VESTNIK UDMURTSKOGO UNIVERSITETA. MATEMATIKA. MEKHANIKA. KOMP'YUTERNYE NAUKI
MATHEMATICS 2025. Vol. 35. Issue 3. Pp. 356–379.

MSC2020: 93B55, 93C05, 93B25, 93B10

© *V. A. Zaitsev, I. G. Kim*

ARBITRARY MATRIX COEFFICIENT ASSIGNMENT FOR BLOCK MATRIX BILINEAR CONTROL SYSTEMS IN THE FROBENIUS FORM

The paper relates to the classical problem of eigenvalue spectrum assignment. We consider this problem in a generalized formulation. The system coefficients are block matrices. It is required to construct a controller that assigns the given block matrix coefficients of the characteristic matrix polynomial to the closed-loop system. For block matrix bilinear control systems, we obtain sufficient conditions for resolving the problem of arbitrary matrix coefficient assignment for the characteristic matrix polynomial when the coefficients of the system have a special form, namely, the state matrix is a lower block Frobenius matrix, and the matrix coefficients at the controller contain some zero blocks. It is proved that, the main result generalizes the corresponding theorem for block matrix linear control system closed-loop by linear static output feedback. It is shown that sufficient conditions are not necessary. Special cases are considered. Examples are presented to illustrate the results.

Keywords: linear autonomous system, eigenvalue spectrum assignment, bilinear control system, block matrix system.

DOI: 10.35634/vm250302

§ 1. Introduction

Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$; $\mathbb{K}^n := \{x=\operatorname{col}(x_1,\dots,x_n)\colon x_i\in\mathbb{K},\ i=\overline{1,n}\}$; $M_{p,q}(\mathbb{K})$ is a space of $p\times q$-matrices with elements of $\mathbb{K}$, $M_q(\mathbb{K}) := M_{q,q}(\mathbb{K})$ (we will denote $M_{p,q} := M_{p,q}(\mathbb{K})$, $M_q := M_q(\mathbb{K})$, if the set $\mathbb{K}$ is predefined); $I_q\in M_q(\mathbb{K})$ is the identity matrix (we will omit the index q in the matrix I_q when it does not cause confusion); T is the transposition of a matrix or a vector; $\chi(A,\lambda)$ is the characteristic polynomial of a matrix $A\in M_r(\mathbb{K})$.

This work continues the research of [1–3]. Consider a linear control system

$$\dot{x}=Fx+Gu,\quad y=Hx. \tag{1}$$

Here $x\in\mathbb{K}^n$ is a state vector, $u\in\mathbb{K}^m$ is a control vector, $y\in\mathbb{K}^k$ is an output vector, $F\in M_n(\mathbb{K})$, $G\in M_{n,m}(\mathbb{K})$, $H\in M_{k,n}(\mathbb{K})$. Suppose that the control in system (1) has the form of linear static output feedback (LSOF):

$$u=Qy. \tag{2}$$

Here $Q\in M_{m,k}(\mathbb{K})$. The closed-loop system has the form

$$\dot{x}=(F+GQH)x,\quad x\in\mathbb{K}^n. \tag{3}$$

If $k=n$ and

$$H=I\in M_n(\mathbb{K}), \tag{4}$$

then $y=x$, that is (2) is a linear static state feedback (LSSF) control

$$u=Qx, \tag{5}$$

and the closed-loop system has the form

$$\dot{x}=(F+GQ)x,\quad x\in\mathbb{K}^n. \tag{6}$$

The problem of eigenvalue spectrum assignment for system (1) by LSOF (2) (or by LSSF (5)) is a classical problem of control theory. The formulation of this problem is as follows. For the case $\mathbb{K} = \mathbb{C}$: let an arbitrary set $\sigma = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$ of numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be given. For the case $\mathbb{K} = \mathbb{R}$: let an arbitrary set $\sigma = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$ of real type be given (that is the set σ is invariant under the operation of complex conjugation). One needs to construct a gain matrix $Q \in M_{m,k}(\mathbb{K})$ such that the eigenvalue spectrum of the matrix of the closed-loop system (3) coincides with the given set σ. There is a bijection $(\gamma_1, \ldots, \gamma_n) \in \mathbb{K}^n \longleftrightarrow \sigma = \{\lambda_1, \ldots, \lambda_n\}$ ($\lambda_j \in \mathbb{K}$) between the ordered set of coefficients of the characteristic polynomial

$$\chi(F + GQH, \lambda) = \lambda^n + \gamma_1\lambda^{n-1} + \ldots + \gamma_{n-1}\lambda + \gamma_n \tag{7}$$

of the matrix of system (3) and the set $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ of roots of this polynomial. Thus, the eigenvalue spectrum assignment problem for system (1) by LSOF (2) is formulated in the form of the problem of assigning arbitrary coefficients for the characteristic polynomial (7) (see [2, Definition 1]).

Definition 1. It is said that, for system (1), the problem of *arbitrary coefficient assignment* (*ACA*) *for the characteristic polynomial* (*CP*) *by linear static output feedback* (*LSOF*) *is resolvable*, if for any $\gamma_i \in \mathbb{K}$, $i = \overline{1, n}$, there exists a gain matrix $Q \in M_{m,k}(\mathbb{K})$ such that the characteristic polynomial $\chi(F + GQH, \lambda)$ of the matrix $F + GQH$ of system (3) satisfies equality (7).

In partial case, when (4) is fulfilled and the closed-loop system has the form (6), it is said that, for system (1), the problem of *ACA for CP by LSSF is resolvable*.

The problem of ACA for CP by LSSF has been solved in [4] for $\mathbb{K} = \mathbb{C}$ and in [5] for $\mathbb{K} = \mathbb{R}$.

The problem of ACA for CP by LSOF is a hard problem in control theory and still has no complete constructive solution in the general case. We mention here the papers [6–9] and the reviews [10–12].

Let $G =: [g_1, \ldots, g_m]$; here $g_\alpha \in \mathbb{K}^n$ ($\alpha = \overline{1, m}$) are the column vectors of the matrix G. Let $H =: \begin{bmatrix} \xi_1 \\ \ldots \\ \xi_k \end{bmatrix}$; here $\xi_\beta \in (\mathbb{K}^n)^T$ $\left(\beta = \overline{1, k}\right)$ are the row vectors of the matrix H. Set $r := mk$. Let us expand the matrix $Q = \{q_{\alpha\beta}\}$, $\alpha = \overline{1, m}$, $\beta = \overline{1, k}$ column by column into a column vector

$$u = \operatorname{col}(u_1, \ldots, u_r) := \operatorname{col}(q_{11}, \ldots, q_{m1}, \ldots, q_{1k}, \ldots, q_{mk}). \tag{8}$$

Let us construct the following matrices $B_\nu \in M_n$ $\left(\nu = \overline{1, r}\right)$:

$$\begin{array}{rrcr} B_1 := g_1\xi_1, & B_2 := g_2\xi_1, & \ldots\ldots, & B_m := g_m\xi_1, \\ B_{m+1} := g_1\xi_2, & B_{m+2} := g_2\xi_2, & \ldots\ldots, & B_{2m} := g_m\xi_2, \\ \ldots\ldots\ldots, & \ldots\ldots\ldots, & \ldots\ldots, & \ldots\ldots\ldots, \\ B_{(k-1)m+1} := g_1\xi_k, & B_{(k-1)m+2} := g_2\xi_k, & \ldots\ldots, & B_r := g_m\xi_k. \end{array} \tag{9}$$

Let $A := F$. Then, by virtue of (8) and (9), system (3) is written as

$$\dot{x} = (A + u_1B_1 + \ldots + u_rB_r)x, \quad x \in \mathbb{K}^n. \tag{10}$$

System (10) is a *bilinear system*. A system of the form (10) with arbitrary matrices $B_\nu \in M_n$ is a system of a more general form than system (3). For a system (10), the eigenvalue spectrum assignment problem (or the problem of ACA for CP) has the following formulation.

Definition 2. It is said that, for system (10), *the eigenvalue spectrum assignment problem* (*or the problem of ACA for CP*) *is resolvable*, if for any $\gamma_i \in \mathbb{K}$, $i = \overline{1,n}$, there exists a constant controller $u \in \mathbb{K}^r$ such that the characteristic polynomial $\chi\big(F + \sum_{\nu=1}^{r} u_\nu B_\nu, \lambda\big)$ of the matrix of system (10) satisfies equality

$$\chi(F + u_1 B_1 + \ldots + u_r B_r, \lambda) = \lambda^n + \gamma_1 \lambda^{n-1} + \ldots + \gamma_{n-1}\lambda + \gamma_n.$$

The eigenvalue spectrum assignment problem for bilinear systems is related to the inverse eigenvalue problem and the matrix completion problem. We mention here the papers [13–20] and the reviews [21–23].

In the papers [2,3], the formulation of the problem of ACA for CP by LSOF was generalized to block matrix systems. Let $s \in \mathbb{N}$ be fixed. Consider an input-output linear control system with block matrix coefficients:

$$\dot{x} = Fx + Gu, \quad x \in \mathbb{K}^{ns}, \quad u \in \mathbb{K}^{ms}, \tag{11}$$

$$y = Hx, \quad y \in \mathbb{K}^{ks}, \tag{12}$$

$$F = \begin{bmatrix} F_{11} & \ldots & F_{1n} \\ \vdots & & \vdots \\ F_{n1} & \ldots & F_{nn} \end{bmatrix}, \quad G = \begin{bmatrix} G_{11} & \ldots & G_{1m} \\ \vdots & & \vdots \\ G_{n1} & \ldots & G_{nm} \end{bmatrix}, \quad H = \begin{bmatrix} H_{11} & \ldots & H_{1n} \\ \vdots & & \vdots \\ H_{k1} & \ldots & H_{kn} \end{bmatrix}. \tag{13}$$

Here $x \in \mathbb{K}^{ns}$ is a state vector, $u \in \mathbb{K}^{ms}$ is a control vector, $y \in \mathbb{K}^{ks}$ is an output vector; $F_{ij}, G_{j\alpha}, H_{\beta i} \in M_s(\mathbb{K})$, $i, j = \overline{1,n}$, $\alpha = \overline{1,m}$, $\beta = \overline{1,k}$. Suppose that the control in system (11), (12), (13) has the form of linear static output feedback (LSOF):

$$u = Qy. \tag{14}$$

Here $Q = \{Q_{\alpha\beta}\} \in M_{ms,ks}(\mathbb{K})$, $Q_{\alpha\beta} \in M_s(\mathbb{K})$, $\alpha = \overline{1,m}$, $\beta = \overline{1,k}$. The closed-loop system has the form

$$\dot{x} = (F + GQH)x, \quad x \in \mathbb{K}^{ns}. \tag{15}$$

Let an nth degree sth order monic matrix polynomial be given:

$$\Psi(\Lambda) = I\Lambda^n + \Gamma_1 \Lambda^{n-1} + \ldots + \Gamma_{n-1}\Lambda + \Gamma_n, \quad I, \Lambda, \Gamma_i \in M_s, \quad i = \overline{1,n}. \tag{16}$$

From the polynomial (16), construct the *block companion matrix* associated to the matrix polynomial $\Psi(\Lambda)$:

$$\Phi = \begin{bmatrix} 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & I \\ -\Gamma_n & -\Gamma_{n-1} & -\Gamma_{n-2} & \ldots & -\Gamma_1 \end{bmatrix}. \tag{17}$$

The following definition was given in [2, Definition 2] and in [3, Definition 2].

Definition 3. We say that, for system (11), (12), (13), the problem of *arbitrary matrix coefficient assignment* (*AMCA*) *for the characteristic matrix polynomial* (*CMP*) *by linear static output feedback* (*LSOF*) *is resolvable* if for any $\Gamma_i \in M_s(\mathbb{K})$, $i = \overline{1,n}$, there exists a gain matrix $Q \in M_{ms,ks}(\mathbb{K})$ such that the closed-loop system (15) is reducible by some change of variables $z = Sx$ to the system

$$\dot{z} = \Phi z, \quad z \in \mathbb{K}^{ns}, \tag{18}$$

with the matrix (17), that is the matrix $F + GQH$ of the system (15) is similar to the matrix (17).

In [3], for system (11), (12), (13), conditions were obtained for the resolvability of the problem of AMCA for CMP by LSOF. In the present paper, we consider a block matrix control system of a more general form, namely, a block matrix bilinear control system.

Consider a block matrix bilinear control system

$$\dot{x} = (A + u_1 B_1 + \ldots + u_r B_r)x, \quad x \in \mathbb{K}^{ns}, \quad u_\nu \in \mathbb{K}, \tag{19}$$

$$A = \begin{bmatrix} A_{11} & \ldots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \ldots & A_{nn} \end{bmatrix}, \quad B_\nu = \begin{bmatrix} B_{\nu 11} & \ldots & B_{\nu 1n} \\ \vdots & & \vdots \\ B_{\nu n1} & \ldots & B_{\nu nn} \end{bmatrix}, \tag{20}$$
$$A_{ij}, B_{\nu ij} \in M_s(\mathbb{K}), \quad i,j = \overline{1,n}, \quad \nu = \overline{1,r}.$$

Here $x \in \mathbb{K}^{ns}$ is a state vector, $u = \operatorname{col}(u_1, \ldots, u_r) \in \mathbb{K}^r$ is a control vector.

Definition 4. We say that, for system (19), (20), the problem of *arbitrary matrix coefficient assignment* (*AMCA*) *for the characteristic matrix polynomial* (*CMP*) *is resolvable* if for any $\Gamma_i \in M_s(\mathbb{K})$, $i = \overline{1,n}$, there exist $u_\nu \in \mathbb{K}$, $\nu = \overline{1,r}$, such that the system (19), (20) is reducible by some non-degenerate change of variables $z = Sx$ to the system (18) with the matrix Φ of (17), that is the matrix of the system (19), (20) is similar to the matrix (17): $S(A+u_1B_1+\ldots+u_rB_r)S^{-1} = \Phi$.

In this paper, the results of the paper [3], obtained for system (15), are generalized (partially) to a more general class of block matrix bilinear control systems (19), (20).

§ 2. Notations, definitions, and auxiliary statements

We will use some notations, definitions, and statements from [1] and [3]. Here and throughout, we suppose that the numbers $s, n, r \in \mathbb{N}$, and $p \in \{1, \ldots, n\}$ are fixed. For any matrix $Z \in M_\omega$, we suppose, by definition, $Z^0 = I \in M_\omega$, where I is the identity matrix; $[e_1, \ldots, e_\omega] := I \in M_\omega$; $\operatorname{Sp} Z$ is the trace of $Z \in M_\omega$. Denote by $\otimes$ the right Kronecker product of matrices $Y = \{y_{ij}\} \in$ $\in M_{\omega,\rho}$, $i = \overline{1,\omega}$, $j = \overline{1,\rho}$, and $Z \in M_{\sigma,\tau}$ [24, Ch. 12] defined by the formula

$$Y \otimes Z := \begin{bmatrix} y_{11}Z & y_{12}Z & \ldots & y_{1\rho}Z \\ y_{21}Z & y_{22}Z & \ldots & y_{2\rho}Z \\ \vdots & \vdots & & \vdots \\ y_{\omega 1}Z & y_{\omega 2}Z & \ldots & y_{\omega\rho}Z \end{bmatrix} \in M_{\omega\sigma,\rho\tau}.$$

Denote $\mathcal{J} := J \otimes I \in M_{ns}$ where $I \in M_s$ and $J := \{\epsilon_{ij}\} \in M_n$, $\epsilon_{ij} = 1$ for $j = i+1$ and $\epsilon_{ij} = 0$ for $j \neq i+1$. We will use the mapping vecc that unroll a matrix $Z = \{z_{ij}\} \in M_{\omega,\rho}(\mathbb{K})$, $i = \overline{1,\omega}$, $j = \overline{1,\rho}$, column-by-column into the column vector :

$$\operatorname{vecc} Z = \operatorname{col}(z_{11}, \ldots, z_{\omega 1}, \ldots, z_{1\rho}, \ldots, z_{\omega\rho}) \in M_{\omega\rho,1}(\mathbb{K}).$$

Definition 5 (see [1, Definition 4]). For the fixed $s \in \mathbb{N}$, let us introduce the operation of the block trace $\operatorname{SP}_s \colon M_{\omega s} \to M_s$ by the following rule: if $Z = \{Z_{ij}\} \in M_{\omega s}$, $Z_{ij} \in M_s$, $i,j = \overline{1,\omega}$, then $\operatorname{SP}_s Z = \sum\limits_{i=1}^{\omega} Z_{ii}$.

Definition 6 (see [1, Definition 5]). Suppose that X and Y are block matrices with $s \times s$-blocks such that the number of the (block) columns of X is equal to the number of the (block) rows of Y:

$$X = \{X_{ij}\} \in M_{\omega s,\rho s}, \quad X_{ij} \in M_s, \quad i = \overline{1,\omega}, \quad j = \overline{1,\rho};$$
$$Y = \{Y_{j\nu}\} \in M_{\rho s,\tau s}, \quad Y_{j\nu} \in M_s, \quad j = \overline{1,\rho}, \quad \nu = \overline{1,\tau}.$$

For the matrices X and Y, let us introduce the operation of the block multiplication by the following rule:

$$Z = X \star Y := \{Z_{i\nu}\}, \quad Z_{i\nu} := \sum_{j=1}^{\rho} X_{ij} \otimes Y_{j\nu}, \quad i = \overline{1,\omega}, \quad \nu = \overline{1,\tau}.$$

We have $Z_{i\nu} \in M_{s^2}$ for all $i = \overline{1,\omega}$, $\nu = \overline{1,\tau}$, therefore, $Z := X \star Y \in M_{\omega s^2, \tau s^2}$.

Definition 7 (see [1, Definition 6]). For the fixed $s \in \mathbb{N}$, let us introduce the operation of the block transposition $\mathcal{T}$ by the following rule: if $Y = \{Y_{ij}\} \in M_{\omega s, \rho s}$, $Y_{ij} \in M_s$, $i = \overline{1,\omega}$, $j = \overline{1,\rho}$, then

$$Y^{\mathcal{T}} := Z = \{Z_{ji}\} \in M_{\rho s, \omega s}, \quad Z_{ji} := Y_{ij}, \quad j = \overline{1,\rho}, \quad i = \overline{1,\omega}.$$

Definition 8 (see [1, Definition 7]). Let X be a block matrix with $s \times s$-blocks:

$$X = \{X_{ij}\} \in M_{\omega s, \rho s}, \quad X_{ij} \in M_s, \quad i = \overline{1,\omega}, \quad j = \overline{1,\rho}.$$

Let us construct the mappings VecCR_s, $\mathrm{VecRR}_s \colon M_{\omega s, \rho s} \to M_{s, \omega\rho s}$ that unroll the matrix $X = \{X_{ij}\} \in M_{\omega s, \rho s}$ by block columns and by block rows respectively into the block row with $s \times s$-blocks:

$$\mathrm{VecCR}_s\, X = [X_{11}, \ldots, X_{\omega 1}, \ldots, X_{1\rho}, \ldots, X_{\omega\rho}],$$
$$\mathrm{VecRR}_s\, X = [X_{11}, \ldots, X_{1\rho}, \ldots, X_{\omega 1}, \ldots, X_{\omega\rho}].$$

Consider a block matrix

$$Z = \{Z_{ij}\} \in M_{ns}, \quad Z_{ij} \in M_s, \quad i,j = \overline{1,n}. \tag{21}$$

It is said that the matrix Z is a *lower block Hessenberg matrix* if $Z_{ij} = 0 \in M_s$, $j > i + 1$. If, in addition, $\det Z_{i,i+1} \neq 0$, then this lower block Hessenberg matrix is called *unreduced*. We will consider only those lower block Hessenberg matrices that are unreduced, therefore, for brevity, we will omit the word "unreduced". If, for the block matrix (21), we have $Z_{i,i+1} = I \in M_s$, $i = \overline{1,n-1}$, and $Z_{ij} = 0 \in M_s$, $i = \overline{1,n-1}$, $j = \overline{1,n}$, $i+1 \neq j$ (that is Z has the form (17)), then it is said that Z is a *lower block Frobenius matrix*.

Lemma 1 (see [3, Lemma 3]). *Suppose that a block matrix (21) is a lower block Hessenberg matrix. Then there exists a non-degenerate lower block triangular matrix S such that the matrix SZS^{-1} is a lower block Frobenius matrix.*

§ 3. Sufficient conditions to solving the problem of AMCA for CMP for block matrix bilinear control systems with a lower block Frobenius matrix

Consider system (19), (20). Suppose that the coefficients of this system have the following special form: for some $p \in \{1, \ldots, n\}$, the first $p-1$ block rows and the last $n-p$ block columns of the matrices B_ν are zero, and the matrix A is a lower block Frobenius matrix, i. e.,

$$A = \begin{bmatrix} 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & I \\ -A_n & -A_{n-1} & -A_{n-2} & \ldots & -A_1 \end{bmatrix}, \qquad 0, I, A_i \in M_s, \quad i = \overline{1,n}, \tag{22}$$

$$B_\nu = \begin{bmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \\ B_{\nu p1} & \dots & B_{\nu pp} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ B_{\nu n1} & \dots & B_{\nu np} & 0 & \dots & 0 \end{bmatrix}, \quad 0, B_{\nu\tau\sigma} \in M_s, \quad \nu = \overline{1,r},\ \tau = \overline{p,n},\ \sigma = \overline{1,p}. \tag{23}$$

Construct the following matrices $\Psi_{i\alpha} \in M_{s^2,1}(\mathbb{K})$, $i = \overline{1,n}$, $\alpha = \overline{1,r}$:

$$\begin{matrix} \Psi_{11} = \operatorname{vecc}\big[\mathrm{SP}_s(B_1)\big], & \dots\dots, & \Psi_{1r} = \operatorname{vecc}\big[\mathrm{SP}_s(B_r)\big], \\ \Psi_{21} = \operatorname{vecc}\big[\mathrm{SP}_s(AB_1)\big], & \dots\dots, & \Psi_{2r} = \operatorname{vecc}\big[\mathrm{SP}_s(AB_r)\big], \\ \dots\dots\dots\dots\dots\dots, & \dots\dots, & \dots\dots\dots\dots\dots\dots, \\ \Psi_{n1} = \operatorname{vecc}\big[\mathrm{SP}_s(A^{n-1}B_1)\big], & \dots\dots, & \Psi_{nr} = \operatorname{vecc}\big[\mathrm{SP}_s(A^{n-1}B_r)\big]. \end{matrix} \tag{24}$$

Construct

$$\Psi = \begin{bmatrix} \Psi_{11} & \dots & \Psi_{1r} \\ \vdots & & \vdots \\ \Psi_{n1} & \dots & \Psi_{nr} \end{bmatrix} \in M_{ns^2,r}(\mathbb{K}). \tag{25}$$

Theorem 1. *For system* (19), (20)*, with coefficients* (22), (23)*, the problem of AMCA for CMP is resolvable, if the following condition holds:*

$$\operatorname{rank}\Psi = ns^2. \tag{26}$$

§ 4. Proof of Theorem 1

The proof of Theorem 1 will follow the scheme of proof of the corresponding theorem for system (11), (12), (13), (14) (see [3, Theorem 3]). Let the matrix A have the form (22). Here and everywhere below we assume that $A_0 := I \in M_s$. From the matrix A, construct the following matrix:

$$P := \begin{bmatrix} A_0 & 0 & \dots & 0 \\ A_1 & A_0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ A_{n-1} & \dots & A_1 & A_0 \end{bmatrix} \in M_{ns}. \tag{27}$$

Let a block matrix $D \in M_{ns}$ with $s \times s$-blocks have the following form: the first $p-1$ block rows and the last $n-p$ block columns of the matrix D are zero, i. e.,

$$D = \begin{bmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \\ D_{p1} & \dots & D_{pp} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ D_{n1} & \dots & D_{np} & 0 & \dots & 0 \end{bmatrix}, \qquad 0, D_{\tau\sigma} \in M_s, \quad \tau = \overline{p,n}, \quad \sigma = \overline{1,p}. \tag{28}$$

Lemma 2 (see [3, Lemma 5]). *There exists a lower block triangular matrix S such that the following holds: the matrix $S(A+D)S^{-1}$ is equal to the matrix Φ of* (17)*, the block matrix coefficients of which are related to the coefficients of the matrices A and D as follows:*

$$\Gamma_i = A_i - \mathrm{SP}_s(\mathcal{J}^{i-1}PD), \quad i = \overline{1,n}.$$

From the matrix A, construct the following matrices:

$$N_0 := I \in M_{ns}, \quad N_\mu := N_{\mu-1} \cdot A + (L \otimes A_\mu) \in M_{ns}, \quad \mu = \overline{1, n-1},$$

where $L = I \in M_n$. Then,

$$\begin{aligned}
&N_0 = I,\\
&N_1 = A + (I \otimes A_1),\\
&N_2 = A^2 + (I \otimes A_1) \cdot A + (I \otimes A_2),\\
&\ldots,\\
&N_\mu = A^\mu + (I \otimes A_1) \cdot A^{\mu-1} + (I \otimes A_2) \cdot A^{\mu-2} + \ldots + (I \otimes A_{\mu-1}) \cdot A + (I \otimes A_\mu),\\
&\ldots,\\
&N_{n-1} = A^{n-1} + (I \otimes A_1) \cdot A^{n-2} + \ldots + (I \otimes A_{n-2}) \cdot A + (I \otimes A_{n-1}).
\end{aligned}$$

Lemma 3 (see [3, Lemma 6]). *Let a block matrix $D \in M_{ns}$ have the form* (28)*. Then,*

$$\mathrm{SP}_s(\mathcal{J}^\mu P D) = \mathrm{SP}_s(N_\mu D)$$

for all $\mu = 0, \ldots, n-1$.

Lemma 4 (see [3, Lemma 7]). *Let matrices A and D have the form* (22) *and* (28)*, respectively. Then, there exists a lower block triangular matrix S such that the following holds: the matrix $S(A + D)S^{-1}$ is equal to the matrix Φ of* (17)*, the block matrix coefficients of which are related to the coefficients of the matrices A and D as follows:*

$$\Gamma_i = A_i - \mathrm{SP}_s(N_{i-1}D), \quad i = \overline{1, n}. \tag{29}$$

Lemma 4 follows from Lemmas 2 and 3.

Denote

$$T_1 := \mathrm{SP}_s(D), \quad T_2 := \mathrm{SP}_s(AD), \quad \ldots, \quad T_n := \mathrm{SP}_s(A^{n-1}D), \tag{30}$$

$$\widehat{\Gamma} := \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_n \end{bmatrix} \in M_{ns,s}, \quad \widehat{A} := \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} \in M_{ns,s}, \quad \widehat{T} := \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix} \in M_{ns,s}. \tag{31}$$

Lemma 5 (see [3, Lemma 8]). *Equalities* (29) *are equivalent to the equality*

$$\widehat{\Gamma} = \widehat{A} - P\widehat{T}. \tag{32}$$

Remark 1. Since P is non-degenerate, then, for any $\widehat{\Gamma}$, one can express $\widehat{T}$ from (32):

$$\widehat{T} = P^{-1}(\widehat{A} - \widehat{\Gamma}). \tag{33}$$

Consider the system (19), (20) with coefficients (22), (23). From (23), it follows that the matrix $\sum_{\nu=1}^{r} u_\nu B_\nu$ has the form (28) of the matrix D. Let us replace the matrix D in the equalities (30) with $\sum_{\nu=1}^{r} u_\nu B_\nu$. Then, equalities (30) take the form

$$\begin{aligned}
T_1 &= \mathrm{SP}_s\left(\sum_{\nu=1}^{r} u_\nu B_\nu\right),\\
T_2 &= \mathrm{SP}_s\left(A \sum_{\nu=1}^{r} u_\nu B_\nu\right),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
T_n &= \mathrm{SP}_s\left(A^{n-1} \sum_{\nu=1}^{r} u_\nu B_\nu\right).
\end{aligned} \tag{34}$$

Equalities (34) are equivalent to

$$\begin{aligned} T_1 &= \sum_{\nu=1}^{r} u_\nu \, \mathrm{SP}_s(B_\nu), \\ T_2 &= \sum_{\nu=1}^{r} u_\nu \, \mathrm{SP}_s(AB_\nu), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ T_n &= \sum_{\nu=1}^{r} u_\nu \, \mathrm{SP}_s(A^{n-1}B_\nu). \end{aligned} \tag{35}$$

Denote

$$u := \mathrm{col}(u_1, \ldots, u_r) \in \mathbb{K}^r, \tag{36}$$

$$w := \mathrm{col}\left(\mathrm{vecc}\,(T_1), \ldots, \mathrm{vecc}\,(T_n)\right) \in \mathbb{K}^{ns^2}. \tag{37}$$

Let us apply the mapping vecc to equalities (35). Then, equalities (35) take the form

$$\Psi u = w, \tag{38}$$

where Ψ is defined by (24), (25).

Lemma 6. *System (38) is resolvable with respect to u for arbitrary w iff condition (26) holds.*

The proof of Lemma 6 is clear. If condition (26) holds, then system (38) has the particular solution

$$u = \Psi^T(\Psi\Psi^T)^{-1}w. \tag{39}$$

Let us carry out the final arguments to prove Theorem 1. Let condition (26) hold. We show that the problem of AMCA for CMP is resovable. Let arbitrary matrices $\Gamma_i \in M_s(\mathbb{K})$ be given. One needs to construct $u_\nu \in \mathbb{K}$, $\nu = \overline{1, r}$, and a matrix $S \in M_{ns}(\mathbb{K})$ such that $S(A + u_1B_1 + \ldots + u_rB_r)S^{-1} = \Phi$ and Φ has the given block coefficients $-\Gamma_{n+1-i}$ in the last block row. By Lemma 4, for this, it is sufficient to construct $u \in \mathbb{K}^r$, which ensures the fulfillment of the equalities

$$\Gamma_i = A_i - \mathrm{SP}_s\left(N_{i-1}\sum_{\nu=1}^{r} u_\nu B_\nu\right), \quad i = \overline{1, n}. \tag{40}$$

Construct the matrices $\widehat{A}$ and $\widehat{\Gamma}$ by using formula (31). By using (33), let us construct the matrix $\widehat{T}$. Construct the vector w by using formula (37). Let us resolve the system (38) by formula (39). From (39), we find the vector $u \in \mathbb{K}^r$ of (36). Then, taking into account Lemma 5, equalities (40) are satisfied. The proof of Theorem 1 is complete. □

Based on the proof of Theorem 1, we present an algorithm for solving the problem of AMCA for CMP, for system (19), (20), with coefficients (22), (23).

Algorithm 1. Let the system (19), (20) with coefficients (22), (23) be given.

1. By usung (24), (25), construct the matrix Ψ.
2. Check the condition (26). If this condition is satisfied, then the problem is solvable.
3. Let arbitrary matrices $\Gamma_1, \ldots, \Gamma_n \in M_s$ be given.
4. Construct the matrices P of (27) and the matrices $\widehat{\Gamma}$ and $\widehat{A}$ of (31).
5. Calculate the matrix $\widehat{T}$ using the equality (33).

6. The matrix $\widehat{T}$ has the form (31). From $\widehat{T}$, find the matrices $T_1, \dots, T_n \in M_s$.
7. From the matrices $T_1, \dots, T_n \in M_s$, construct the vector w using the formula (37).
8. Solve the system (38) with respect to the vector u; for example, using the formula (39).
9. Denote the matrix $A + u_1 B_1 + \ldots + u_r B_r$ of the closed-loop system by Z. This matrix is a lower block Hessenberg matrix. By Lemma 1 (see [3, Lemma 3]), from the matrix Z, construct the matrices $S_1, \dots, S_n$ and the matrix S. Then, this matrix S reduces the matrix $Z = A + u_1 B_1 + \ldots + u_r B_r$ of the closed-loop system to the matrix Φ of (17), i. e., $S(A + u_1 B_1 + + \ldots + u_r B_r) S^{-1} = \Phi$.

We will demonstrate this algorithm below in Section 8.

§ 5. Theorem 1 generalizes the theorem on AMCA for CMP by LSOF for system (11)–(13)

Consider system (11), (12), (13). Suppose that the coefficients of this system have the following special form: for some $p \in \{1, \dots, n\}$, the first $p-1$ block rows of the matrix G are zero, the last $n-p$ block columns of the matrix H are zero, the matrix F is a lower block Frobenius matrix, i. e.,

$$F = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \\ -A_n & -A_{n-1} & -A_{n-2} & \dots & -A_1 \end{bmatrix}, \qquad 0, I, A_i \in M_s, \quad i = \overline{1,n}, \tag{41}$$

$$G = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ G_{p1} & \dots & G_{pm} \\ \vdots & & \vdots \\ G_{n1} & \dots & G_{nm} \end{bmatrix}, \qquad 0, G_{j\alpha} \in M_s, \quad j = \overline{p,n}, \quad \alpha = \overline{1,m}, \tag{42}$$

$$H = \begin{bmatrix} H_{11} & \dots & H_{1p} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ H_{k1} & \dots & H_{kp} & 0 & \dots & 0 \end{bmatrix}, \qquad 0, H_{\beta i} \in M_s, \quad \beta = \overline{1,k}, \quad i = \overline{1,p}. \tag{43}$$

Consider the matrices

$$(H^{\mathcal{T}})^T \star G, \quad (H^{\mathcal{T}})^T \star (FG), \quad \dots, \quad (H^{\mathcal{T}})^T \star (F^{n-1}G).$$

We have $(H^{\mathcal{T}})^T \in M_{ks,ns}$, $F^{i-1}G \in M_{ns,ms}$, hence, $(H^{\mathcal{T}})^T \star (F^{i-1}G) \in M_{ks^2,ms^2}$ for all $i = \overline{1,n}$. Let us construct the matrices $\mathrm{VecRR}_{s^2}\big((H^{\mathcal{T}})^T \star (F^{i-1}G)\big) \in M_{s^2,kms^2}$, $i = \overline{1,n}$, and the matrix

$$\Theta = \begin{bmatrix} \mathrm{VecRR}_{s^2}\big((H^{\mathcal{T}})^T \star G\big) \\ \mathrm{VecRR}_{s^2}\big((H^{\mathcal{T}})^T \star (FG)\big) \\ \dots\dots\dots\dots\dots\dots \\ \mathrm{VecRR}_{s^2}\big((H^{\mathcal{T}})^T \star (F^{n-1}G)\big) \end{bmatrix} \in M_{ns^2,kms^2}. \tag{44}$$

The following theorem was proved in [3, Theorem 3].

Theorem 2. *For system* (11), (12), *with coefficients* (41), (42), (43), *the problem of AMCA for CMP by LSOF is resolvable, if the following condition holds:*

$$\operatorname{rank} \Theta = ns^2. \tag{45}$$

We will show here that Theorem 1 is a generalization of Theorem 2.

Suppose that system (11), (12) has coefficients (41), (42), (43). Consider the closed-loop system

$$\dot{x} = (F + GQH)x, \quad x \in \mathbb{K}^{ns}. \tag{46}$$

Let us represent system (46) in the form (19).

We have $Q = \{Q_{\alpha\beta}\} \in M_{ms,ks}(\mathbb{K})$, $Q_{\alpha\beta} \in M_s(\mathbb{K})$, $\alpha = \overline{1,m}$, $\beta = \overline{1,k}$. Consider formula [3, (73)]. We have

$$\mathrm{VecCR}_s Q = [Q_{11}, Q_{21}, \ldots, Q_{m1}, \ldots, Q_{1k}, \ldots, Q_{mk}] =: \Delta \in M_{s,mks}(\mathbb{K}). \tag{47}$$

Denote

$$Q_{\alpha\beta} =: \begin{bmatrix} q_{\alpha\beta}^{11} & \ldots & q_{\alpha\beta}^{1s} \\ \vdots & & \vdots \\ q_{\alpha\beta}^{s1} & \ldots & q_{\alpha\beta}^{ss} \end{bmatrix} \in M_s(\mathbb{K}), \quad \alpha = \overline{1,m}, \quad \beta = \overline{1,k}. \tag{48}$$

Then,

$$\begin{gathered} \mathrm{vecc}\,\Delta = \mathrm{col}\,(q_{11}^{11}, \ldots, q_{11}^{s1}, \ldots, q_{11}^{1s}, \ldots, q_{11}^{ss}, q_{21}^{11}, \ldots, q_{21}^{s1}, \ldots, q_{21}^{1s}, \ldots, q_{21}^{ss}, \ldots, \\ q_{m1}^{11}, \ldots, q_{m1}^{s1}, \ldots, q_{m1}^{1s}, \ldots, q_{m1}^{ss}, \ldots, \\ q_{1k}^{11}, \ldots, q_{1k}^{s1}, \ldots, q_{1k}^{1s}, \ldots, q_{1k}^{ss}, \ldots, q_{mk}^{11}, \ldots, q_{mk}^{s1}, \ldots, q_{mk}^{1s}, \ldots, q_{mk}^{ss}). \end{gathered} \tag{49}$$

Set $r := mks^2$. Set

$$u := \mathrm{vecc}\,\Delta \in \mathbb{K}^r. \tag{50}$$

Let $G =: [G_1, \ldots, G_m]$; here $G_\alpha \in M_{ns,s}(\mathbb{K})$ $(\alpha = \overline{1,m})$. Denote $G_\alpha =: [g_\alpha^1, \ldots, g_\alpha^s]$, $g_\alpha^\rho \in \mathbb{K}^{ns}$, $\alpha = \overline{1,m}$, $\rho = \overline{1,s}$.

Let $H =: \begin{bmatrix} \Xi_1 \\ \ldots \\ \Xi_k \end{bmatrix}$; here $\Xi_\beta \in M_{s,ns}(\mathbb{K})$ $(\beta = \overline{1,k})$. Denote $\Xi_\beta =: \begin{bmatrix} \xi_\beta^1 \\ \ldots \\ \xi_\beta^s \end{bmatrix}$, $\xi_\beta^\tau \in \left(\mathbb{K}^{ns}\right)^T$, $\beta = \overline{1,k}$, $\tau = \overline{1,s}$.

Let us construct the following matrices $B_\nu \in M_{ns}(\mathbb{K})$, $\nu = \overline{1,r}$:

$$\begin{array}{cccc}
B_1 := g_1^1\xi_1^1, & \ldots\ldots, & B_s := g_1^s\xi_1^1, & \ldots\ldots, \\
 & B_{s^2-s+1} := g_1^1\xi_1^s, & \ldots\ldots, & B_{s^2} := g_1^s\xi_1^s, \\
B_{s^2+1} := g_2^1\xi_1^1, & \ldots\ldots, & B_{s^2+s} := g_2^s\xi_1^1, & \ldots\ldots, \\
 & B_{2s^2-s+1} := g_2^1\xi_1^s, & \ldots\ldots, & B_{2s^2} := g_2^s\xi_1^s, \\
\ldots\ldots, & \ldots\ldots, & \ldots\ldots, & \ldots\ldots, \\
 & \ldots\ldots, & \ldots\ldots, & \ldots\ldots, \\
B_{(m-1)s^2+1} := g_m^1\xi_1^1, & \ldots\ldots, & B_{(m-1)s^2+s} := g_m^s\xi_1^1, & \ldots\ldots, \\
 & B_{ms^2-s+1} := g_m^1\xi_1^s, & \ldots\ldots, & B_{ms^2} := g_m^s\xi_1^s, \\
\ldots\ldots, & \ldots\ldots, & \ldots\ldots, & \ldots\ldots, \\
 & \ldots\ldots, & \ldots\ldots, & \ldots\ldots, \\
B_{(k-1)ms^2+1} := g_1^1\xi_k^1, & \ldots\ldots, & B_{(k-1)ms^2+s} := g_1^s\xi_k^1, & \ldots\ldots, \\
 & B_{((k-1)m+1)s^2-s+1} := g_1^1\xi_k^s, & \ldots\ldots, & B_{((k-1)m+1)s^2} := g_1^s\xi_k^s, \\
\ldots\ldots, & \ldots\ldots, & \ldots\ldots, & \ldots\ldots, \\
 & \ldots\ldots, & \ldots\ldots, & \ldots\ldots, \\
B_{(km-1)s^2+1} := g_m^1\xi_k^1, & \ldots\ldots, & B_{(km-1)s^2+s} := g_m^s\xi_k^1, & \ldots\ldots, \\
 & B_{kms^2-s+1} := g_m^1\xi_k^s, & \ldots\ldots, & B_{kms^2} := g_m^s\xi_k^s.
\end{array} \tag{51}$$

By virtue of equalities (49), (50), and (51), we have

$$GQH = u_1B_1 + \ldots + u_rB_r. \tag{52}$$

By virtue of equalities (42), (43), and (51), the matrices (51) have the form (23).

Set

$$A := F. \tag{53}$$

Then, system (46) has the form of system (19), (20) with matrices (22), (23).

By virtue of (52) and (53), the equalities (34), for the system (46), take the form

$$\begin{aligned} &T_1 = \mathrm{SP}_s(GQH), \\ &T_2 = \mathrm{SP}_s(FGQH), \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \\ &T_n = \mathrm{SP}_s(F^{n-1}GQH). \end{aligned} \tag{54}$$

(See [3, (72)].) Denote

$$\mathfrak{v} := \mathrm{vecc}\,(\mathrm{VecCR}_s Q) \in \mathbb{K}^{kms^2}, \tag{55}$$

$$\mathfrak{w} := \mathrm{col}\,\big(\mathrm{vecc}\,(T_1), \ldots, \mathrm{vecc}\,(T_n)\big) \in \mathbb{K}^{ns^2}. \tag{56}$$

By [3, Lemma 9], equalities (54) are equivalent to the equality

$$\Theta\mathfrak{v} = \mathfrak{w}, \tag{57}$$

where Θ is defined by (44), $\mathfrak{v}$ is defined by (55), and $\mathfrak{w}$ is defined by (56). In its turn, equalities (34) are equivalent to (38). From (37) and (56), it follows that

$$w = \mathfrak{w}. \tag{58}$$

From (47), (48), (49), (50), and (55), it follows that

$$u = \mathfrak{v}. \tag{59}$$

Thus, we get: equality (38), after applying the vecc^{-1} operation, takes the form (34); equalities (34) coincide with (54); equalities (54), after applying the vecc operation, by [3, Lemma 9], take the form (57). Due to equalities (58) and (59), we obtain that

$$\Psi = \Theta. \tag{60}$$

Thus, if the bilinear system has the form (46) with matrices (41), (42), (43), then the condition (26) of Theorem 1 turns into the condition (45) of Theorem 2, due to the equality (60).

Thus, we have the following proposition.

Theorem 3. *Theorem 1 is a generalization of Theorem 2, for systems (11), (12) with matrices (41), (42), (43), to block matrix bilinear control systems (19), (20) with matrices (22), (23).*

§ 6. Special cases

In [3], for system (11), (12), (13) with matrices (41), (42), (43), in special cases when the matrix blocks are scalar matrices, it was shown that rank condition (45) can be reduced (see [3, Section 7]).

Let us consider system (19), (20) with coefficients (22), (23). Suppose that the blocks of the matrices (22), (23) are scalar matrices, i. e.,

$$A_i = a_i I, \quad a_i \in \mathbb{K}, \quad I \in M_s, \quad i = \overline{1,n},$$
$$B_{\nu\tau\sigma} = b_{\nu\tau\sigma} I, \quad b_{\nu\tau\sigma} \in \mathbb{K}, \quad I \in M_s, \quad \nu = \overline{1,r}, \quad \tau = \overline{p,n}, \quad \sigma = \overline{1,p}.$$

Denote

$$\mathcal{A} = J + e_n\varphi \in M_n, \quad \varphi = [-a_n, \dots, -a_1] \in M_{1,n},$$
$$\mathcal{B}_\nu = \{b_{\nu\tau\sigma}\}_{\tau,\sigma=1}^{n} \in M_n, \quad b_{\nu\tau\sigma} := 0 \ \big(\tau = \overline{1,p-1} \vee \sigma = \overline{p+1,n}\big), \quad \nu = \overline{1,r}.$$

Then,

$$A = \mathcal{A} \otimes I, \quad B_\nu = \mathcal{B}_\nu \otimes I, \quad I \in M_s, \quad \nu = \overline{1,r}.$$

Then, for all $i = 0, \dots, n-1$ and $\nu = 1, \dots, r$,

$$\mathrm{SP}_s\big(A^i B_\nu\big) = \mathrm{SP}_s\big((\mathcal{A} \otimes I)^i (\mathcal{B}_\nu \otimes I)\big) = \mathrm{SP}_s\big(\mathcal{A}^i \mathcal{B}_\nu \otimes I\big) = \mathrm{Sp}(\mathcal{A}^i \mathcal{B}_\nu) \cdot I. \tag{61}$$

Consider the first block row in matrix (25):

$$L_1 := \begin{bmatrix} \Psi_{11} & \dots & \Psi_{1r} \end{bmatrix}.$$

The first row of the matrix L_1 is $\psi_1 = [b_{1pp}, \dots, b_{rpp}]$. Since all the matrices B_ν $\big(\nu = \overline{1,r}\big)$ are scalar, therefore the remaining rows of the matrix L_1 are either zero or coincide with ψ_1. We obtain that all rows of the matrix L_1 linearly depend on the first row, that is, the rank of the matrix L_1 does not exceed 1. Due to the equality (61), similar properties hold for other block rows of the matrix (25). Thus, the rank of the matrix Ψ does not exceed n. Therefore, if $s > 1$, then the condition (26) of Theorem 1 cannot be satisfied. Thus, the propositions of [3, Section 7] that hold for system (11), (12), (13) with matrices (41), (42), (43) cannot be extended to block matrix bilinear control systems in the general case when $s > 1$.

Remark 2. The case $s = 1$ was fully investigated in [19] and [20]. The conditions of Theorem 1 are transformed into the conditions of [20, Theorem 2]. Moreover, these conditions are both sufficient and necessary.

§ 7. The converse of Theorem 1 is not true in general

The question of the converse of Theorem 2 was raised in [3], and it was pointed out that in some partial cases the converse is true, but in general the question remained open.

We will show here that, for Theorem 1, the converse is not true in general.

In $M_s(\mathbb{K})$, consider the subspace $UT_s(\mathbb{K}) \subset M_s(\mathbb{K})$ of the upper triangular matrices. By analogy with [2, Definition 3], we give the following definition.

Definition 9. We say that, for system (19), (20), the problem of *arbitrary upper triangular matrix coefficient assignment* (*AUTMCA*) *for CMP is resolvable* if for any $\Delta_i \in UT_s(\mathbb{K})$, $i = \overline{1,n}$, there exist $u_\nu \in \mathbb{K}$, $\nu = \overline{1,r}$, such that the system (19), (20) is reducible by some non-degenerate change of variables $z = Sx$ to the system

$$\dot z = \Omega z, \quad z \in \mathbb{K}^{ns},$$

with the matrix

$$\Omega = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \\ -\Delta_n & -\Delta_{n-1} & -\Delta_{n-2} & \dots & -\Delta_1 \end{bmatrix}, \tag{62}$$

that is the matrix of the system (19), (20) is similar to the matrix (62):

$$S(A + u_1B_1 + \ldots + u_rB_r)S^{-1} = \Omega. \tag{63}$$

The following theorem takes place (see [2, Theorem 3]).

Theorem 4. *Let $\mathbb{K} = \mathbb{C}$, $n = 2$, and $s = 2$. For any matrix Φ of (17), where $0, I, \Gamma_i \in M_s(\mathbb{K})$, $i = \overline{1, n}$, there exists a matrix Ω of (62), where $0, I, \Delta_i \in UT_s(\mathbb{K})$, $i = \overline{1, n}$, such that $\Omega \sim \Phi$.*

Theorem 4 implies the following theorem.

Theorem 5. *Let $\mathbb{K} = \mathbb{C}$, $n = 2$, and $s = 2$. Suppose that, for system (19), (20), the problem of AUTMCA for CMP is resolvable. Then, for system (19), (20), the problem of AMCA for CMP is resolvable.*

Now, consider the following example. Let $n = 2$, $s = 2$, $r = 6$, $\mathbb{K} = \mathbb{C}$. Suppose that the system (19), (20) has the following block matrix coefficients:

$$A := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ B_1 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ B_2 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ B_3 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$
$$B_4 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_5 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_6 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{64}$$

So, the coefficients have the form (22), (23), where $p = 2$. Let arbitrary upper triangular matrices be given:

$$\Delta_1 = \begin{bmatrix} \delta_1 & \delta_2 \\ 0 & \delta_3 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} \delta_4 & \delta_5 \\ 0 & \delta_6 \end{bmatrix}. \tag{65}$$

Let the matrix Ω have the form (62) with the matrices (65). Set $u_1 := -\delta_4$, $u_2 := -\delta_5$, $u_3 := -\delta_6$, $u_4 := -\delta_1$, $u_5 := -\delta_2$, $u_6 := -\delta_3$. Set $S := I \in M_4$. Then, obviously, the equality (63) is satisfied. Hence, for system (19), (20) with the matrices (64), the problem of AUTMCA for CMP is resolvable. Therefore, by Theorem 5, for system (19), (20) with the matrices (64), the problem of AMCA for CMP is resolvable.

Now, for system (19), (20) with the matrices (64), construct the matrix Ψ of (25). Then

$$\Psi = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{66}$$

From (66), it follows that $\operatorname{rank}\Psi = 6$. So, $\operatorname{rank}\Psi < 8 = ns^2$. Thus, condition (26) does not hold. So, for the case $\mathbb{K} = \mathbb{C}$, $n = 2$, and $s = 2$, the converse of Theorem 1 is not true.

Remark 3. The question of the validity of Theorems 4 and 5, for the case $\mathbb{K} = \mathbb{C}$ and ($n > 2$ or $s > 2$), remains open.

Remark 4. The question of a similar counterexample for the case $\mathbb{K} = \mathbb{R}$ remains open.

§ 8. Example

Example 1. Consider an example illustrating Theorem 3 and Theorem 1. Consider system (11), (12) with $n = 3$, $s = 2$, $m = k = p = 2$ with the following matrices:

$$F = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 3 & 1 & -1 \\ 2 & 1 & 0 & -1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{67}$$

The matrices (67) have the form (41), (42), (43).

Constructing matrix (44) (see [3, Example 1]), we get

$$\Theta = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 3 & 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 2 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 3 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 3 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 3 & 0 & -1 & 1 & -1 & -1 & 0 \\ 2 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & 1 & -2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -3 & 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let us represent system (46) in the form (19).

We have

$$g_1^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \quad g_1^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \quad g_2^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad g_2^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$$

$$\begin{aligned} \xi_1^1 &= [1, 0, 0, 0, 0, 0], \\ \xi_1^2 &= [0, 0, 0, 1, 0, 0], \\ \xi_2^1 &= [1, 0, 0, 0, 0, 0], \\ \xi_2^2 &= [0, -1, 1, 0, 0, 0]. \end{aligned}$$

Let us construct matrices (51). We get

$$B_1 = g_1^1\xi_1^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = g_1^2\xi_1^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{68}$$

$$B_3 = g_1^1\xi_1^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix}, \quad B_4 = g_1^2\xi_1^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \tag{69}$$

$$B_5 = g_2^1\xi_1^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_6 = g_2^2\xi_1^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{70}$$

$$B_7 = g_2^1\xi_1^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_8 = g_2^2\xi_1^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \tag{71}$$

$$B_9 = g_1^1\xi_2^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_{10} = g_1^2\xi_2^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{72}$$

$$B_{11} = g_1^1\xi_2^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 & 0 \end{bmatrix}, \quad B_{12} = g_1^2\xi_2^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{73}$$

$$B_{13} = g_2^1\xi_2^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_{14} = g_2^2\xi_2^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{74}$$

$$B_{15} = g_2^1\xi_2^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B_{16} = g_2^2\xi_2^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}, \tag{75}$$

Define the matrix A according to (53). Then,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}. \tag{76}$$

Let us calculate the matrices A^jB_ν, $j = 1, 2$, $\nu = \overline{1, 16}$. We get

$$AB_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad AB_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$AB_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \end{bmatrix}, \qquad AB_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$AB_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad AB_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$AB_7 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad AB_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$AB_9 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad AB_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} |!,$$

$$AB_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 & 0 \end{bmatrix}, \qquad AB_{12} = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$AB_{13} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad AB_{14} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$AB_{15} = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad AB_{16} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^2B_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A^2B_2 = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^2B_3 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A^2B_4 = \begin{bmatrix} 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix},$$

$$A^2B_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A^2B_6 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^2B_7 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A^2B_8 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$A^2B_9 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A^2B_{10} = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^2B_{11} = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A^2B_{12} = \begin{bmatrix} 0 & -3 & 3 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix},$$

$$A^2B_{13} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A^2B_{14} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^2B_{15} = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A^2B_{16} = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

By using (24), (25), we construct the matrix Ψ for system (19), (20) with the matrices (76), (68)–(75). So, we get

$$\Psi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 3 & 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 2 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 3 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 3 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 3 & 0 & -1 & 1 & -1 & -1 & 0 \\ 2 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & 1 & -2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -3 & 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We obtain that the matrix Ψ coincides with Θ. Thus, the statement of Theorem 3 is confirmed.

Now, using the example of the system (19), (20) with the matrices (76), (68)–(75) we will demonstrate Theorem 1 and Algorithm 1.

Calculating the rank of the matrix Ψ, we obtain that $\operatorname{rank}\Psi = 12$. So, conditions of Theorem 1 are fulfilled. Hence, by Theorem 1, for system (19), (20) with the matrices (76), (68)–(75), the problem of AMCA for CMP is resolvable. Let us construct this feedback control. Suppose, for example, that

$$\Gamma_1 = \begin{bmatrix} 7 & 2 \\ 0 & 9 \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} 14 & 9 \\ 0 & 23 \end{bmatrix}, \qquad \Gamma_3 = \begin{bmatrix} 8 & 7 \\ 0 & 15 \end{bmatrix}.$$

We have

$$A = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -A_3 & -A_2 & -A_1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Construct the matrix $P = \begin{bmatrix} I & 0 & 0 \\ A_1 & I & 0 \\ A_2 & A_1 & I \end{bmatrix}$ of (27) and $\widehat{\Gamma} = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{bmatrix}$ и $\widehat{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$ of (31).

Calculating (33), we obtain that

$$\widehat{T} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}, \quad T_1 = \begin{bmatrix} -7 & -2 \\ 0 & -8 \end{bmatrix}, \quad T_2 = \begin{bmatrix} -13 & -9 \\ 0 & -14 \end{bmatrix}, \quad T_3 = \begin{bmatrix} -3 & -5 \\ 0 & 7 \end{bmatrix}.$$

Construct w by formula (37). Then, $w = \operatorname{col}(-7, 0, -2, -8, -13, 0, -9, -14, -3, 0, -5, 7)$.

Now, resolving the system $\Theta u = w$ of (38) by the formula (39), we obtain that

$$u = \operatorname{col}\left(0, -\frac{5}{2}, -11, -33, -\frac{15}{2}, -10, 31, -41, 0, -\frac{5}{2}, \frac{15}{2}, \frac{43}{2}, -\frac{15}{2}, -10, -\frac{57}{2}, \frac{43}{2}\right).$$

Denote by Z the matrix $A + u_1 B_1 + \ldots + u_{16} B_{16}$. We get

$$Z = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -20 & 7 & -7 & -2 & 1 & 0 \\ -15 & 0 & 0 & -8 & 0 & 1 \\ -8 & -7 & 6 & -16 & 0 & 0 \\ 15 & -15 & 15 & -15 & 0 & -1 \end{bmatrix}.$$

Applying step 9 of Algorithm 1, we obtain that

$$S = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ S_{31} & S_{32} & I \end{bmatrix}, \quad S_{31} = \begin{bmatrix} -20 & 7 \\ -15 & 0 \end{bmatrix}, \quad S_{32} = \begin{bmatrix} -7 & -2 \\ 0 & -8 \end{bmatrix}.$$

Let us calculate SZS^{-1}. We get

$$SZS^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -8 & -7 & -14 & -9 & -7 & -2 \\ 0 & -15 & 0 & -23 & 0 & -9 \end{bmatrix}.$$

So, the statement of Theorem 1 is confirmed.

Remark 5. The property of AMCA allows simultaneously assign eigenvalues and eigenvectors, i. e., eigenstructure (see [1, Theorem 11] and [2, Theorem 8]). System (18) with the matrix (17) is equivalent to the linear multidimensional differential equation of higher order

$$x^{(n)} + \Gamma_1 x^{(n-1)} + \ldots + \Gamma_n x = 0, \quad x \in \mathbb{K}^s. \tag{77}$$

Consider the following task. It is required to construct a controller $u \in \mathbb{K}^r$ such that the closed-loop system (19), (20) is equivalent to the differential equation (77) having a given basis of solutions. In [1,2], it is shown that this basis can be quite arbitrary.

Let an arbitrary set of linearly independent vectors $h_1, \ldots, h_s \in \mathbb{K}^s$ be given and an arbitrary list $\Omega = (\lambda_1, \lambda_2, \ldots, \lambda_{ns})$ of ns numbers $\lambda_\xi \in \mathbb{K}$ be given such that following vector functions are linearly independent:

$$\begin{array}{llll} \psi_{1,1}(t) = h_1 e^{\lambda_1 t}, & \psi_{1,2}(t) = h_2 e^{\lambda_2 t}, & \ldots\ldots & \psi_{1,s}(t) = h_s e^{\lambda_s t}, \\ \psi_{2,1}(t) = h_1 e^{\lambda_{s+1} t}, & \psi_{2,2}(t) = h_2 e^{\lambda_{s+2} t}, & \ldots\ldots & \psi_{2,s}(t) = h_s e^{\lambda_{2s} t}, \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots, \\ \psi_{n,1}(t) = h_1 e^{\lambda_{(n-1)s+1} t}, & \psi_{n,2}(t) = h_2 e^{\lambda_{(n-1)s+2} t}, & \ldots\ldots & \psi_{n,s}(t) = h_s e^{\lambda_{ns} t}. \end{array} \tag{78}$$

Then, the set (78) may serve as such a basis.

For example, let $n = 3$, $s = 2$, $h_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $h_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\Omega = (-1, -1, -2, -3, -4, -5)$. Then, the set (78) is

$$\begin{aligned} \psi_{1,1}(t) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t}, \quad \psi_{1,2}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}, \\ \psi_{2,1}(t) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2t}, \quad \psi_{2,2}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t}, \\ \psi_{3,1}(t) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-4t}, \quad \psi_{3,2}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-5t}. \end{aligned} \tag{79}$$

The vector functions (79) are linearly independent. Using the proof of [1, Theorem 11] and [2, Theorem 8], construct a differential equation (77) that has the set (79) as its basis of solutions. We have (see [1, (96)])

$$N_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad N_2 = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}, \quad N_3 = \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix}.$$

Construct $S := [h_1, h_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then (see [1, (97)]),

$$L_1 = SN_1S^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad L_2 = SN_2S^{-1} = \begin{bmatrix} -2 & -1 \\ 0 & -3 \end{bmatrix}, \quad L_3 = SN_3S^{-1} = \begin{bmatrix} -4 & -1 \\ 0 & -5 \end{bmatrix}.$$

Constructing Γ_i, $i = 1, 2, 3$, by the formulas (see [1, (98)])

$$\Gamma_1 = -(L_1 + L_2 + L_3), \quad \Gamma_2 = L_1L_2 + L_1L_3 + L_2L_3, \quad \Gamma_3 = -L_1L_2L_3,$$

we obtain that

$$\Gamma_1 = \begin{bmatrix} 7 & 2 \\ 0 & 9 \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} 14 & 9 \\ 0 & 23 \end{bmatrix}, \qquad \Gamma_3 = \begin{bmatrix} 8 & 7 \\ 0 & 15 \end{bmatrix}. \tag{80}$$

In Example 1, we constructed a controller that provides the similarity of the matrix of system (19), (20) with the coefficients (76), (68)–(75) to the matrix (17) with the coefficients (80). Thus, in Example 1, for system (19), (20) with the coefficients (76), (68)–(75), a controller is constructed, which ensures the following property: The closed-loop system (19) is equivalent to the differential equation (77) having the basis of solutions (79). In particular, the system (19) (and the equation (77)) is exponentially stable.

Conclusion

In this work, we have introduced the formulation of the problem of AMCA for CMP, for block matrix bilinear control systems. This problem is a generalization of the problem of assigning the scalar spectrum. Sufficient conditions have been obtained for resolving the problem of AMCA for CMP, when the state matrix is a lower block Frobenius matrix, and the matrix coefficients at the controller contain some zero blocks. The main result is a generalization of the corresponding results of [3] obtained for block matrix linear control system closed-loop by linear static output feedback. Special case is considered when blocks of the matrix coefficients are scalar matrices. It is proved that the obtained sufficient conditions are generally not necessary. The main results are demonstrated by an example. It is shown how the property of AMCA allows simultaneously assign eigenvalues and eigenvectors for a linear multidimensional differential equation of higher order equivalent to the original block matrix system.

Funding. The research was carried out with a grant from the Russian Science Foundation No. 24–21–00311, https://rscf.ru/project/24-21-00311/.

REFERENCES

1. Zaitsev V., Kim I. Matrix eigenvalue spectrum assignment for linear control systems by static output feedback, *Linear Algebra and its Applications*, 2021, vol. 613, pp. 115–150. https://doi.org/10.1016/j.laa.2020.12.017
2. Zaitsev V. A. On arbitrary matrix coefficient assignment for the characteristic matrix polynomial of block matrix linear control systems, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, 2024, vol. 34, issue 3, pp. 339–358. https://doi.org/10.35634/vm240303
3. Zaitsev V., Kim I. Arbitrary matrix coefficient assignment for block matrix linear control systems by static output feedback, *arXiv:2509.03040 [math.OC]*, 2025. https://doi.org/10.48550/arXiv.2509.03040
4. Popov V. Hyperstability and optimality of automatic systems with several control functions, *Revue Roumaine des Sciences Techniques. Série Électrotechnique et Énergétique*, 1964, vol. 9, no. 4, pp. 629–690.
5. Wonham W. On pole assignment in multi-input controllable linear systems, *IEEE Transactions on Automatic Control*, 1967, vol. 12, issue 6, pp. 660–665. https://doi.org/10.1109/tac.1967.1098739
6. Brockett R., Byrnes C. Multivariable Nyquist criteria, root loci, and pole placement: a geometric viewpoint, *IEEE Transactions on Automatic Control*, 1981, vol. 26, issue 1, pp. 271–284. https://doi.org/10.1109/tac.1981.1102571
7. Wang X. Pole placement by static output feedback, *Journal of Mathematical Systems, Estimation, and Control*, 1992, vol. 2, no. 2, pp. 205–218.
8. Wang Xiaochang A. Grassmannian, central projection, and output feedback pole assignment of linear systems, *IEEE Transactions on Automatic Control*, 1996, vol. 41, issue 6, pp. 786–794. https://doi.org/10.1109/9.506231
9. Rosenthal J., Schumacher J. M., Willems J. C. Generic eigenvalue assignment by memoryless real output feedback, *Systems and Control Letters*, 1995, vol. 26, issue 4, pp. 253–260. https://doi.org/10.1016/0167-6911(95)00019-6
10. Syrmos V. L., Abdallah C. T., Dorato P., Grigoriadis K. Static output feedback — A survey, *Automatica*, 1997, vol. 33, issue 2, pp. 125–137. https://doi.org/10.1016/s0005-1098(96)00141-0
11. Sadabadi M. S., Peaucelle D. From static output feedback to structured robust static output feedback: A survey, *Annual Reviews in Control*, 2016, vol. 42, pp. 11–26. https://doi.org/10.1016/j.arcontrol.2016.09.014
12. Shumafov M. M. Stabilization of linear control systems and pole assignment problem: A survey, *Vestnik St. Petersburg University, Mathematics*, 2019, vol. 52, issue 4, pp. 349–367. https://doi.org/10.1134/S1063454119040095
13. Friedland S. Inverse eigenvalue problems, *Linear Algebra and its Applications*, 1977, vol. 17, issue 1, pp. 15–51. https://doi.org/10.1016/0024-3795(77)90039-8

14. Biegler-König F. W. Sufficient conditions for the solubility of inverse eigenvalue problems, *Linear Algebra and its Applications*, 1981, vol. 40, pp. 89–100. https://doi.org/10.1016/0024-3795(81)90142-7
15. Byrnes C. I., Wang Xiaochang. The additive inverse eigenvalue problem for Lie perturbations, *SIAM Journal on Matrix Analysis and Applications*, 1993, vol. 14, issue 1, pp. 113–117. https://doi.org/10.1137/0614009
16. Helton W., Rosenthal J., Wang Xiaochang. Matrix extensions and eigenvalue completions, the generic case, *Transactions of the American Mathematical Society*, 1997, vol. 349, no. 8, pp. 3401–3408. https://doi.org/10.1090/S0002-9947-97-01975-2
17. Ikramov Kh. D., Chugunov V. N. Inverse matrix eigenvalue problems, *Journal of Mathematical Sciences*, 2000, vol. 98, issue 1, pp. 51–136. https://doi.org/10.1007/bf02355380
18. Kim Meeyoung, Rosenthal J., Wang Xiaochang Alex. Pole placement and matrix extension problems: a common point of view, *SIAM Journal on Control and Optimization*, 2004, vol. 42, issue 6, pp. 2078–2093. https://doi.org/10.1137/s0363012999354429
19. Zaitsev V. A. Control of spectrum in bilinear systems, *Differential Equations*, 2010, vol. 46, no. 7, pp. 1071–1075. https://doi.org/10.1134/s0012266110070153
20. Zaitsev V. A. Necessary and sufficient conditions in a spectrum control problem, *Differential Equations*, 2010, vol. 46, no. 12, pp. 1789–1793. https://doi.org/10.1134/s0012266110120128
21. Chu Moody T. Inverse eigenvalue problems, *SIAM Review*, 1998, vol. 40, issue 1, pp. 1–39. https://doi.org/10.1137/s0036144596303984
22. Ghanbari K. A survey on inverse and generalized inverse eigenvalue problems for Jacobi matrices, *Applied Mathematics and Computation*, 2008, vol. 195, issue 2, pp. 355–363. https://doi.org/10.1016/j.amc.2007.05.035
23. Cravo G. Matrix completion problems, *Linear Algebra and its Applications*, 2009, vol. 430, issues 8–9, pp. 2511–2540. https://doi.org/10.1016/j.laa.2008.12.029
24. Lancaster P., Tismenetsky M. *The theory of matrices*, Academic Press, 1985.

Received 31.05.2025
Accepted 05.09.2025

Vasilii Aleksandrovich Zaitsev, Doctor of Physics and Mathematics, Head of the Laboratory of Mathematical Control Theory, Udmurt State University, ul. Universitetskaya, 1, Izhevsk, 426034, Russia.
ORCID: https://orcid.org/0000-0002-0482-4520
E-mail: verba@udm.ru

Inna Geral'dovna Kim, Candidate of Physics and Mathematics, Researcher, Laboratory of Mathematical Control Theory, Udmurt State University, ul. Universitetskaya, 1, Izhevsk, 426034, Russia.
ORCID: https://orcid.org/0000-0003-0565-0742
E-mail: kimingeral@gmail.com

Citation: V. A. Zaitsev, I. G. Kim. Arbitrary matrix coefficient assignment for block matrix bilinear control systems in the Frobenius form, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, 2025, vol. 35, issue 3, pp. 356–379.

В. А. Зайцев, И. Г. Ким

Назначение произвольных матричных коэффициентов для блочных матричных билинейных систем управления в форме Фробениуса

Ключевые слова: линейная автономная система, назначение спектра собственных значений, билинейная система управления, блочная матричная система.

УДК 517.977, 512.643

DOI: 10.35634/vm250302

Работа относится к классической задаче назначения спектра собственных значений. Мы рассматриваем эту задачу в обобщенной постановке. Коэффициенты системы являются блочными матрицами. Требуется построить регулятор, который назначает замкнутой системе заданные блочные матричные коэффициенты характеристического матричного полинома. Для блочных матричных билинейных систем управления получены достаточные условия разрешимости задачи назначения произвольных матричных коэффициентов характеристического матричного полинома, когда коэффициенты системы имеют специальный вид, а именно, матрица состояния является нижней блочной матрицей Фробениуса, а матричные коэффициенты при регуляторе содержат некоторые нулевые блоки. Доказано, что основной результат обобщает соответствующую теорему для блочной матричной линейной системы управления, замкнутой линейной статической обратной связью по выходу. Показано, что достаточные условия не являются необходимыми. Рассмотрены частные случаи. Приведены примеры, иллюстрирующие полученные результаты.

Финансирование. Исследование выполнено за счет гранта Российского научного фонда № 24–21–00311, https://rscf.ru/project/24-21-00311/.

СПИСОК ЛИТЕРАТУРЫ

1. Zaitsev V., Kim I. Matrix eigenvalue spectrum assignment for linear control systems by static output feedback // Linear Algebra and its Applications. 2021. Vol. 613. P. 115–150. https://doi.org/10.1016/j.laa.2020.12.017
2. Zaitsev V. A. On arbitrary matrix coefficient assignment for the characteristic matrix polynomial of block matrix linear control systems // Вестник Удмуртского университета. Математика. Механика. Компьютерные науки. 2024. Т. 34. Вып. 3. С. 339–358. https://doi.org/10.35634/vm240303
3. Zaitsev V., Kim I. Arbitrary matrix coefficient assignment for block matrix linear control systems by static output feedback // arXiv:2509.03040 [math.OC]. 2025. https://doi.org/10.48550/arXiv.2509.03040
4. Popov V. Hyperstability and optimality of automatic systems with several control functions // Revue Roumaine des Sciences Techniques. Série Électrotechnique et Énergétique. 1964. Vol. 9. No. 4. P. 629–690.
5. Wonham W. On pole assignment in multi-input controllable linear systems // IEEE Transactions on Automatic Control. 1967. Vol. 12. Issue 6. P. 660–665. https://doi.org/10.1109/tac.1967.1098739
6. Brockett R., Byrnes C. Multivariable Nyquist criteria, root loci, and pole placement: a geometric viewpoint // IEEE Transactions on Automatic Control. 1981. Vol. 26. Issue 1. P. 271–284. https://doi.org/10.1109/tac.1981.1102571
7. Wang X. Pole placement by static output feedback // Journal of Mathematical Systems, Estimation, and Control. 1992. Vol. 2. No. 2. P. 205–218.
8. Wang Xiaochang A. Grassmannian, central projection, and output feedback pole assignment of linear systems // IEEE Transactions on Automatic Control. 1996. Vol. 41. Issue 6. P. 786–794. https://doi.org/10.1109/9.506231
9. Rosenthal J., Schumacher J. M., Willems J. C. Generic eigenvalue assignment by memoryless real output feedback // Systems and Control Letters. 1995. Vol. 26. Issue 4. P. 253–260. https://doi.org/10.1016/0167-6911(95)00019-6

10. Syrmos V. L., Abdallah C. T., Dorato P., Grigoriadis K. Static output feedback — A survey // Automatica. 1997. Vol. 33. Issue 2. P. 125–137. https://doi.org/10.1016/s0005-1098(96)00141-0
11. Sadabadi M. S., Peaucelle D. From static output feedback to structured robust static output feedback: A survey // Annual Reviews in Control. 2016. Vol. 42. P. 11–26. https://doi.org/10.1016/j.arcontrol.2016.09.014
12. Шумафов М. М. Стабилизация линейных систем управления. Проблема назначения полюсов. Обзор // Вестник Санкт-Петербургского университета. Математика. Механика. Астрономия. 2019. Т. 6. № 4. С. 564–591. https://www.elibrary.ru/item.asp?id=41431608
13. Friedland S. Inverse eigenvalue problems // Linear Algebra and its Applications. 1977. Vol. 17. Issue 1. P. 15–51. https://doi.org/10.1016/0024-3795(77)90039-8
14. Biegler-König F. W. Sufficient conditions for the solubility of inverse eigenvalue problems // Linear Algebra and its Applications. 1981. Vol. 40. P. 89–100. https://doi.org/10.1016/0024-3795(81)90142-7
15. Byrnes C. I., Wang Xiaochang. The additive inverse eigenvalue problem for Lie perturbations // SIAM Journal on Matrix Analysis and Applications. 1993. Vol. 14. Issue 1. P. 113–117. https://doi.org/10.1137/0614009
16. Helton W., Rosenthal J., Wang Xiaochang. Matrix extensions and eigenvalue completions, the generic case // Transactions of the American Mathematical Society. 1997. Vol. 349. No. 8. P. 3401–3408. https://doi.org/10.1090/S0002-9947-97-01975-2
17. Ikramov Kh. D., Chugunov V. N. Inverse matrix eigenvalue problems // Journal of Mathematical Sciences. 2000. Vol. 98. Issue 1. P. 51–136. https://doi.org/10.1007/bf02355380
18. Kim Meeyoung, Rosenthal J., Wang Xiaochang Alex. Pole placement and matrix extension problems: a common point of view // SIAM Journal on Control and Optimization. 2004. Vol. 42. Issue 6. P. 2078–2093. https://doi.org/10.1137/s0363012999354429
19. Зайцев В. А. Управление спектром в билинейных системах // Дифференциальные уравнения. 2010. Т. 46. № 7. С. 1061–1064. https://www.elibrary.ru/item.asp?id=15109818
20. Зайцев В. А. Необходимые и достаточные условия в задаче управления спектром // Дифференциальные уравнения. 2010. Т. 46. № 12. С. 1789–1793. https://www.elibrary.ru/item.asp?id=15524694
21. Chu Moody T. Inverse eigenvalue problems // SIAM Review. 1998. Vol. 40. Issue 1. P. 1–39. https://doi.org/10.1137/s0036144596303984
22. Ghanbari K. A survey on inverse and generalized inverse eigenvalue problems for Jacobi matrices // Applied Mathematics and Computation. 2008. Vol. 195. Issue 2. P. 355–363. https://doi.org/10.1016/j.amc.2007.05.035
23. Cravo G. Matrix completion problems // Linear Algebra and its Applications. 2009. Vol. 430. Issues 8–9. P. 2511–2540. https://doi.org/10.1016/j.laa.2008.12.029
24. Lancaster P., Tismenetsky M. The theory of matrices. Academic Press, 1985.

Поступила в редакцию 31.05.2025
Принята к публикации 05.09.2025

Зайцев Василий Александрович, д. ф.-м. н., заведующий лабораторией математической теории управления, Удмуртский государственный университет, 426034, Россия, г. Ижевск, ул. Университетская, 1.
ORCID: https://orcid.org/0000-0002-0482-4520
E-mail: verba@udm.ru

Ким Инна Геральдовна, к. ф.-м. н., научный сотрудник, лаборатория математической теории управления, Удмуртский государственный университет, 426034, Россия, г. Ижевск, ул. Университетская, 1.
ORCID: https://orcid.org/0000-0003-0565-0742
E-mail: kimingeral@gmail.com

Цитирование: В. А. Зайцев, И. Г. Ким. Назначение произвольных матричных коэффициентов для блочных матричных билинейных систем управления в форме Фробениуса // Вестник Удмуртского университета. Математика. Механика. Компьютерные науки. 2025. Т. 35. Вып. 3. С. 356–379.