, in [9–15], various boundary value problems were investigated and solved for loaded ordinary differential equations and systems of loaded partial differential equations of hyperbolic type, when the loads are given by a variable t. In contrast to the works [11–13], this paper considers the system of loaded hyperbolic equations when the load is given by a variable x. The problems of solvability of the problem with data on characteristics for a loaded hyperbolic equation are considered in [2].

The aim of the paper is to develop a constructive method for solving the problem with data on the characteristics for the loaded system of hyperbolic equations (0.1)–(0.3) and propose algorithms for finding its solutions. We establish criteria for an existence and uniqueness of the classical solution of the considered problem, as well as the continuity dependence of the solution on the initial data.

For this we use the method of introduction new functions [16].

Introducing a new unknown functions, we reduce problem (0.1)–(0.3) to an equivalent family of Cauchy problems for the loaded system of differential equations with parameters and integral relations. We propose an algorithm for finding an approximate solution to problem (0.1)–(0.3) and prove its convergence to the exact solution. Conditions for the existence and uniqueness of the classical solution are established in the terms of the initial data.

§ 1. Scheme of the method and family of Cauchy problems

Let us introduce a new functions in the following form:

$$v(t, x) = \frac{\partial u(t, x)}{\partial x}, \qquad w(t, x) = \frac{\partial u(t, x)}{\partial t} \text{ for all } (t, x) \in \Omega.$$

Then problem (0.1)–(0.3) is reduced to an equivalent problem

$$\frac{\partial w}{\partial x} = B(t, x)w(t, x) + D(t, x)w(t, x_0) + A(t, x)v(t, x) + C(t, x)u(t, x) + f(t, x), \tag{1.1}$$

$$w(t, 0) = \dot{\psi}(t), \qquad t \in [0, T], \tag{1.2}$$

$$u(t,x)=\varphi(x)+\int_0^t w(\tau,x)d\tau, \qquad v(t,x)=\dot{\varphi}(x)+\int_0^t \frac{\partial w(\tau,x)}{\partial x}d\tau, \qquad (t,x)\in\Omega. \quad (1.3)$$

In the problem (1.1)–(1.3) the condition $u(0,x)=\varphi(x)$ is taken into account in relation (1.3).

A triple $\{w(t,x),v(t,x),u(t,x)\}$ of continuous on Ω functions is called a *solution* to problem (1.1)–(1.3) if the function $w(t,x)$ belonging to $C(\Omega,\mathbb{R}^n)$ has a continuous derivative with respect to x on Ω and satisfies the one-parameter family of Cauchy problems for a loaded differential equations (1.1), (1.2), where the functions $u(t,x)$ and $v(t,x)$ are connected with $w(t,x)$ and $\dfrac{\partial w(t,x)}{\partial x}$ by the integral relation (1.3).

Let $u^*(t,x)$ be a classical solution of problem (0.1)–(0.3). Then the triple $\{w^*(t,x),\,v^*(t,x),\,u^*(t,x)\}$, where $w^*(t,x)=\dfrac{\partial u^*(t,x)}{\partial t}$, $v^*(t,x)=\dfrac{\partial u^*(t,x)}{\partial x}$, is a solution of problem (1.1)–(1.3). Conversely, if a triple $\{\widetilde{w}(t,x),\widetilde{v}(t,x),\widetilde{u}(t,x)\}$ is a solution of problem (1.1)–(1.3), then $\widetilde{u}(t,x)$ is a classical solution of problem (0.1)–(0.3).

For fixed $v(t,x)$, $u(t,x)$ in problem (1.1)–(1.3) it is necessary to find a solution of a one-parameter family of Cauchy problems for the system of loaded ordinary differential equations with an integral condition [16].

We represent the solution of the family of Cauchy problems for the system of loaded differential equations (1.1), (1.2) using a fundamental matrix $\Phi(t,x)$ of the family system of homogeneous differential equations $\dfrac{\partial w}{\partial x}=B(t,x)w(t,x)$. The fundamental matrix $\Phi(t,x)$ is continuously differentiable on Ω, invertible for all $(t,x)\in\Omega$, and $\Phi(t,0)=I$, where I is the unit matrix on dimension n.

For fixed $v(t,x)$ and $u(t,x)$, the family of Cauchy problems (1.1), (1.2) has the solution in the following form

$$\begin{gathered}w(t,x)=\Phi(t,x)\dot{\psi}(t)+\Phi(t,x)\int_0^x \Phi^{-1}(t,\xi)D(t,\xi)d\xi\cdot w(t,x_0)+\\ +\Phi(t,x)\int_0^x \Phi^{-1}(t,\xi)[A(t,\xi)v(t,\xi)+C(t,\xi)u(t,\xi)]d\xi+\Phi(t,x)\int_0^x \Phi^{-1}(t,\xi)f(t,\xi)d\xi.\end{gathered} \quad (1.4)$$

From here, for $x=x_0$, we obtain

$$\begin{gathered}\Big[I-\Phi(t,x_0)\int_0^{x_0}\Phi^{-1}(t,\xi)D(t,\xi)\,d\xi\Big]w(t,x_0)=\Phi(t,x_0)\int_0^{x_0}\Phi^{-1}(t,\xi)f(t,\xi)\,d\xi+\\ +\Phi(t,x_0)\dot{\psi}(t)+\Phi(t,x_0)\int_0^{x_0}\Phi^{-1}(t,\xi)[A(t,\xi)v(t,\xi)+C(t,\xi)u(t,\xi)]\,d\xi,\end{gathered} \quad (1.5)$$

where I is the identity matrix of dimension n.

Suppose that the $(n\times n)$-matrix $Q(t,x_0)=I-\Phi(t,x_0)\displaystyle\int_0^{x_0}\Phi^{-1}(t,\xi)D(t,\xi)\,d\xi$ is invertible for all $t\in[0,T]$. Then, from (1.5), we have

$$\begin{gathered}w(t,x_0)=[Q(t,x_0)]^{-1}\Phi(t,x_0)\int_0^{x_0}\Phi^{-1}(t,\xi)f(t,\xi)\,d\xi+[Q(t,x_0)]^{-1}\Phi(t,x_0)\dot{\psi}(t)+\\ +[Q(t,x_0)]^{-1}\Phi(t,x_0)\int_0^{x_0}\Phi^{-1}(t,\xi)[A(t,\xi)v(t,\xi)+C(t,\xi)u(t,\xi)]\,d\xi.\end{gathered} \quad (1.6)$$

Using (1.6), we can rewrite (1.4) in the following form

$$
\begin{gathered}
w(t,x) = \Phi(t,x)\Big[I + \int_0^x \Phi^{-1}(t,\xi)D(t,\xi)\,d\xi[Q(t,x_0)]^{-1}\Phi(t,x_0)\Big]\dot{\psi}(t) + \\
+ \Phi(t,x)\int_0^x \Phi^{-1}(t,\xi)D(t,\xi)\,d\xi[Q(t,x_0)]^{-1}\Phi(t,x_0)\int_0^{x_0}\Phi^{-1}(t,\xi)f(t,\xi)\,d\xi + \\
+ \Phi(t,x)\int_0^x \Phi^{-1}(t,\xi)f(t,\xi)\,d\xi + \Phi(t,x)\int_0^x \Phi^{-1}(t,\xi)[A(t,\xi)v(t,\xi) + C(t,\xi)u(t,\xi)]\,d\xi + \\
+ \Phi(t,x)\int_0^x \Phi^{-1}(t,\xi)D(t,\xi)\,d\xi[Q(t,x_0)]^{-1} \times \\
\times \Phi(t,x_0)\int_0^{x_0}\Phi^{-1}(t,\xi)[A(t,\xi)v(t,\xi) + C(t,\xi)u(t,\xi)]\,d\xi.
\end{gathered}
\tag{1.7}
$$

If the $(n \times n)$-matrix $Q(t,x_0)$ is invertible for all $t \in [0,T]$, then the family of Cauchy problems for the system of loaded differential equations (1.1), (1.2) has a unique solution. The solution and its derivative satisfy the following inequalities:

$$
\begin{gathered}
||w(t,x)|| \leqslant \phi(t)a(t,x)||\dot{\psi}(t)|| + e^{\beta(t)x}xa(t,x_0)\max_{x\in[0,\omega]}||f(t,x)|| + \\
+ e^{\beta(t)x}xa(t,x_0)\Big(\alpha(t)\max_{x\in[0,\omega]}||v(t,x)|| + \gamma(t)\max_{x\in[0,\omega]}||u(t,x)||\Big),
\end{gathered}
\tag{1.8}
$$

$$
\begin{gathered}
\Big|\Big|\frac{\partial w(t,x)}{\partial x}\Big|\Big| \leqslant \phi(t)(\beta(t)a(t,x) + \delta(t)\theta(t))||\dot{\psi}(t)|| + (\beta(t)e^{\beta(t)x}x + 1)a(t,x_0)\max_{x\in[0,\omega]}||f(t,x)|| + \\
+ (\beta(t)e^{\beta(t)x}x + 1)a(t,x_0)\Big(\alpha(t)\max_{x\in[0,\omega]}||v(t,x)|| + \gamma(t)\max_{x\in[0,\omega]}||u(t,x)||\Big),
\end{gathered}
\tag{1.9}
$$

where

$$
\begin{gathered}
\alpha(t) = \max_{x\in[0,\omega]}||A(t,x)||, \quad \beta(t) = \max_{x\in[0,\omega]}||B(t,x)||, \quad \delta(t) = \max_{x\in[0,\omega]}||D(t,x)||, \\
\gamma(t) = \max_{x\in[0,\omega]}||C(t,x)||, \quad \phi(t) = \max_{x\in[0,\omega]}||\Phi(t,x)||, \quad \theta(t) = ||[Q(t,x_0)]^{-1}||, \\
a(t,x) = 1 + \delta(t)\theta(t)e^{\beta(t)x}x.
\end{gathered}
$$

From the integral relations (1.3), we have

$$
||u(t,x)|| \leqslant ||\varphi(x)|| + \int_0^t ||w(\tau,x)||\,d\tau, \tag{1.10}
$$

$$
||v(t,x)|| \leqslant ||\dot{\varphi}(x)|| + \int_0^t \Big|\Big|\frac{\partial w(\tau,x)}{\partial x}\Big|\Big|\,d\tau. \tag{1.11}
$$

Taking into account (1.8) and (1.9), from the inequalities (1.10) and (1.11), we obtain

$$
\begin{gathered}
\max\Big\{\max_{x\in[0,\omega]}||v(t,x)||, \max_{x\in[0,\omega]}||u(t,x)||\Big\} \leqslant \max\Big\{\max_{x\in[0,\omega]}||\varphi(x)||, \max_{x\in[0,\omega]}||\dot{\varphi}(x)||\Big\} + \\
+ \int_0^t a_1(\tau)||\dot{\psi}(\tau)||\,d\tau + \int_0^t a_2(\tau)\max_{x\in[0,\omega]}||f(\tau,x)||\,d\tau + \\
+ \int_0^t a_2(\tau)a_3(\tau)\max\Big\{\max_{x\in[0,\omega]}||v(\tau,x)||, \max_{x\in[0,\omega]}||u(\tau,x)||\Big\}\,d\tau \leqslant \\
\leqslant \max\Big\{\max_{x\in[0,\omega]}||\varphi(x)||, \max_{x\in[0,\omega]}||\dot{\varphi}(x)||\Big\} + \int_0^T a_1(\tau)||\dot{\psi}(\tau)||\,d\tau + \int_0^T a_2(\tau)\max_{x\in[0,\omega]}||f(\tau,x)||\,d\tau +
\end{gathered}
$$

$$+\int_0^t a_2(\tau)a_3(\tau)\max\Big\{\max_{x\in[0,\omega]}||v(\tau,x)||,\max_{x\in[0,\omega]}||u(\tau,x)||\Big\}\,d\tau,$$

where

$$a_1(t)=\phi(t)\max\Big\{\max_{x\in[0,\omega]}a(t,x),\beta(t)\max_{x\in[0,\omega]}a(t,x)+\delta(t)\theta(t)\Big\},$$

$$a_2(t)=\max_{x\in[0,\omega]}\Big\{e^{\beta(t)x}x,\beta(t)e^{\beta(t)x}x+1\Big\}a(t,x_0),\qquad a_3(t)=\alpha(t)+\gamma(t).$$

From here, using the Gronwall–Bellman inequality, we set

$$\begin{aligned}\max\Big\{\max_{x\in[0,\omega]}||v(t,x)||,\max_{x\in[0,\omega]}||u(t,x)||\Big\}\leqslant e^{b(t)}\Big\{\max\Big\{\max_{x\in[0,\omega]}||\varphi(x)||,\max_{x\in[0,\omega]}||\dot{\varphi}(x)||\Big\}+\\+\int_0^T a_1(\tau)||\dot{\psi}(\tau)||\,d\tau+\int_0^T a_2(\tau)\max_{x\in[0,\omega]}||f(\tau,x)||\,d\tau\Big\},\end{aligned}\tag{1.12}$$

where

$$b(t)=\int_0^t a_2(\tau)a_3(\tau)d\tau.$$

§ 2. Algorithm and main result

We propose an algorithm for finding a solution of the problem (1.1)–(1.3) and show its convergence.

If we know $v(t,x)$, $u(t,x)$, then from the family of Cauchy problems (1.1), (1.2) we find $w(t,x)$ for all $(t,x)\in\Omega$. Conversely, if we know $w(t,x)$ and its derivative $\dfrac{\partial w(t,x)}{\partial x}$, then from integral relations (1.3) we can determine $v(t,x)$ and $u(t,x)$ for all $(t,x)\in\Omega$. Since the $v(t,x)$, $u(t,x)$ and $w(t,x)$ are unknown to find a solution of problem (1.1)–(1.3), we use the iterative method. A triple $\{w^*(t,x),v^*(t,x),u^*(t,x)\}$ is determined as a limit sequence $\{w^{(k)}(t,x)$, $v^{(k)}(t,x)$, $u^{(k)}(t,x)\}$, and $k=0,1,2,\dots$, by the following algorithm.

Step 0.

1. Let the $(n\times n)$-matrix $Q(t,x_0)=I-\Phi(t,x_0)\displaystyle\int_0^{x_0}\Phi^{-1}(t,\xi)D(t,\xi)\,d\xi$ be invertible for all $t\in[0,T]$. Assuming that $v(t,x)=\dot{\varphi}(x)$, $u(t,x)=\varphi(x)$ for all $(t,x)\in\Omega$ in the system (1.1), we solve the family of Cauchy problems for the loaded system of differential equations (1.1), (1.2), and find the zero approximation $w^{(0)}(t,x)$ and its derivative $\dfrac{\partial w^{(0)}(t,x)}{\partial x}$ for all $(t,x)\in\Omega$.

2. Assuming $w(t,x)=w^{(0)}(t,x)$, $\dfrac{\partial w(t,x)}{\partial x}=\dfrac{\partial w^{(0)}(t,x)}{\partial x}$ in the integral relations (1.3), we determine $v^{(0)}(t,x)$ and $u^{(0)}(t,x)$ for all $(t,x)\in\Omega$.

Step k.

1. Assuming that $v(t,x)=v^{(k-1)}(t,x)$, $u(t,x)=u^{(k-1)}(t,x)$ for all $(t,x)\in\Omega$ in the system (1.1), we solve the family of Cauchy problems for the loaded system of differential equations (1.1), (1.2), and find the k-th approximation $w^{(k)}(t,x)$ and its derivative $\dfrac{\partial w^{(k)}(t,x)}{\partial x}$ for all $(t,x)\in\Omega$.

2. Assuming $w(t,x) = w^{(k)}(t,x)$, $\dfrac{\partial w(t,x)}{\partial x} = \dfrac{\partial w^{(k)}(t,x)}{\partial x}$ in the integral relations (1.3), we determine $v^{(k)}(t,x)$ and $u^{(k)}(t,x)$ for all $(t,x) \in \Omega$; $k = 1, 2, \dots$.

At each step of the algorithm: 1) solve the family of Cauchy problems for the loaded system of differential equations (1.1), (1.2) with respect to the function $w(t,x)$; 2) determine from the integral relations (1.3) the functions $v(t,x)$ and $u(t,x)$.

The convergence conditions of the algorithm provide assumptions about the initial data.

Theorem 1. *Let the following conditions be met:*

(a) *the $(n \times n)$-matrices $A(t,x)$, $B(t,x)$, $C(t,x)$, $D(t,x)$, and the n-vector function $f(t,x)$ are continuous on Ω;*

(b) *the n-vector function $\varphi(x)$ is continuously differentiable on $[0,\omega]$, the n-vector function $\psi(t)$ is continuously differentiable on $[0,T]$, and the compatibility condition is valid:*

$$\varphi(0) = \psi(0);$$

(c) *the $(n \times n)$-matrix $Q(t,x_0) = I - \Phi(t,x_0) \displaystyle\int_0^{x_0} \Phi^{-1}(t,\xi) D(t,\xi)\, d\xi$ is invertible for all $t \in [0,T]$, where $0 < x_0 \leqslant \omega$.*

Then the problem with data on the characteristics for the system of loaded hyperbolic equations (0.1)–(0.3) has a unique classical solution $u^*(t,x)$ defined as the limit $u^{(k)}(t,x)$ as $k \to \infty$, where the sequences of functions $u^{(k)}(t,x)$ and its derivatives $v^{(k)}(t,x)$, $w^{(k)}(t,x)$ are found from the algorithm constructed above for all $(t,x) \in \Omega$.

P r o o f. By virtue of the equivalence of problems (0.1)–(0.3) and (1.1)–(1.3), it suffices to justify the unique solvability of problem (1.1)–(1.3). We find a solution $\{w(t,x), v(t,x), u(t,x)\}$ of problem (1.1)–(1.3) by the algorithm proposed above. As the initial approximation $v(t,x)$ and $u(t,x)$ we take $\dot{\varphi}(x)$ and $\varphi(x)$, respectively, and then find $w^{(0)}(t,x)$ from the family of Cauchy problems in the next form

$$\frac{\partial w}{\partial x} = B(t,x)w(t,x) + D(t,x)w(t,x_0) + f(t,x) + A(t,x)\dot{\varphi}(x) + C(t,x)\varphi(x), \tag{2.1}$$

$$w(t,0) = \dot{\psi}(t), \qquad t \in [0,T]. \tag{2.2}$$

Let condition (c) be fulfilled. Then, the family of Cauchy problems for the loaded system of differential equations (2.1), (2.2) has a unique solution $v^{(0)}(t,x)$. We find the solution $w^{(0)}(t,x)$ of the family of Cauchy problems for the loaded system of differential equations (2.1), (2.2) in the form (1.7), where $v(t,x) = \dot{\varphi}(x)$ and $u(t,x) = \varphi(x)$.

For $w^{(0)}(t,x)$ and its derivative $\dfrac{\partial w^{(0)}(t,x)}{\partial x}$, the following estimate is valid:

$$\max\Big\{\max_{x\in[0,\omega]} ||w^{(0)}(t,x)||,\ \max_{x\in[0,\omega]} \Big|\Big|\frac{\partial w^{(0)}(t,x)}{\partial x}\Big|\Big|\Big\} \leqslant a_1(t)||\dot{\psi}(t)|| + a_2(t)\max_{x\in[0,\omega]} ||f(t,x)|| +$$
$$+ a_2(t)a_3(t)\max\Big\{\max_{x\in[0,\omega]} ||\dot{\varphi}(x)||,\ \max_{x\in[0,\omega]} ||\varphi(x)||\Big\}.$$

Then, from the integral relations (1.3), we determine $v^{(0)}(t,x)$ and $u^{(0)}(t,x)$:

$$u^{(0)}(t,x) = \varphi(x) + \int_0^t w^{(0)}(\tau,x)\, d\tau, \qquad v^{(0)}(t,x) = \dot{\varphi}(x) + \int_0^t \frac{\partial w^{(0)}(\tau,x)}{\partial x}\, d\tau,$$

for all $(t,x) \in \Omega$ and the following estimate is true:

$$\max\Big\{\max_{x\in[0,\omega]}||v^{(0)}(t,x)||, \max_{x\in[0,\omega]}||u^{(0)}(t,x)||\Big\} \leqslant \max\Big\{\max_{x\in[0,\omega]}||\dot{\varphi}(x)||, \max_{x\in[0,\omega]}||\varphi(x)||\Big\} + \\ + \int_0^t \max\Big\{\max_{x\in[0,\omega]}||w^{(0)}(\tau,x)||, \max_{x\in[0,\omega]}\Big|\Big|\frac{\partial w^{(0)}(\tau,x)}{\partial x}\Big|\Big|\Big\}\,d\tau.$$

Suppose $v^{(k-1)}(t,x)$ and $u^{(k-1)}(t,x)$ are known. Then $w^{(k)}(t,x)$ can be found from the family of Cauchy problems in the next form

$$\frac{\partial w}{\partial x} = B(t,x)w(t,x) + D(t,x)w(t,x_0) + f(t,x) + A(t,x)v^{(k-1)}(t,x) + C(t,x)u^{(k-1)}(t,x), \tag{2.3}$$

$$w(t,0) = \dot{\psi}(t), \qquad t \in [0,T]. \tag{2.4}$$

We find the solution $w^{(k)}(t,x)$ of the family of Cauchy problems for the loaded system of differential equations (2.3), (2.4) in the form (1.7), where $v(t,x) = v^{(k-1)}(t,x)$ and $u(t,x) = u^{(k-1)}(t,x)$.

For $w^{(k)}(t,x)$ and its derivative $\dfrac{\partial w^{(k)}(t,x)}{\partial x}$ the following estimate is true:

$$\max\Big\{\max_{x\in[0,\omega]}||w^{(k)}(t,x)||, \max_{x\in[0,\omega]}\Big|\Big|\frac{\partial w^{(k)}(t,x)}{\partial x}\Big|\Big|\Big\} \leqslant a_1(t)||\dot{\psi}(t)|| + a_2(t)\max_{x\in[0,\omega]}||f(t,x)|| + \\ + a_2(t)a_3(t)\max\Big\{\max_{x\in[0,\omega]}||v^{(k-1)}(t,x)||, \max_{x\in[0,\omega]}||u^{(k-1)}(t,x)||\Big\}.$$

Then, from the integral relations (1.3), we determine $v^{(k)}(t,x)$ and $u^{(k)}(t,x)$:

$$u^{(k)}(t,x) = \varphi(x) + \int_0^t w^{(k)}(\tau,x)\,d\tau, \qquad v^{(k)}(t,x) = \dot{\varphi}(x) + \int_0^t \frac{\partial w^{(k)}(\tau,x)}{\partial x}\,d\tau$$

for all $(t,x) \in \Omega$ and the following estimate is valid:

$$\max\Big\{\max_{x\in[0,\omega]}||v^{(k)}(t,x)||, \max_{x\in[0,\omega]}||u^{(k)}(t,x)||\Big\} \leqslant \max\Big\{\max_{x\in[0,\omega]}||\dot{\varphi}(x)||, \max_{x\in[0,\omega]}||\varphi(x)||\Big\} + \\ + \int_0^t \max\Big\{\max_{x\in[0,\omega]}||w^{(k)}(\tau,x)||, \max_{x\in[0,\omega]}\Big|\Big|\frac{\partial w^{(k)}(\tau,x)}{\partial x}\Big|\Big|\Big\}\,d\tau, \qquad k = 1,2,\dots.$$

Analogously, for differences

$$\Delta w^{(k)}(t,x) = w^{(k+1)}(t,x) - w^{(k)}(t,x), \qquad \frac{\partial \Delta w^{(k)}(t,x)}{\partial x} = \frac{\partial w^{(k+1)}(t,x)}{\partial x} - \frac{\partial w^{(k)}(t,x)}{\partial x}, \\ \Delta v^{(k)}(t,x) = v^{(k+1)}(t,x) - v^{(k)}(t,x), \qquad \Delta u^{(k)}(t,x) = u^{(k+1)}(t,x) - u^{(k)}(t,x),$$

we obtain the following estimates

$$\max\Big\{\max_{x\in[0,\omega]}||\Delta w^{(k)}(t,x)||, \max_{x\in[0,\omega]}\Big|\Big|\frac{\partial \Delta w^{(k)}(t,x)}{\partial x}\Big|\Big|\Big\} \leqslant \\ \leqslant a_2(t)a_3(t)\max\Big\{\max_{x\in[0,\omega]}||\Delta v^{(k-1)}(t,x)||, \max_{x\in[0,\omega]}||\Delta u^{(k-1)}(t,x)||\Big\}, \tag{2.5}$$

$$\max\Big\{\max_{x\in[0,\omega]}||\Delta v^{(k)}(t,x)||, \max_{x\in[0,\omega]}||\Delta u^{(k)}(t,x)||\Big\} \leqslant \\ \leqslant \int_0^t \max\Big\{\max_{x\in[0,\omega]}||\Delta w^{(k)}(\tau,x)||, \max_{x\in[0,\omega]}\Big|\Big|\frac{\partial \Delta w^{(k)}(\tau,x)}{\partial x}\Big|\Big|\Big\}\,d\tau. \tag{2.6}$$

Taking into account (2.5) in (2.6), we set

$$\max\Big\{\max_{x\in[0,\omega]}||\Delta v^{(k)}(t,x)||,\max_{x\in[0,\omega]}||\Delta u^{(k)}(t,x)||\Big\}\leqslant$$
$$\leqslant\int_0^t a_2(\tau)a_3(\tau)\max\Big\{\max_{x\in[0,\omega]}||\Delta v^{(k-1)}(\tau,x)||,\max_{x\in[0,\omega]}||\Delta u^{(k-1)}(\tau,x)||\Big\}\,d\tau. \tag{2.7}$$

And using (2.6) in (2.5), we also obtain

$$\max\Big\{\max_{x\in[0,\omega]}||\Delta w^{(k)}(t,x)||,\max_{x\in[0,\omega]}\Big|\Big|\frac{\partial\Delta w^{(k)}(t,x)}{\partial x}\Big|\Big|\Big\}\leqslant$$
$$\leqslant a_2(t)a_3(t)\int_0^t\max\Big\{\max_{x\in[0,\omega]}||\Delta w^{(k-1)}(\tau,x)||,\max_{x\in[0,\omega]}\Big|\Big|\frac{\partial\Delta w^{(k-1)}(\tau,x)}{\partial x}\Big|\Big|\Big\}\,d\tau. \tag{2.8}$$

From (2.8) it follows that the sequences $\{w^{(k)}(t,x)\}$ and $\{\dfrac{\partial w^{(k)}(t,x)}{\partial x}\}$ are convergent in the space $C(\Omega,\mathbb{R}^n)$ as $k\to\infty$. Then the uniform convergence on Ω of the sequences $\{v^{(k)}(t,x)\}$ and $\{u^{(k)}(t,x)\}$ follows from the estimate (2.6) (or (2.7)).

In this case, the limit functions $w^*(t,x)$, $\dfrac{\partial w^*(t,x)}{\partial x}$, $v^*(t,x)$ and $u^*(t,x)$ are continuous on Ω, and the triple $\{w^*(t,x),\ v^*(t,x),\ u^*(t,x)\}$ is a solution to problem (1.1)–(1.3). Using the estimates (1.8), (1.12), we obtain

$$\max\Big\{||w^*||_0,||v^*||_0,||u^*||_0\Big\}\leqslant K\cdot\max\Big\{||f||_0,\max_{t\in[0,T]}||\dot\psi(t)||,\max_{x\in[0,\omega]}||\varphi(x)||,\max_{x\in[0,\omega]}||\dot\varphi(x)||\Big\}, \tag{2.9}$$

where

$$K=\max\Big\{\max_{(t,x)\in\Omega}\phi(t)a(t,x),\int_0^T a_1(\tau)d\tau\Big\}+\max\Big\{\max_{(t,x)\in\Omega}e^{\beta(t)x}xa(t,x_0),\int_0^T a_2(\tau)d\tau\Big\}+e^{b(T)}$$

and independents of f, and ψ, and φ.

Now let $\{\widetilde{w}(t,x),\ \widetilde{v}(t,x),\ \widetilde{u}(t,x)\}$ be a solution to problem (1.1)–(1.3), where $f(t,x)=0$, $\psi(t)=0$, and $\varphi(x)=0$ for all $(t,x)\in\Omega$. Then the unique solvability of family of Cauchy problems (1.1), (1.2) together with (1.3) imply that $\widetilde{w}(t,x)=0$, $\widetilde{v}(t,x)=0$, and $\widetilde{u}(t,x)=0$ for all $(t,x)\in\Omega$. Thus, it follows from the estimate (2.9) that problem (0.1)–(0.3) is uniquely solvable. The proof of Theorem 1 is complete. □

Remark 1. For the case $D(t,x)=0$, we have the problem on the characteristics for the system of hyperbolic equations. In this case the matrix $Q(t,x_0)$ has the form $D(t,x_0)=I$ and is always invertible for all $t\in[0,T]$.

Remark 2. We can establish a conditions for unique solvability problem (0.1)–(0.3) without fundamental matrix $\Phi(t,x)$. For this, we use Dzhumabaev's parameterization method [17]. This method was proposed to solve boundary value problems for system of ordinary differential equations. By using this method, necessary and sufficient conditions for a unique solvability to considered problem were obtained. Algorithms for finding approximate solution of problem were offered and their convergence to exact solution were proved.

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Задача с данными на характеристиках для нагруженной системы гиперболических уравнений

Ключевые слова: нагруженные системы гиперболических уравнений, задача с данными на характеристиках, семейства задач Коши, алгоритм, критерий разрешимости.

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Рассматривается задача с данными на характеристиках для нагруженной системы гиперболических уравнений второго порядка в прямоугольной области. Исследуются вопросы существования и единственности классического решения рассматриваемой задачи, а также непрерывной зависимости решения от исходных данных. Предлагается новый подход к решению задачи с данными на характеристиках для нагруженной системы гиперболических уравнений второго порядка на основе введения новых функций. Путем введения новых неизвестных функций задача сводится к эквивалентному семейству задач Коши для нагруженной системы дифференциальных уравнений с параметрами и интегральным соотношениям. Предложен алгоритм нахождения приближенного решения эквивалентной задачи и доказана его сходимость. Установлены условия однозначной разрешимости задачи с данными на характеристиках для нагруженной системы гиперболических уравнений второго порядка в терминах коэффициентов системы.

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