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PURE PHASES OF THE FERROMAGNETIC POTTS MODEL WITH q STATES ON THE CAYLEY TREE OF ORDER THREE

One of the main issues in statistical mechanics is the phase transition phenomenon. It happens when there are at least two distinct Gibbs measures in the model. It is known that the ferromagnetic Potts model with q states possesses, at sufficiently low temperatures, at most $2^q - 1$ translation-invariant splitting Gibbs measures. For continuous Hamiltonians, in the space of probability measures, the Gibbs measures form a non-empty, convex, compact set. Extremal measures, which corresponds to the extreme points of this set, determines pure phases. We study the extremality of the translation-invariant splitting Gibbs measures for the ferromagnetic q-state Potts model on the Cayley tree of order three. We define the regions where the translation-invariant Gibbs measures for this model are extreme or not. We reduce description of Gibbs measures to solving a non-linear functional equation, each solution of which corresponds to one Gibbs measure.

Keywords: Cayley tree, configuration, Potts model, Gibbs measure, translation-invariant measure.

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Introduction

Dobrushin's groundbreaking work established the existence of Gibbs measures for a broad class of Hamiltonians (see, e. g., [1–4]). Nonetheless, it is frequently a challenging task to fully description the collection of Gibbs measures for a particular Hamiltonian.

The Gibbs measure is often unique for small values of β (high temperatures) (see [3–5]), reflecting the physical fact that at high temperatures there is no phase transition. It is important to note that the analysis at low temperatures requires certain assumptions about the form of the Hamiltonian.

It is known that the Gibbs measures for continuous Hamiltonians form a non-empty, convex, compact subset in the space of probability measures (see Chapter 7 in [2]). Pure phases correspond to extremal measures. Extremal measures, which corresponds the extreme points of this set, determine pure phases. Notably, the set of splitting Gibbs measures includes extremal measures (see [2, Chapter 11]). Nevertheless, at even lower temperatures [6–9], the Gibbs measure corresponding to the free boundary condition loses its extremality. It remains extremal inside an intermediate temperature interval below the transition temperature. As a result, each of this set's extremal elements must be identified in order to fully characterize it.

The main purpose of this work is to study the extreme measures which correspond to pure phases of the set of Gibbs measures for the Potts model. Extremal Gibbs measures are crucial for understanding all possible local behaviors of the biological and physical system (see [10, 11]).

In [12, 13], phase diagrams of the q-state Potts models on Cayley trees (Bethe lattices) were studied, and pure phases of the ferromagnetic Potts model were identified. Applying these results, uncountably many pure phases of the 3-state Potts model were constructed in [14–16]. These investigations were based on a measure-theoretic approach developed in [5]. In [17], the translation-invariant splitting Gibbs measures (TISGMs) for the ferromagnetic Potts model with q states were thoroughly described, and it was demonstrated that their number is equal to $2^q - 1$. The problem of the extremality of these measures in the case on the Cayley tree of order two was studied in [18].

The works [19] and [20] were devoted to the study of Gibbs measures for the Potts model, in particular, they give explicit formulas for TISGMs for the ferromagnetic Potts model with three and four states on the Cayley tree of the third order; and in [20, 21] regions were found the (not) extremes of these measures. In the present paper, we generalize these results to arbitrary spin values.

The TISGM of the antiferromagnetic Potts model with an external field was shown to be unique in [22]. The Potts model with a nonzero external field and a countable number of states was the focus of [23]. It was established that this model possesses a unique TISGM.

In the papers [21, 24–30] periodic and weakly periodic Gibbs measures were considered. In particular, in [29] it was shown that all periodic Gibbs measures are translation-invariant (TI) under certain conditions. In [27] it was shown that all periodic Gibbs measures are TI for the ferromagnetic Potts model with q states on the Cayley tree of order k.

In this paper, we generalize the results of [20, 21]. In particular, explicit formulas for the TISGM are obtained for the ferromagnetic Potts model with q states on the Cayley tree of order three, and the regions of (non) extremality of these measures are found.

§ 1. Notations and definitions

The Cayley tree $\Im^k = (V, L)$ of order $k \geq 1$ with the root x^0 is an infinite tree, i. e., a graph without cycles such that exactly $k+1$ edges originate from each vertex. Here V is the set of vertices and L is the set of edges. The vertices x and y are called nearest neighbors if there exists an edge $l \in L$ connecting them where $l = \langle x, y\rangle$. The distance on the Cayley tree, denoted by $d(x, y)$, is defined as the number of edges of the shortest path between the vertices x and y.

We consider the following sets

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \qquad V_n = \bigcup_{m=0}^{n} W_m.$$

For $x \in W_n$, we define the following set

$$S(x) = \{y_i \in W_{n+1} \mid d(x, y_i) = 1,\ i = 1, 2, \ldots, k\}.$$

The set $S(x)$ is called the set of direct successors of x.

The (formal) Hamiltonian of the Potts model is defined as

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \delta_{\sigma(x)\sigma(y)}, \tag{1.1}$$

where $J \in \mathbb{R}$ is the coupling constant, the spins $\sigma(x)$ take values in the set $\Phi = \{1, 2, \ldots, q\}$, $q \geq 2$, and δ_{ij} is the Kronecker symbol (see [3]).

In this paper, we restrict ourselves to the case of ferromagnetic interaction $J > 0$.

For $x \in V \backslash \{x^0\} \mapsto \tilde{h}_x = (\tilde{h}_{1,x}, \ldots, \tilde{h}_{q,x}) \in \mathbb{R}^q$, $x \in V$, we define the (finite-dimensional) Gibbs distributions by the following formula

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\left\{-\beta H_n(\sigma_n) + \sum_{x \in W_n} \tilde{h}_{\sigma(x),x}\right\}, \tag{1.2}$$

where $\beta = 1/T$, $T > 0$ is a temperature, Z_n^{-1} is the partition function and $H_n(\sigma_n)$ is the restriction of Hamiltonian on V_n.

The probability distributions (1.2) are compatible if for all $\sigma_{n-1} \in \Phi^{V_{n-1}}$ one has

$$\sum_{\omega_n \in \Phi^{W_n}} \mu_n(\sigma_{n-1} \vee \omega_n) = \mu_{n-1}(\sigma_{n-1}), \tag{1.3}$$

where $\sigma_{n-1} \vee \omega_n$ is the concatenation of the configurations. Under condition (1.3), by the well-known Kolmogorov's extension theorem, there exists a unique measure μ on Φ^V such that, for all $n \in \mathbb{N}$ and $\sigma_n \in \Phi^{V_n}$,

$$\mu(\{\sigma \mid_{V_n} = \sigma_n\}) = \mu_n(\sigma_n),$$

and we call it a *splitting Gibbs measure* (SGM)[1] corresponding to the the Hamiltonian (1.1) and vector-valued function $\tilde{h}_x, x \in V$ (see [14]).

The compatibility condition is satisfied if and only if, for any $x \in V \backslash \{x^0\}$, the following vector identity holds (for more details we refer the reader to [3]):

$$h_x = \sum_{y \in S(x)} F(h_y, \theta), \tag{1.4}$$

where $F\colon h = (h_1, \dots, h_{q-1}) \in \mathbb{R}^{q-1} \to F(h, \theta) = (F_1, \dots, F_{q-1}) \in \mathbb{R}^{q-1}$ is defined as

$$F_i = \ln \left(\frac{(\theta - 1) \exp h_i + \sum\limits_{j=1}^{q-1} \exp h_j + 1}{\theta + \sum\limits_{j=1}^{q-1} \exp h_j} \right),$$

$\theta = \exp(J\beta)$, $S(x)$ is the set of direct successors of x, and $h_x = (h_{1,x}, \dots, h_{q-1,x}) \in \mathbb{R}^{q-1}$ with

$$h_{i,x} = \tilde{h}_{i,x} - \tilde{h}_{q,x}, \quad i = 1, \dots, q-1.$$

For any $h = \{h_x,\, x \in V\}$ satisfying (1.4), there exists a unique SGM μ, and vice versa. It is known [2, Theorem 12.6] that any extremal Gibbs measure of a Hamiltonian with nearest-neighbor interactions is a splitting Gibbs measure (or tree-indexed Markov chain [2]). This implies that a phase transition occurs if and only if equation (1.4) has multiple solutions.

§ 2. Description of translation-invariant Gibbs measures

A translation-invariant splitting Gibbs measure (TISGM) corresponds to a solution h_x of (1.4) with $h_x = h = (h_1, \dots, h_{q-1}) \in \mathbb{R}^{q-1}$ for all $x \in V$. In this case Eq. (1.4) reads $h = kF(h, \theta)$. Setting $z_i = \exp(h_i),\ i = 1, \dots, q-1$, we can rewrite the last equation as

$$z_i = \left(\frac{(\theta - 1) z_i + \sum\limits_{j=1}^{q-1} z_j + 1}{\theta + \sum\limits_{j=1}^{q-1} z_j} \right)^k, \quad i = 1, \dots, q-1. \tag{2.1}$$

The following results were obtained in [17].

1. The solution of Eq. (2.1) allows us to completely describe the set of TISGMs and shows that any TISGM of the Potts model corresponds to the solution of the equation

$$z = f_m(z) \equiv \left(\frac{(\theta + m - 1) z + q - m}{mz + q - m - 1 + \theta} \right)^k, \tag{2.2}$$

for some $m = 1, \dots, [q/2]$.

[1] The measure is called a *Markov chain* in Ref. [2]

2. There are values $\theta_m = \theta_m(k)$ that are critical for changing the number of TISGMs $\big($at $k=2$, there is an explicit equation $\theta_m(2) = 1+2\sqrt{q\cdot m - m^2}\big)$, $m = 1,\dots,[q/2]$. If $\theta < \theta_1$, then for $k \geq 2$ and $J > 0$, there exists a unique TISGM.

3. If $z(m_1)$ is a solution to (2.2) for $m = m_1$, then $z^{-1}(m_1)$ is also a solution to (2.2) for $m = q - m_1$ (see [17, Lemma 1, p. 193]).

Note that $z_0 = 1$ is a solution to Equation (2.2). For $k = 3,\ \theta > 1$, dividing (2.2) by $z - 1$ and introducing the notation $\sqrt[3]{z} = x$, we have

$$\varphi(x) = mx^3 - (\theta-1)x^2 - (\theta-1)x + q - m = 0. \tag{2.3}$$

In [19] it is shown that the equation (2.3) does not have a solution for $\theta < \theta_{cr}(m,q)$, has one solution except $z_0 = 1$ for $\theta = \theta_{cr}(m,q)$ and has two solutions different from $z_0 = 1$ for $\theta > \theta_{cr}(m,q)$. Here $\theta_{cr}(m,q)$ is a solution of $\varphi'(x^*) = 0$ and

$$x^*(\theta,m) = \frac{\theta - 1 + \sqrt{(\theta-1)^2 + 3m(\theta-1)}}{3m}.$$

Formulas for calculating $\theta_{cr}(m,q) = \theta_m$ were presented in [17] (see formulas (3.17), (3.18)), and in [19]. Using these formulas for $k = 3$, it was obtained

$$\theta_{cr}(m,q) = \frac{mx_0^3 + q - m}{x_0^2 + x_0} + 1, \tag{2.4}$$

where

$$x_0 = \frac{\sqrt[4]{8\alpha_0^3} + \sqrt{(3-2\alpha_0)\sqrt{2\alpha_0} - 6 + \frac{4q}{m}}}{2\sqrt[4]{2\alpha_0}} - \frac{1}{2}, \qquad \alpha_0 = \frac{\sqrt[3]{m(8m^2 - 12mq + 4q^2)} + m}{2m}.$$

Remark 2.1. We note that

- in [19], all solutions of equation (2.2) are described for $k = 3$, $q = 3$ (see [19, Proposition 1, p. 1655]);
- in [20], all solutions of equation (2.2) are described for $k = 3$, $q = 4$ (see [20, Proposition 3.2, p. 121]).

Let $k = 3$ and $\theta > 1$. For each fixed m and q, we find solutions of Eq. (2.3) using Cardano's formula. To do this, we set $x = y + (\theta-1)/3m$ and rewrite (2.3) in the form

$$y^3 - p(t)y + s(t) = 0, \tag{2.5}$$

where

$$p(t) = \frac{t}{3m^2}(t+3m), \quad s(t) = -\frac{t^2}{27m^3}(2t+9m) + \frac{q-m}{m}, \quad \theta - 1 = t \quad (t>0).$$

According to Cardano's formula, Eq. (2.5) has three real roots for $\theta > \theta_m$:

$$y_1 = 2\sqrt{\frac{p(t)}{3}}\cdot\cos\frac{\phi(t)}{3}, \quad y_2 = 2\sqrt{\frac{p(t)}{3}}\cdot\cos\frac{\phi(t)+2\pi}{3}, \quad y_3 = 2\sqrt{\frac{p(t)}{3}}\cdot\cos\frac{\phi(t)+4\pi}{3},$$

where

$$\cos\phi(t) = -\frac{s(t)}{2r(t)}, \qquad r(t) = \frac{\sqrt{t^3(t+3m)^3}}{27m^3}, \qquad \phi(t) \in \left[\frac{\pi}{2};\pi\right].$$

It is easy to see that y_2 is negative for $t > 0$. Then the equation (2.3) has only three positive solutions

$$x_0 = 1, \qquad x_1(t, q, m) = \frac{t}{3m} + y_1, \qquad x_2(t, q, m) = \frac{t}{3m} + y_3.$$

If $\theta = \theta_m$, then $x_1(t, q, m) = x_2(t, q, m)$, and the equation (2.5) has two solutions.

Summarize, we have three translation-invariant solutions $z_0 = 1$ and

$$z_1(t, q, m) = \frac{1}{27m^3}\Big(t + 2\sqrt{t(t+3m)} \cdot \cos\frac{\phi(t)}{3}\Big)^3,$$
$$z_2(t, q, m) = \frac{1}{27m^3}\Big(t + 2\sqrt{t(t+3m)} \cdot \cos\frac{\phi(t)+4\pi}{3}\Big)^3.$$

Remark 2.2. Note that the above solution is a particular case of Theorem 1 in [17]. Here, we give explicit formulas for the solutions corresponding to TISGM in the case $k = 3,\ m \le q/2$.

Denote $t_m = \theta_m - 1 > 0$.

Proposition 2.1. *Let $k = 3$ and $q \ge 2m$. Then the inequality $t_m \ge m$ holds for t_m.*

P r o o f. Due to (2.4), we have

$$\theta_m - 1 = t_m = \frac{mx^3 + q - m}{x^2 + x}.$$

After some algebras, we can obtain $t_m \ge m$. Equality is fulfilled only for $\theta = \theta_{cr}$ $(t = t_{cr})$, that is, it is achieved for $q = 2m$ and $x^* = 1$. □

Proposition 2.2. *Let $k = 3,\ q \ge 4,\ t > 0$ and $t_{cr} = q/2$. Then, for the above-mentioned solutions $z_1(t, q, m)$ and $z_2(t, q, m)$, the following statements hold:*

(i) $1 < z_1(t, q, m) = z_2(t, q, m)$, if $t = t_m$ $(t_m \ne t_{cr})$;

(ii) $z_2(t, q, m) = 1$, if $t = t_{cr}$;

(iii) $1 < z_2(t, q, m) < z_1(t, q, m)$, if $t_m < t < t_{cr}$;

(iv) $z_1(t, q, m)$ is an increasing function, and $z_2(t, q, m)$ is a decreasing function with respect to t.

P r o o f. (i) Let $t = t_m$. In this case, it is known that equation (2.5) has a unique positive solution, i. e., $z_1(t_m, q, m) = z_2(t_m, q, m)$. Then,

$$\cos\frac{\phi(t_m)}{3} = \cos\frac{\phi(t_m)+4\pi}{3} = \frac{1}{2}, \qquad \phi(\theta_1) = \pi.$$

It follows that

$$z_1(t_m, q, m) = z_2(t_m, q, m) = \frac{1}{27m^3}\Big(t_m + \sqrt{t_m(t_m+3m)}\Big)^3.$$

Due to Proposition 2.1, we have $t_m > m$ for $t \ne t_{cr}$. It follows that

$$\frac{t_m + \sqrt{t_m(t_m+3m)}}{3m} > \frac{m + \sqrt{m(m+3m)}}{3m} = 1.$$

(ii) For $q \ge 4$ and $t = q/2$ (i. e., $\theta = q/2 + 1$), we rewrite equation (2.3) as follows:

$$m(x-1)(x^2+x+1) - \frac{q}{2}(x^2-1) - \frac{q}{2}(x-1) = (x-1)\Big(mx^2 + \Big(m - \frac{q}{2}\Big)x + m - q\Big) = 0.$$

Hence, we have two positive solutions

$$x = 1 \ \big(\text{i. e., } \ z_2(\theta_{cr}) = 1\big) \ \text{ and } \ x = \frac{q/2 - m + \sqrt{q^2/4 + 3qm - 3m^2}}{2m} \ \big(\text{i.e., } \ z_1(\theta_{cr}) = x^3\big).$$

(iii) At first, we prove that $z_2(t, q, m) < z_1(t, q, m)$ for $t_m < t < t_{cr}$. We have

$$\sqrt[3]{z_1(t, q, m)} - \sqrt[3]{z_2(t, q, m)} = \frac{2\sqrt{3t(t + 3m)}}{3m} \cdot \sin\frac{\phi(t) + 2\pi}{3} > 0,$$

since $\dfrac{\phi + 2\pi}{3} \in \big[5\pi/6; \pi\big)$.

On the other hand, $z_2(t, q, m)$ is a decreasing function for $t > t_m$ with respect to t (see below (iv)). Taking into account $z_2(t_{cr}) = 1$, we have that $1 < z_2(t, q, m)$ for $t_m < t < t_{cr}$.

(iv) To prove that $z_1(t, q, m)$ (resp. $z_2(t, q, m)$) is an increasing (resp. decreasing) function with respect to t, it suffices to show that $z_1'(t, q, m) > 0$ (resp. $z_2'(t, q, m) < 0$).

First, let us show that the function $\phi(t) = \arccos\Big(-\dfrac{s(t)}{2r(t)}\Big)$ is decreasing. Indeed,

$$\phi'(t) = \frac{r(t)}{\sqrt{4r^2(t) - s^2(t)}} \cdot \frac{s'(t)r(t) - r'(t)s(t)}{r^2(t)},$$

where

$$s(t) = -\frac{t^2}{27m^3}\big(2t + 9m\big) + \frac{q - m}{m}, \qquad r(t) = \frac{\sqrt{t^3(t + 3m)^3}}{27m^3}.$$

Then, $\phi'(t) < 0$, since

$$s'(t)r(t) - s(t)r'(t) = -\frac{\big[(9m + 6t)(q - m) + t^2\big]\sqrt{t(t + 3m)}}{54m^4} < 0.$$

Therefore, $\phi(t)$ is decreasing. Consider the derivative $z_1'(t, q, m)$:

$$z_1'(t, q, m) = \frac{z_1(t, q, m)^{2/3}}{m} \cdot \left(1 + \frac{2t + 3m}{\sqrt{t^2 + 3mt}}\cos\frac{\phi(t)}{3} - \frac{2}{3}\sin\frac{\phi(t)}{3}\phi'(t)\sqrt{t^2 + 3mt}\right).$$

Note that $\phi'(t) < 0$. It implies that $z_1'(t, q, m) > 0$, i. e., $z_1(t, q, m)$ is an increasing function with respect to t.

Now consider the derivative $z_2'(t, q, m)$:

$$z_2'(t, q, m) = \frac{z_2(t, q, m)^{2/3}}{m} \cdot \left(1 + \frac{2t + 3m}{\sqrt{t^2 + 3mt}}\cos\frac{\phi(t) + 4\pi}{3} - \frac{2}{3}\sin\frac{\phi(t) + 4\pi}{3}\phi'(t)\sqrt{t^2 + 3mt}\right).$$

We have $\frac{\phi + 4\pi}{3} \in \Big[3\pi/2; 5\pi/3\Big]$. As a result, $z_2(t, q, m)$ is a decreasing function with respect to t since $\phi'(t) < 0$ and $\sin\dfrac{\phi(t) + 4\pi}{3} < 0$ (see Fig. 1). □

Remark 2.3. From Prorosition 2.2 it follows that $z_1(\theta, q, m) > 1$ (resp., $z_2(\theta, q, m) > 1$) for any $\theta \geq \theta_m$ (resp., $\theta_{cr} > \theta \geq \theta_m$), and $z_2(\theta, q, m) < 1$ for $\theta > \theta_{cr}$.

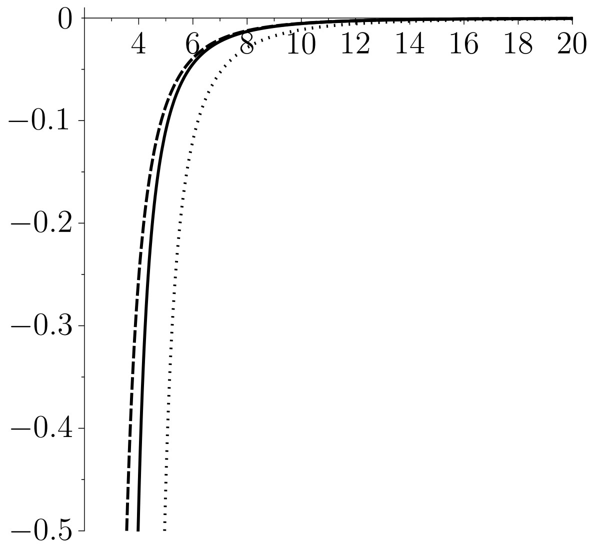

Fig. 1. Graph of the function $z_2'(\theta, q, m)$ for $q = 4,\ m = 1$ (dashed), $q = 5,\ m = 2$ (solid) and $q = 7,\ m = 3$ (dotted)

§ 3. The problem of extremality of the translation-invariant measures when $k = 3$

Following [18], to check the extremity of the Gibbs measure, we apply arguments of a reconstruction on trees [31–33].

By the symmetry of the Potts model, for each fixed $m \le [q/2]$, we reduce the study of extremality to at most three TISGMs: μ_0, which corresponds to the solution $h_x = (0, 0, 0, \dots, 0)$, and two TISGMs $\mu_{m1}(\theta, q, m)$, $\mu_{m2}(\theta, q, m)$, which correspond to the vectors

$$h_x = h_1 = (\underbrace{\ln z_1, \ln z_1, \dots, \ln z_1}_{m}, \underbrace{0, 0, \dots, 0}_{q-m}), \qquad h_x = h_2 = (\underbrace{\ln z_2, \ln z_2, \dots, \ln z_2}_{m}, \underbrace{0, 0, \dots, 0}_{q-m}),$$

where $z_1 = z_1(\theta, q, m)$, $z_2 = z_2(\theta, q, m)$ are solutions of the equation (2.1).

Thus, for each fixed value of m, all TISGMs can be categorized into three distinct classes.

— The first class contains only μ_0.

— The second class includes all measures that match vectors derived from permutations of h_1 coordinates.

— The second class includes all measures that match vectors derived from permutations of h_2 coordinates.

Remark 3.1. Note that, in case $m = q/2$, the solutions h_1 and h_2 define the same TISGMs after a relabeling of indices and a re-normalization of h_1.

For $l = (\underbrace{z, z, \dots, z}_{m}, \underbrace{1, 1, \dots, 1}_{q-m})$ the TISGM corresponding to a vector $l \in \mathbb{R}^q$ is a Markov chain with the states $\{1, 2, \dots, q\}$ (see [18]) and the transition probability matrix $\mathbb{P} = (P_{ij})$ with

$$P_{ij} = \frac{l_j \exp(J\beta\delta_{ij})}{\sum_{r=1}^{q} l_r \exp(J\beta\delta_{ir})}.$$

This expression gives

$$P_{ij}=\begin{cases}\theta z/Z_1, & \text{if} \quad i=j, \quad i\in\{1,\dots,m\}\\ z/Z_1, & \text{if} \quad i\neq j, \quad i,j\in\{1,\dots,m\}\\ 1/Z_1, & \text{if} \quad i\in\{1,\dots,m\}, \quad j\in\{m+1,\dots,q\}\\ z/Z_2, & \text{if} \quad i\in\{m+1,\dots,q\}, j\in\{1,\dots,m\}\\ \theta/Z_2, & \text{if} \quad i=j, \quad i\in\{m+1,\dots,q\}\\ 1/Z_2, & \text{if} \quad i\neq j, \quad i,j\in\{m+1,\dots,q\},\end{cases} \tag{3.1}$$

where

$$Z_1=(\theta+m-1)z+q-m, \qquad Z_2=mz+\theta+q-m-1. \tag{3.2}$$

Let us first give some necessary definitions from [34]. We consider the finite complete subtree $\mathcal{T}$, containing all initial points of the semitree $\Gamma^k_{x^0}$. The boundary $\partial\mathcal{T}$ of the subtree of its vertices, which are in $\Gamma^k_{x^0}\setminus\mathcal{T}$. We identify the subtree $\mathcal{T}$ with the set of its vertices. The set of all edges A and ∂A is denoted by $E(A)$.

In [34], the key values are κ and γ. These parameters define the properties of the set of the Gibbs measures $\{\mu^\tau_{\mathcal{T}}\}$, where the boundary condition τ is fixed and $\mathcal{T}$ is the arbitrary initial complete and finite subtree in $\Gamma^k_{x^0}$. For a given initial subtree $\mathcal{T}$ of the tree $\Gamma^k_{x^0}$ and the vertex $x\in\mathcal{T}$, we write $\mathcal{T}_x$ for the (maximum) subtree $\mathcal{T}$ with the initial point at x. If x is not the initial point of the $\mathcal{T}$, then the Gibbs measure is denoted by $\mu^s_{\mathcal{T}_x}$ where the "ancestor" x has the spin s and the configuration at the lower boundary $\mathcal{T}_x$ (i.e., on $\partial\mathcal{T}_x\setminus\{x\}$) is given in terms of τ.

For two measures μ_1 and μ_2 on Ω, we introduce the distance

$$\|\mu_1-\mu_2\|_x=\frac{1}{2}\sum_{i=1}^{q}\big|\mu_1(\sigma(x)=i)-\mu_2(\sigma(x)=i)\big|.$$

Let $\eta^{x,s}$ be the configuration η with the spin at x equal to s.

Following [34], we define

$$\kappa\equiv\kappa(\mu)=\sup_{x\in\Gamma^k}\max_{x,s,s'}\|\mu^s_{\mathcal{T}_x}-\mu^{s'}_{\mathcal{T}_x}\|_x; \qquad \gamma\equiv\gamma(\mu)=\sup_{A\subset\Gamma^k}\max\|\mu^{\eta^{y,s}}_A-\mu^{\eta^{y,s'}}_A\|_x,$$

where the maximum is taken over all boundary conditions η, all $y\in\partial A$, all neighbors $x\in A$ of the vertex y, and all spins $s,s'\in\{1,\dots,q\}$.

A sufficient condition for the Gibbs measure μ to be extreme is the inequality (see [34, Theorem 9.3])

$$k\kappa(\mu)\gamma(\mu)<1.$$

Note that κ has the simple form (see [34])

$$\kappa=\frac{1}{2}\max_{i,j}\sum_l|P_{il}-P_{jl}|. \tag{3.3}$$

The constant γ does not have a clean general formula, but can be estimated in specific models (as Ising, Potts, hard-core, etc.). For example, if $\mathbb{P}$ is a symmetric matrix of the Potts model (or the matrix corresponding to the solution $z=1$), then $\gamma\leq\frac{\theta-1}{\theta+1}$ (see [34, Theorem 8.1]).

Using formulas (3.1) for $i\neq j$ and (3.3), we obtain (see [18])

$$\frac{1}{2}\sum_{l=1}^{q}|P_{il}-P_{jl}|=\begin{cases}a, & \text{if } i,j=1,\dots,m,\\ b, & \text{if } i,j=m+1,\dots,q,\\ c_m, & \text{otherwise,}\end{cases}$$

where a, b, and c_m are defined as

$$a = \frac{(\theta-1)z}{Z_1}, \quad b = \frac{\theta-1}{Z_2}, \quad c_m = \frac{1}{2Z_1}\Big(z\big|\theta-\sqrt[k]{z}\big|+\big|1-\theta\sqrt[k]{z}\big|+(z(m-1)+q-m-1)\big|1-\sqrt[k]{z}\big|\Big),$$

and Z_1 and Z_2 are given by (3.2). Obviously,

$$\kappa = \begin{cases} \max\{b, c_1\}, & \text{if} \quad m=1, \\ \max\{a, b, c_m\}, & \text{if} \quad 2 \leq m \leq [q/2]. \end{cases}$$

We consider the case $z \neq 1$ (where $z = x^3$ and x are solutions of (2.1)). We take $(\underbrace{z, z, \dots, z}_{m}, 1, \dots, 1)$ as the fixed solution of Eq. (2.1) and let $\mathbb{P}$ be the corresponding matrix.

Now, let us simplify c_m based on $z > 1$. From (2.3), we obtain

$$z = \frac{\theta-1}{m} z^{2/3} + \frac{\theta-1}{m} z^{1/3} - \frac{q-m}{m}. \tag{3.4}$$

Using (3.4) for $z > 1$, after simple algebras we get

$$c_m = \frac{(m-1)(\theta-1)^2 z^{2/3} + \Big(m(\theta^2-\theta+q-m)-(\theta-1)^2\Big)z^{1/3} - (q-m)\big((m-1)\theta+1\big)}{m^2 Z_1}.$$

In particular, for $m = 1$ and $z > 1$,

$$c_1 = \frac{(\theta+q-2)z^{1/3}-(q-1)}{Z_1}.$$

Using (3.4) for $z < 1$, after simple algebras we get

$$c_m = \frac{\theta z^{1/3}-1}{Z_1}.$$

Let us show that $c_1 > b$ for $m = 1$ and $z > 1$. Indeed, $c_1 > b$ is equivalent to $z^{1/3} > 1$. Now we show that $b > c_1$ for $m \geq 1$ and $z < 1$. Indeed, $b > c_1$ is equivalent to $1 > z^{1/3}$. As a result, at $m = 1$ we have

$$\kappa = \begin{cases} c_1, & \text{if} \quad z > 1, \\ b, & \text{if} \quad z < 1. \end{cases} \tag{3.5}$$

If $m \geq 2$ and $z > 1$ ($z > z^{1/3}$), then $a > b$; if $z < 1$ ($z < z^{1/3}$), then $b > a$. Let us show that $a > c_m$. Indeed, under the condition (3.4) we get

$$\phi(\theta,q,m) := (\theta-1)^2 z^{2/3} + \Big(\theta^2-(m+2)\theta-m(q-m-1)+1\Big)z^{1/3} - (q-m)(\theta-m-1) > 0.$$

Note that, for any $2 \leq m \leq q/2$ and $q \geq 4$, the function $\phi(\theta, q, m) > 0$ (see Fig. 2).

As a result, for $m \geq 2$, we have:

$$\kappa = \begin{cases} a, & \text{if} \quad z > 1, \\ b, & \text{if} \quad z < 1. \end{cases} \tag{3.6}$$

At $z = 1$, we get

$$\kappa = \frac{\theta-1}{\theta+q-1}.$$

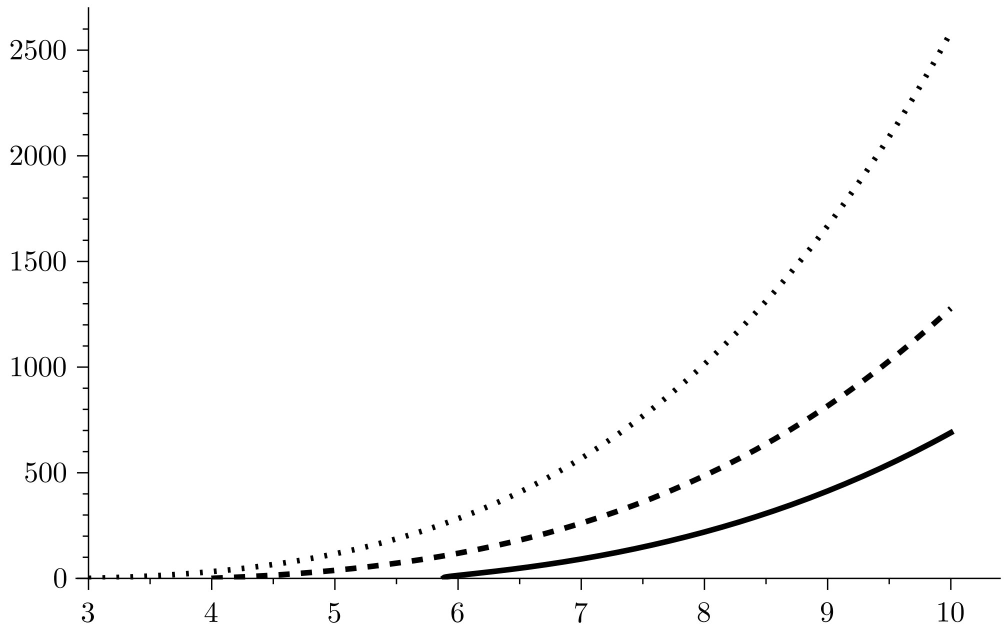

Fig. 2. Graph of the function $\phi(\theta, q, m)$ for $q = 4,\ m = 2$ (dotted), $q = 6,\ m = 3$ (dashed) and $q = 10,\ m = 4$ (solid)

3.1. Conditions under which the measure is not extreme

It is known that, for the Gibbs measure corresponding to $\mathbb{P}$ to be non-extremal, it is sufficient that the inequality $k\breve{\lambda}^2 > 1$ (the sufficient Kesten–Stigum condition) is satisfied where $\breve{\lambda}$ is the second eigenvalue in absolute value of the matrix $\mathbb{P}$ (see [31]).

Corollary 3.1 (see [18]). *Let $m \leq [q/2]$. Then the matrix $\mathbb{P}$ determined by formula (3.1) has the following eigenvalues:*

$$\begin{cases} \{1, b, \lambda_2(P)\}, & \textit{if} \quad m = 1, \\ \{1, a, b, \lambda_2(P)\}, & \textit{if} \quad m \geq 2, \end{cases} \tag{3.7}$$

where

$$a = \frac{(\theta - 1)z}{Z_1}, \quad b = \frac{(\theta - 1)\sqrt[3]{z}}{Z_1}, \quad \lambda_2(P) = \frac{\left[\theta - 1 + (1 - \sqrt[3]{z})m\right]z}{Z_1}. \tag{3.8}$$

Denote by $\breve{\lambda}$ the second eigenvalue in absolute value of the matrix $\mathbb{P}$.
For $m = 1$, from (3.7) and (3.8), after some algebras we obtain

$$\breve{\lambda} = \begin{cases} b, & \text{if} \quad z < 1, \\ \lambda_2(P), & \text{if} \quad z > 1. \end{cases} \tag{3.9}$$

For $m \geq 2$, we obtain

$$\breve{\lambda} = \begin{cases} b, & \text{if} \quad z < 1, \\ a, & \text{if} \quad z > 1. \end{cases} \tag{3.10}$$

In particular, for $z = 1$ we have

$$\breve{\lambda} = \frac{\theta - 1}{\theta + q - 1}. \tag{3.11}$$

Remark 3.2. The explicit forms of $z_1(\theta, q, m)$ and $z_2(\theta, q, m)$ are bulky. Therefore, below we use some estimations for inequalities.

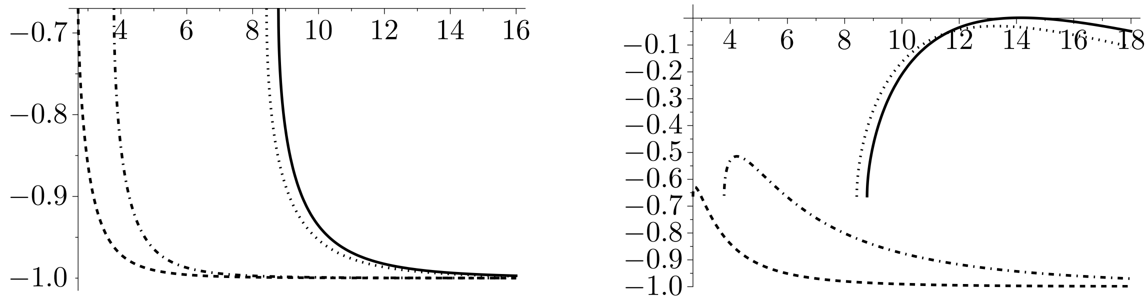

Fig. 3. Graph of the function $3\breve{\lambda}_2 - 1$ for $z_1(\theta)$ (left) and for $z_2(\theta)$ (right), in the cases $q = 4$ (dashed), $q = 10$ (dash-doted), $q = 100$ (doted) and $q = 113$ (solid)

Conditions of non-extremality of the measure μ_0**.** Let us first derive the condition of non-extremality. For this, we use the Kesten–Stigum condition ($3\breve{\lambda}^2 > 1$). From (3.11), we have

$$\breve{\lambda} = \frac{\theta - 1}{\theta + q - 1}.$$

As a result, we get

$$3\Big(\frac{\theta - 1}{\theta + q - 1}\Big)^2 > 1.$$

The positive solution to the last inequality is

$$\theta > \breve{\theta} = \frac{\sqrt{3} + 1}{2} q + 1.$$

Thus, the measure μ_0 is non-extreme for $\theta > \breve{\theta}$.

Conditions of non-extremality of the measures μ_{11} **and** μ_{12} **for** $m = 1$**.**

Case $z > 1$. According to Proposition 2.2, $z_1(\theta) > 1$ for $\theta > \theta_1$, and $z_2(\theta) > 1$ for $\theta_1 < \theta < \theta_{cr}$. Then, from (3.9), we get $\breve{\lambda} = \lambda_2(P)$ and the corresponding inequality $3\breve{\lambda}^2 > 1$ has no solution. We assume that the converse is true, i.e., $3\breve{\lambda}^2 < 1$. The last inequality is equivalent to

$$\frac{\sqrt{3}(\theta - \sqrt[3]{z})z}{Z_1} < 1.$$

Using (3.4), we rewrite the last equation as follows:

$$(\theta^2 - \theta)z^{2/3} + \big((\theta - \sqrt{3})(\theta - 1) - \sqrt{3}(q - 1)\big)z^{1/3} + (\sqrt{3} + 1 - \theta)(q - 1) > 0. \tag{3.12}$$

It is clear that (3.12) is valid for $4 \le q \le \theta$. For $\theta_1 < \theta < q$ and $z = z_1(\theta)$, the inequality (3.12) always holds. At the same time, for $4 \le q < 113$ and $z = z_2(\theta)$, the inequality (3.12) is appropriate (see Fig. 3). It means that the measure μ_{11} is obviously extreme for $\theta > \theta_1$ and the measure μ_{12} is obviously extreme for $\theta_1 < \theta < \theta_{cr}$ and $4 \le q \le 113$.

Case $z < 1$. In this case, inequality $3\breve{\lambda}^2 > 1$ is equivalent to

$$h_1(\theta) = (\sqrt{3} - 1)(\theta - 1) - q + 1 - z_2(\theta) > 0.$$

The explicit form of z is bulky. Thus, we use some estimations. If we replace z with 1, then we get that $h_1(\theta) > 0$ for $\theta \ge \breve{\theta}$. Therefore, the measure μ_{12} is non-extreme for $\theta \ge \breve{\theta}$.

Conditions of non-extremality of the measures μ_{m1} **and** μ_{m2} **for** $m \ge 2$**.**

Case $z > 1$. According to Proposition 2.2, $z_1(\theta) > 1$ for $\theta > \theta_m$, and $z_2(\theta) > 1$ for $\theta_m < \theta < \theta_{cr}$. For $m \geq 2$, from (3.10), we find $\breve{\lambda} = a$. Then, inequality $3a^2 > 1$ reads

$$\Big((\sqrt{3}-1)(\theta-1)-m\Big)z > q-m.$$

If we replace z with 1, then we obtain $(\sqrt{3}-1)(\theta-1) > q$. This implies $\theta \geq \breve{\theta}$. Thus, the measure μ_{m1} is non-extreme for $\theta \geq \breve{\theta}$.

Case $z < 1$. In this case, from (3.10), we find $\breve{\lambda} = b$. Then, inequality $3b^2 > 1$ reads

$$\sqrt{3}(\theta-1) > mz+\theta+q-m-1.$$

If we replace z with 1, then we have $(\sqrt{3}-1)(\theta-1) > q$. This implies $\theta > \breve{\theta}$, i.e., the measure μ_{m2} is non-extreme for $\theta > \breve{\theta}$.

3.2. Conditions of extremality of measures

Conditions of extremality of the measure μ_0. Now let us check the condition of extremality ($k\kappa(\mu)\gamma(\mu) < 1$). In this case, we obtain $\kappa = \frac{\theta-1}{\theta+q-1}$ for $z = 1$. Then, for $\gamma \leq \frac{\theta-1}{\theta+1}$, the condition $k\kappa(\mu)\gamma(\mu) < 1$ reads

$$2\theta^2-(q+6)\theta+4-q < 0.$$

The solution of the above inequality is

$$\frac{1}{4}\Big(q+6-\sqrt{q^2+20q+4}\Big) < \theta < \frac{1}{4}\Big(q+6+\sqrt{q^2+20q+4}\Big).$$

Note that, for $q \geq 4$, inequality $q+6-\sqrt{q^2+20q+4} \leq 0$ holds. As a result, the measure μ_0 is extreme for $1 < \theta < \theta^{(1)}$, where

$$\theta^{(1)} = \frac{1}{4}\Big(q+6+\sqrt{q^2+20q+4}\Big).$$

Conditions of extremality of the measures μ_{11} and μ_{12}.

Case $z > 1$. In this case, from (3.5), we get $\kappa = c$. Then, the condition $3\kappa(\mu)\gamma(\mu) < 1$ reads:

$$3(\theta-1)((\theta+q-2)z^{1/3}-(q-1)) < (\theta+1)(\theta z+q-1).$$

Using (3.4), for $m = 1$ and after some algebras, we have

$$\theta(\theta+1)z^{2/3}+(\theta^2-2\theta+6-3q)z^{1/3}-(q-1)(\theta-2) > 0.$$

If we replace z with 1, then we have $2\theta^2-\theta q+4-q > 0$, and the positive solution to the last inequality is

$$\theta > \theta^{(2)} = \frac{1}{4}\Big(q+\sqrt{q^2+8q-32}\Big).$$

As a result, the measure μ_{11} is extreme for $\theta > \theta^{(2)}$ and the measure μ_{12} is extreme for $\theta^{(2)} < \theta < \theta_{cr}$.

Case $z < 1$. In this case, we obtain $\kappa = b$. Then, inequality $3\kappa(\mu)\gamma(\mu) < 1$ reads

$$3(\theta-1)^2 < (\theta+1)(z+\theta+q-2).$$

If we replace z with 0, then we have $2\theta^2-(q+5)\theta-(q-5)<0$, and the positive solution to the last inequality is

$$\theta_{cr}<\theta<\theta^{(3)}=\frac{1}{4}\Big(q+5+\sqrt{q^2+18q-15}\Big).$$

Therefore, the measure μ_{12} is extreme for $\theta_1<\theta<\theta^{(3)}$. This implies that the measure μ_{12} is extreme for $\theta^{(2)}<\theta<\theta^{(3)}$.

Conditions of extremality of measures μ_{m1} and μ_{m2} for $m\geq 2$.

Case $z>1$. In this case, from (3.6), we find that $\kappa=a$. Then, from (3.5), we find that $\kappa=a$. As a result, the extremality condition $3a\gamma<1$ is equivalent

$$(2\theta^2-(m+6)\theta+4-m)z<(q-m)(\theta+1).$$

If the leading coefficient in the LHS of last inequality is negative, then the inequality is satisfied. In this case, we get

$$\theta<\theta^{(4)}=\frac{1}{4}\Big(m+6+\sqrt{m^2+20m+4}\Big).$$

Therefore, the measures μ_{m1} and μ_{m2} are extreme for $\theta_m<\theta<\theta^{(4)}$.

Case $z<1$. In this case from (3.6), we get that $\kappa=b$. As a result, the extremality condition $3b\gamma<1$ is equivalent to

$$3(\theta-1)^2<(\theta+1)(mz+\theta+q-m-1).$$

If we replace z with 0, then after some algebras we obtain

$$\theta<\theta^{(5)}=\frac{1}{4}\Big(q-m+6+\sqrt{m^2-2mq+q^2-20m+20q+12}\Big).$$

This implies that the measure μ_{m2} is extreme for $\theta_m<\theta<\theta^{(4)}$.

Thus, the following theorem holds.

Theorem 3.1. *Let $k=3$, $q\geq 4$ and $J>0$. Then the following statements hold:*

(1) *the measure μ_0 is extreme for $1<\theta<\theta^{(1)}$ and is non-extreme for $\theta>\breve{\theta}$;*

(2) *the measure μ_{11} is extreme for $\theta>\theta^{(2)}$;*

(3) *the measure μ_{12} is extreme for $\theta^{(2)}<\theta<\theta^{(3)}$ and is non-extreme for $\theta\geq\breve{\theta}$;*

(4) *the measure μ_{m1} is extreme for $\theta_m<\theta<\theta^{(4)}$ and is non-extreme for $\theta\geq\breve{\theta}$;*

(5) *the measure μ_{m2} is extreme for $\theta_m<\theta<\theta^{(5)}$ and is non-extreme for $\theta\geq\breve{\theta}$,*

where

$$\breve{\theta}=\frac{\sqrt{3}+1}{2}q+1,\quad \theta^{(1)}=\frac{1}{4}\Big(q+6+\sqrt{q^2+20q+4}\Big),\quad \theta^{(2)}=\frac{1}{4}\Big(q+\sqrt{q^2+8q-32}\Big),$$
$$\theta^{(3)}=\frac{1}{4}\Big(q+5+\sqrt{q^2+18q-15}\Big),\quad \theta^{(4)}=\frac{1}{4}\Big(m+6+\sqrt{m^2+20m+4}\Big),$$
$$\theta^{(5)}=\frac{1}{4}\Big(q-m+6+\sqrt{m^2-2mq+q^2-20m+20q+12}\Big).$$

Remark 3.3. In Theroem 3.1, we obtain the regions of extremality and non-extremality for ferromagnetic Potts model with q states. Due to technical complexity, these results may be not sharp.

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Чистые фазы ферромагнитной модели Поттса с q состояниями на дереве Кэли третьего порядка

Ключевые слова: дерево Кэли, конфигурация, модель Поттса, мера Гиббса, трансляционно-инвариантная мера.

УДК 517.98

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Изучение фазового перехода является одной из центральных проблем статистической механики. Он происходит, когда для модели существуют по крайней мере две различные меры Гиббса. Известно, что для ферромагнитной модели Поттса с q состояниями при достаточно низких температурах существуют не более $2^q - 1$ трансляционно-инвариантных расщепленных мер Гиббса. Для непрерывных гамильтонианов меры Гиббса образуют непустое, выпуклое, компактное подмножество в пространстве всех вероятностных мер. Экстремальные меры, которые соответствуют крайним точкам этого множества, определяют чистые фазы. Мы изучаем экстремальность трансляционно-инвариантных расщепленных мер Гиббса для ферромагнитной модели Поттса с q состояниями на дереве Кэли третьего порядка. Мы определяем области, в которых изучаемые трансляционно-инвариантные меры Гиббса для этой модели являются экстремальными или не являются экстремальными. Мы сводим описание мер Гиббса к решению нелинейного функционального уравнения, каждое решение которого соответствует одной предельной мере Гиббса.

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