VESTNIK UDMURTSKOGO UNIVERSITETA. MATEMATIKA. MEKHANIKA. KOMP'YUTERNYE NAUKI
MATHEMATICS 2026. Vol. 36. Issue 1. Pp. 82–98.

MSC2020: 16S34, 16U60, 20C07

© ***D. T. Trang***

RINGS WITH ∇-NILPOTENT UNIT DEVIATIONS

We introduce and investigate a new class of associative rings, called n-∇U rings, characterized by the condition that for every unit u in the ring, the element $u^n - 1$ belongs to a distinguished subset $\nabla(R)$ of ∇-nilpotent elements. This subset consists of elements $x \in R$ such that $1 - ux$ is invertible for all units u commuting with x. We explore the structural properties of n-∇U and π-∇U rings, provide illustrative examples, and examine their behavior under various ring-theoretic constructions including direct products, quotient rings, and trivial extensions. Our results establish connections between the n-∇U condition and classical notions such as regularity, cleanness, and Dedekind-finiteness, offering new insights into the interplay between unit powers and nilpotency in ring theory.

Keywords: n-∇U ring, ∇-nilpotent element, Jacobson radical, Dedekind-finite ring, clean ring, exchange ring, regular ring, π-regular ring.

DOI: 10.35634/vm260105

§ 1. Introduction

In ring theory, the behavior of unit elements often serves as a powerful lens through which the structure and properties of a ring can be understood. Units not only govern invertibility but also interact intricately with other algebraic elements such as idempotents, nilpotents, and radicals. Over the years, various classes of rings have been defined based on how units behave: examples include clean rings, exchange rings, and UJ-rings (see [1–4, 8–12]). These constructions have led to rich theories and applications across algebra and module theory.

This paper introduces and investigates a new class of rings, called n-∇U rings, which are defined via a subtle interaction between unit powers and ∇-nilpotent elements. The central idea is to examine how far a unit element $u \in U(R)$ deviates from the identity when raised to a fixed power n, and whether this deviation lies within a special subset of the ring known as $\nabla(R)$.

Let R be an associative ring with identity, not necessarily commutative. We denote:

- $U(R)$: the group of units in R,
- $Z(R)$: the center of R,
- $J(R)$: the Jacobson radical,
- $\mathrm{Nil}(R)$: the set of nilpotent elements,
- $\mathrm{Id}(R)$: the set of idempotents.

We consider the set

$$\begin{aligned} J(R) \subseteq \Delta(R) &= \{x \in R \colon x + u \in U(R) \ \text{ for all } \ u \in U(R)\} \\ &= \{x \in R \colon 1 - xu \ \text{ is invertible for all } u \in U(R)\} \\ &= \{x \in R \colon 1 - ux \ \text{ is invertible for all } u \in U(R)\} \end{aligned}$$

that was handled by Lam [6, Exercise 4.24].

An element $x \in R$ is called ∇-*nilpotent* if $1 - ax \in U(R)$ for every $a \in U(R)$ such that $xa = ax$ (see [9]). We define

$$\nabla(R) = \{x \in R \mid 1 - ux \in U(R) \ \text{ for all } \ u \in U(R) \ \text{ with } \ ux = xu\}.$$

This set generalizes the notion of nilpotency by focusing on invertibility conditions under commutativity constraints, and it includes elements that behave "almost nilpotently" with respect to units.

A ring R is called a ∇U *ring* (or ΔNU *ring*) if every unit $u \in U(R)$ satisfies $u - 1 \in \nabla(R)$ (see [9]). With this setup, we define two key classes.

- Let $n \geq 2$ be a fixed integer. A ring R is called an n-∇U if every unit $u \in U(R)$ satisfies $u^n - 1 \in \nabla(R)$.
- A ring R is called a π-∇U *ring* if for each unit $u \in U(R)$, there exists an integer $i \geq 2$ (depending on u) such that $u^i - 1 \in \nabla(R)$.

These definitions capture a wide spectrum of ring behaviors. For example, in division rings, the n-∇U condition forces every unit to be an n-th root of unity, leading to strong finiteness constraints. In more general settings, the condition interacts with the ring's radical, its idempotents, and its direct product structure.

The motivation behind studying n-∇U rings stems from a desire to unify and extend existing concepts in ring theory that involve unit elements and nilpotency. By focusing on the deviation $u^n - 1$ and its membership in $\nabla(R)$, we obtain a flexible framework that encompasses several known ring classes and reveals new structural phenomena.

In this paper, we develop the foundational theory of n-∇U and π-∇U rings. We begin by presenting illustrative examples that demonstrate the diversity of rings satisfying these conditions. We then explore their behavior under standard ring constructions such as direct products, quotient rings, and trivial extensions. Furthermore, we establish connections between the n-∇U property and classical notions such as regularity, cleanness, and Dedekind-finiteness. Our results show that the n-∇U condition is not only algebraically meaningful but also structurally rich, offering new insights into the interplay between unit powers and ∇-nilpotency in ring theory.

§ 2. n-∇U rings

In this section, we introduce the notion of n-∇U rings, examine their structural properties, and illustrate the theory with examples. For clarity, the presentation is divided into several subsections.

2.1. Definitions and examples

Definition 2.1. Let $n \geq 2$ be a fixed integer. A ring R is called an n-∇U if every unit $u \in U(R)$ satisfies $u^n - 1 \in \nabla(R)$.

Definition 2.2. A ring R is called π-∇U if, for any $u \in U(R)$, there exists $i \geq 2$ depending on u such that $u^i - 1 \in \nabla(R)$.

From the above definitions, we consider the following examples of n-∇U rings and π-∇U rings.

Example 2.1. (1) Let $R = \mathbb{Z}_8$. Then $U(R) = \{1, 3, 5, 7\}$, and for each unit u, we have $u^2 - 1 = 0 \in \nabla(R)$. It follows that $\mathbb{Z}_8$ is a 2-∇U ring.

(2) Let $R = \mathbb{Z}_{14} \cong \mathbb{Z}_2 \times \mathbb{Z}_7$. The unit group is $U(R) = \{1, 3, 5, 9, 11, 13\}$. For each unit u, we have $u^6 - 1 = 0 \in \nabla(R)$. Hence, $\mathbb{Z}_{14}$ is a 6-∇U ring.

(3) Let $R = \mathbb{F}_2[[x]]$, the ring of formal power series over $\mathbb{F}_2$. Then R is a local ring with maximal ideal (x), and every unit has the form $1 + f(x)$, where $f(x) \in x\mathbb{F}_2[[x]]$. Since $(1 + f(x))^2 - 1 = f(x)^2 + 2f(x) \in x\mathbb{F}_2[[x]]$, which is topologically nilpotent, we conclude that R is a 2-∇U ring. However, $u^2 - 1$ is not ∇-nilpotent in general.

(4) Let $R = \mathbb{F}_3\langle x, y\rangle/(x^2)$, the noncommutative polynomial ring over $\mathbb{F}_3$ modulo the relation $x^2 = 0$. Then R is a 2-∇U ring.

Example 2.2. (1) Let $R = \mathrm{M}_2(\mathbb{F}_3)$, the ring of 2×2 matrices over the finite field $\mathbb{F}_3$. Then, the unit group $U(R) = \mathrm{GL}_2(\mathbb{F}_3)$ has order $(3^2 - 1)(3^2 - 3) = 48$. It follows that for each invertible matrix u, there exists some $i \geq 2$ (depending on the order of u) such that $u^i = I_2$, and so $u^i - I_2 = 0 \in \nabla(R)$. Thus, $\mathrm{M}_2(\mathbb{F}_3)$ is a π-∇U ring.

(2) Let $R = \mathbb{Z}_2[x]/(x^2 - x)$. This is a Boolean ring, so every element is idempotent. The only unit is 1, and, for $u = 1$, we have $u^i - 1 = 0$ for all $i \geq 2$. Hence, R is a π-∇U ring.

2.2. Basic properties

In this subsection, we present the basic properties of the classes of rings just defined, and at the same time examine under what conditions they are division rings.

We now examine how the n-∇U property behaves under direct product constructions. The following proposition shows that this condition is componentwise.

Proposition 2.1. *Let $\{R_i\}_{i\in I}$ be a family of rings. Then the direct product $\prod_{i\in I} R_i$ is n-∇U if and only if each component ring R_i is n-∇U.*

P r o o f. It is well known that the unit group and the quasi-nilpotent set of a direct product decompose as

$$U\left(\prod_{i\in I} R_i\right) = \prod_{i\in I} U(R_i), \quad \nabla\left(\prod_{i\in I} R_i\right) = \prod_{i\in I} \nabla(R_i).$$

Let $u = (u_i)_{i\in I} \in U\left(\prod R_i\right)$. Then

$$u^n - 1 = (u_i^n - 1)_{i\in I}.$$

Thus, $u^n - 1 \in \nabla\left(\prod R_i\right)$ if and only if $u_i^n - 1 \in \nabla(R_i)$ for all $i \in I$. Hence, the product ring is n-∇U if and only if each R_i is n-∇U . □

Example 2.3. Consider the rings $R_1 = \mathbb{Z}_2$ and $R_2 = \mathbb{Z}_3$.

- For $R_1 = \mathbb{Z}_2$, the unit group is $U(R_1) = \{1\}$. For any $n \geq 2$, we have $1^n - 1 = 0 \in \nabla(R_1)$. Hence, $\mathbb{Z}_2$ is n-∇U for all n.
- For $R_2 = \mathbb{Z}_3$, the unit group is $U(R_2) = \{1, 2\}$. For $n = 2$, we compute: $1^2 - 1 = 0 \in \nabla(R_2)$, and $2^2 - 1 = 3 \equiv 0 \in \nabla(R_2)$. Thus, $\mathbb{Z}_3$ is a 2-∇U ring.

Now consider the direct product $R = R_1 \times R_2 = \mathbb{Z}_2 \times \mathbb{Z}_3$. Its unit group is $U(R) = U(R_1)\times U(R_2) = \{1\}\times\{1, 2\}$. For $n = 2$, each unit satisfies $u^2 = 1$, hence $u^2 - 1 = 0 \in \nabla(R)$.

Therefore, $R = \mathbb{Z}_2 \times \mathbb{Z}_3$ is a 2-∇U ring, illustrating Proposition 2.1.

Proposition 2.2. *Let R be an n-∇U ring. If n is an odd integer, then $2 \in \Delta(R)$.*

P r o o f. Since n is odd, we have $(-1)^n = -1 \in 1 + \nabla(R)$, which implies $-2 \in \nabla(R)$. Therefore, $2 \in \nabla(R)$. As $2 \in Z(R)$, and ∇-nilpotent central elements lie in the $\Delta(R)$, it follows that $2 \in \Delta(R)$, as claimed. □

Example 2.4. For Proposition 2.2, the assumption that n is odd is essential. For instance, although $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ is a 2-∇U ring, a direct computation shows that $2 \notin \Delta(\mathbb{Z}_6)$. This illustrates that the conclusion does not hold when n is even.

Now we provide a necessary and sufficient condition for an n-∇U ring to be ∇U.

However, first we must prove the following lemma to use it in proving the subsequent results.

Lemma 2.1. *Let R be a ring and $a \in R$. If $a^m \in \nabla(R)$ with $m \geq 1$, then $a \in \nabla(R)$.*

P r o o f. Let $u \in U(R)$ with $au = ua$. Then we have $u^m a^m = a^m u^m$. Define $b = 1 + au + (au)^2 + \cdots + (au)^{m-1}$. It follows that

$$b(1 - au) = (1 - au)b = 1 - (au)^m = 1 - a^m u^m \in U(R).$$

Since $a^m \in \nabla(R)$, so $1 - au \in U(R)$. This proves that $a \in \nabla(R)$. □

Proposition 2.3. *Suppose $k \geq 1$. Then, a ring R is ∇U if and only if*

(1) $2 \in \Delta(R)$;

(2) *R is a 2^k-∇U ring for all $k \geq 1$;*

(3) *$ab \in \nabla(R)$ for all $a, b \in \nabla(R)$ with $ab = ba$.*

P r o o f. "⇒" (1) is obvious.

(2) Let $u \in U(R)$, and so $u^{2^k} \in U(R)$. Since R is ∇U, then $u^{2^k} - 1 \in \nabla(R)$. Thus, R is a 2^k-∇U ring.

(3) As $a, b \in \nabla(R)$ and $ab = ba$, $1 + a \in U(R)$ and $(1 + a)^{-1}b = b(1 + a)^{-1}$. It follows that

$$1 + a + b = (1 + a)\left[1 + (1 + a)^{-1}b\right] \in U(R) = 1 + \nabla(R),$$

so, $a + b \in \nabla(R)$.

"⇐". Let $u \in U(R)$. By (2), we have $u^{2^k} \in 1 + \nabla(R)$ and so $u^{2^k} = 1 + q$ for some $q \in \nabla(R)$. Therefore, combining with (1), we conclude that $(u - 1)^{2^k} = 1 + u^{2^k} + j$ for some $j = 2f(u) \in \Delta(R)$ with $f(x) \in \mathbb{Z}[x]$. So, $(u - 1)^{2^k} = (2 + j) + q$ for some $j \in \Delta(R)$. Call $y = 2 + j \in \Delta(R)$. Note that $y(u - 1)^{2^k} = (u - 1)^{2^k} y$. Then, we have $qy = yq$. By (3), it infers that $(u - 1)^{2^k} = y + q \in \nabla(R)$. Thus, we conclude that $u - 1 \in \nabla(R)$ by Lemma 2.1, which ensures that R is a ∇U ring, as required. □

We now turn to the structure of n-∇U division rings, where the ∇-nilpotent condition exhibits notable rigidity owing to the absence of nontrivial radical elements. The proposition below offers a complete characterization.

Proposition 2.4. *Let $\mathbb{F}$ be a division ring.*

(1) *$\mathbb{F}$ is n-∇U if and only if $u^n = 1$ for every $u \in U(\mathbb{F})$.*

(2) *If $\mathbb{F}$ is a field, then $\mathbb{F}$ is n-∇U if and only if $\mathbb{F}$ is finite and $(|\mathbb{F}| - 1) \mid n$.*

(3) *If $n \geq 2$ and $\mathbb{F}$ is n-∇U, then $\mathbb{F}$ is a finite field and $(|\mathbb{F}| - 1) \mid n$.*

P r o o f. (1) In any division ring $\mathbb{F}$, the ∇-nilpotent element is trivial, namely $\nabla(\mathbb{F}) = \{0\}$. Accordingly, the requirement $u^n - 1 \in \nabla(\mathbb{F})$ simplifies to $u^n = 1$. Thus, $\mathbb{F}$ is n-∇U precisely when every unit is an n-th root of unity.

(2) Suppose $\mathbb{F}$ is n-∇U. Then for every unit $u \in \mathbb{F}^*$, we have $u^n = 1 + r$, where $r \in \nabla(\mathbb{F})$. Since $\mathbb{F}$ is a field, $\nabla(\mathbb{F}) = \{0\}$, so $u^n = 1$. Thus, every element of $\mathbb{F}^*$ is an n-th root of unity.

Let $f(x) = x^n - 1 \in \mathbb{F}[x]$. Then all elements of $\mathbb{F}^*$ are roots of $f(x)$, so $\mathbb{F}^* \subseteq \mathrm{Root}(f)$, and hence $|\mathbb{F}^*| \le n$. Since $\mathbb{F}^*$ is a finite cyclic group, say $\mathbb{F}^* = \langle a \rangle$, then $a^n = 1$, so the order of a, denoted $o(a)$, divides n. But $o(a) = |\mathbb{F}^*|$, so, $|\mathbb{F}^*| \mid n$, i. e., $(|\mathbb{F}| - 1) \mid n$.

Conversely, if $\mathbb{F}$ is finite and $(|\mathbb{F}| - 1) \mid n$, then every element $u \in \mathbb{F}^*$ satisfies $u^n = 1$, so $u^n - 1 = 0 \in \nabla(\mathbb{F})$. Hence, $\mathbb{F}$ is n-∇U .

(3) Since $\mathbb{F}$ is a division ring, we have $\nabla(\mathbb{F}) = \{0\}$. Thus, for any unit $a \in \mathbb{F}^*$, the condition $a^n - 1 \in \nabla(\mathbb{F})$ implies $a^n = 1$. In particular, $a = a^{n+1}$, so every element satisfies a power identity.

By Jacobson's Theorem (see [5, 12.10]), any division ring in which every element is algebraic over its center and satisfies such a power identity must be commutative. Hence, $\mathbb{F}$ is a field. The second assertion follows directly from (2). □

Corollary 2.1. *If $\mathbb{F}$ is a division ring which is π-∇U, then $\mathbb{F}$ is a field.*

Example 2.5. (1) Let $\mathbb{F} = \mathbb{Z}_5$. Then $U(\mathbb{F}) = \{1, 2, 3, 4\}$, which is a cyclic group of order 4. Therefore, $\mathbb{Z}_5$ is n-∇U if and only if $4 \mid n$. For instance, when $n = 4$, every unit satisfies $u^4 = 1$, so $\mathbb{Z}_5$ is 4-∇U.

(2) As a counterexample, consider $\mathbb{F} = \mathbb{Q}$, the field of rational numbers. Here $U(\mathbb{Q}) = \mathbb{Q}\backslash\{0\}$, which is infinite. Not every unit is an n-th root of unity, so $\mathbb{Q}$ is not n-∇U for any $n \ge 2$.

2.3. Homomorphisms and subrings

In this subsection, we study the properties of subrings and quotient rings of n-∇U rings.

Proposition 2.5. *Let R be an n-∇U ring.*

1. *If $\nabla(R)$ is a subring of R and $k \in \mathbb{N}$ is such that $n \mid k$, then R is a k-∇U ring.*
2. *For any unital subring $S \subseteq R$, if $S \cap \nabla(R) \subseteq \nabla(S)$, then S is an n-∇U ring. In particular, the center $Z(R)$ of R is an n-∇U ring.*

P r o o f. (1) Since R is an n-∇U ring, for any unit $u \in U(R)$, we have $u^n = 1 + r$ for some $r \in \nabla(R)$. Given that $n \mid k$, there exists $t \in \mathbb{N}$ such that $k = tn$. By the hypothesis, we have that $\nabla(R)$ is a subring of R (i. e., $\nabla(R)$ is closed under addition and multiplication), and so

$$u^k = (u^n)^t = (1 + r)^t = 1 + r',$$

where $r' = (1 + r)^t - 1 \in \nabla(R)$. Therefore, $u^k - 1 \in \nabla(R)$, and R satisfies the k-∇U condition.

(2) Let $v \in U(S) \subseteq U(R)$. Since R is n-∇U, we have $v^n - 1 \in \nabla(R)$. As $v \in S$, it follows that $v^n - 1 \in S \cap \nabla(R) \subseteq \nabla(S)$. Hence, $v^n - 1 \in \nabla(S)$, and so S satisfies the n-∇U condition.

For the particular case, note that the center $Z(R)$ is a unital subring of R, and since $\nabla(R) \subseteq \subseteq Z(R)$ (as ∇-nilpotent elements commute with all elements), we have $Z(R) \cap \nabla(R) = \nabla(R) \subseteq \subseteq \nabla(Z(R))$. Thus, the condition is satisfied, and $Z(R)$ is n-∇U. □

Recall that a ring homomorphism $f\colon S \to R$ is said to be *local* if every non-unit element of S maps to a non-unit in R. A subring $S \subseteq R$ is called *rationally closed* if $U(S) = U(R) \cap S$, which is equivalent to the inclusion map $\iota\colon S \hookrightarrow R$ being a local homomorphism.

Since a rationally closed subring $S \subseteq R$ satisfies $S \cap \nabla(R) \subseteq \nabla(S)$, we may apply Proposition 2.5 to obtain the following consequence.

Corollary 2.2. *Let R be an n-∇U ring and $S \subseteq R$ be a rationally closed subring. Then S is also an n-∇U ring.*

Example 2.6. (1) Consider $R = \mathbb{Z}_8$. We know that R is a 2-∇U ring since for each unit $u \in U(R) = \{1,3,5,7\}$ we have $u^2 - 1 = 0 \in \nabla(R)$. Moreover, $\nabla(R)$ is a subring of R. Since $2 \mid 4$, by Proposition 2.5 (1), it follows that R is also a 4-∇U ring.

(2) Consider $R = \mathbb{Z}_2 \times \mathbb{Z}_3$. Then R is a 2-∇U ring because each unit $(a,b) \in U(R)$ satisfies $(a,b)^2 = (1,1)$, hence, $(a,b)^2 - (1,1) = (0,0) \in \nabla(R)$. Now take the unital subring $S = \mathbb{Z}_2 \times \{0,1\} \subseteq R$. We have $S \cap \nabla(R) \subseteq \nabla(S)$, so by Proposition 2.5 (2), S is also a 2-∇U ring. In particular, the center $Z(R)$ coincides with R itself, so $Z(R)$ is a 2-∇U ring.

Proposition 2.6. *Let R be an n-∇U ring. Suppose $f\colon R \to T$ is a ring epimorphism such that $f(\nabla(R)) \subseteq \nabla(T)$ and every unit of T lifts to a unit of R. Then T is an n-∇U ring.*

P r o o f. Let $v \in U(T)$. By assumption, there exists $u \in U(R)$ such that $f(u) = v$. Since R is n-∇U, we have $u^n = 1 + r$ for some $r \in \nabla(R)$. Applying f, we obtain:

$$v^n = f(u)^n = f(u^n) = f(1+r) = f(1) + f(r) = 1 + f(r).$$

By hypothesis, $f(r) \in \nabla(T)$, so $v^n - 1 \in \nabla(T)$. Hence, T satisfies the n-∇U condition. □

Example 2.7. Let $R = \mathbb{Z}_{14}$ and consider the natural epimorphism

$$f\colon R \to T = \mathbb{Z}_7, \quad f(x) = x \pmod 7.$$

- First, note that $R = \mathbb{Z}_{14}$ is a 6-∇U ring. Indeed, $U(R) = \{1,3,5,9,11,13\}$ and for each unit u, we have $u^6 - 1 = 0 \in \nabla(R)$.
- The map f is a ring epimorphism. Moreover, $f(\nabla(R)) \subseteq \nabla(T)$ because $\nabla(R)$ maps into $\nabla(\mathbb{Z}_7)$ under reduction modulo 7.
- Every unit of $T = \mathbb{Z}_7$ lifts to a unit of R. For example, $U(T) = \{1,2,3,4,5,6\}$, and each of these has a representative in $U(R)$ (e. g., $2 \mapsto 9$, $3 \mapsto 3$, $4 \mapsto 11$, $5 \mapsto 5$, $6 \mapsto 13$).

Therefore, by Proposition 2.6, $T = \mathbb{Z}_7$ is also a 6-∇U ring.

Our first major assertion establishes a necessary and sufficient condition connecting the quasi-nilpotent elements of a ring and its quotient modulo a radical ideal. The following lemma is straightforward to verify.

Lemma 2.2 (see [9]). *Let R be a ring, $I \subseteq J(R)$ an ideal of R, and let $\overline{R} = R/I$. Then the following statements hold.*

(1) *For every $\bar{q} \in \nabla(\overline{R})$, we have $q \in \nabla(R)$.*

(2) *For every $\bar{q} \in \nabla(\overline{R})$ and $p \in I$, we have $q + p \in \nabla(R)$.*

We proceed to formulate a lifting result for the n-∇U property through quotient rings modulo radical ideals, offering a useful criterion for confirming the n-∇U condition in more general settings.

Theorem 2.1. *Let $I \subseteq J(R)$ be an ideal of a ring R. If the quotient ring R/I is n-∇U, then R is also n-∇U.*

P r o o f. Assume R/I is an n-∇U ring, and let $u \in U(R)$. Then the image $\bar{u} = u + I$ lies in $U(R/I)$, and by the n-∇U property of R/I, we have $\bar{u}^n = \bar{1} + \bar{r}$ for some $\bar{r} \in \nabla(R/I)$. This implies that $u^n + I = 1 + r + I$ for some $r \in \nabla(R)$, so $u^n - (1 + r) \in I$. Hence, we can write $u^n = 1 + (r + r')$ for some $r' \in I$. Since $I \subseteq J(R)$, Lemma 2.2 guarantees that $r + r' \in \nabla(R)$. Therefore, $u^n \in 1 + \nabla(R)$, and thus, R is an n-∇U ring. □

Corollary 2.3. *Let R be a ring. If the quotient $R/J(R)$ is n-∇U, then R is n-∇U.*

Example 2.8. Let $R = \mathbb{Z}_9$ and $I = 3\mathbb{Z}_9 = \{0, 3, 6\}$. Note that $I \subseteq J(R)$, since $J(\mathbb{Z}_9) = 3\mathbb{Z}_9$.

- The quotient ring is $R/I \cong \mathbb{Z}_3$. In $\mathbb{Z}_3$, the unit group is $U(\mathbb{Z}_3) = \{1, 2\}$, which is cyclic of order 2. Hence, $\mathbb{Z}_3$ is n-∇U if and only if $2 \mid n$. For example, when $n = 2$, both units satisfy $u^2 = 1$, so $\mathbb{Z}_3$ is 2-∇U.
- By Theorem 2.1, it follows that $R = \mathbb{Z}_9$ is also a 2-∇U ring.

Thus, this example shows that if R/I is 2-∇U, then R itself is also 2-∇U.

2.4. Examples of n-∇U matrix rings

In this subsection, we study (examples of) certain classes of matrix rings that are n-∇U rings, and this depends on n.

First, we investigate the corner rings of n-∇U rings.

Proposition 2.7. *Let R be an n-∇U ring and e be an idempotent of R. Then eRe is an n-∇U ring.*

P r o o f. Let $u \in U(eRe)$. Call $u^{-1} \in U(eRe)$ with $uu^{-1} = u^{-1}u = e$. Then, we have that $u(1-e) = 0 = (1-e)u$, $u^{-1}(1-e) = 0 = (1-e)u^{-1}$, and

$$[u + (1-e)][u^{-1} + (1-e)] = e + 1 - e = 1,$$
$$[u^{-1} + (1-e)][u + (1-e)] = e + 1 - e = 1.$$

Therefore, $u + (1-e) \in U(R)$. Since R is an n-∇U ring, there exists $q \in \nabla(R)$ such that

$$(u + (1-e))^n = u^n + (1-e) = 1 + q \in 1 + \nabla(R).$$

Thus, it follows that $u^n - e = q \in \nabla(R)$. Now, we show that $u^n - e \in \nabla(eRe)$. Let v be an arbitrary element of $U(eRe)$ such that

$$v(u^n - e) = (u^n - e)v.$$

Since $u^n - e \in \nabla(R)$, we have $1 - (1 - e + v)(u^n - e) \in U(R)$, and so $1 - v(u^n - e) \in U(R)$ meaning there exists $c \in R$ such that

$$c[1 - v(u^n - e)] = 1 = [1 - v(u^n - e)]c. \tag{2.1}$$

We have $u, v \in eRe$ and so $u^n = u^n e$, $v = ev$. From (2.1), we obtain that

$$ec[1 - v(u^n - e)]e = e = e[1 - v(u^n - e)]ce,$$
$$ec[e - v(u^n e - e)] = e = [e - ev(u^n - e)]ce,$$
$$ec[e^2 - ev(u^n - e)] = e = [e^2 - v(u^n e - e^2)]ce,$$
$$ec[e^2 - ev(u^n - e)] = e = [e^2 - v(u^n - e)e]ce,$$
$$ece[e - v(u^n - e)] = e = [e - v(u^n - e)]ece.$$

It follows that $e - v(u^n - e) \in U(eRe)$, leading to the conclusion that $u^n - e \in \nabla(eRe)$. Hence, we deduce that $u^n \in e + \nabla(eRe)$, which implies that eRe is an n-∇U ring. □

Example 2.9. Let $R = \mathbb{Z}_6$, which is known to be a 2-∇U ring. Consider the idempotent $e = 4 \in R$, since $4^2 = 16 \equiv 4 \pmod 6$. Then the corner ring $eRe = \{ere\colon r \in R\} = \{0, 2, 4\}$ is also a 2-∇U ring.

Hence, this example illustrates Proposition 2.7: if R is an n-∇U ring and e is an idempotent, then eRe inherits the n-∇U property.

Example 2.10. Suppose $R = \mathrm{M}_2(\mathbb{Z}_2)$ and $n \equiv 0$ (mod 3). Then, we claim that R is an n-∇U ring.

In fact, since $U^6(R) = \mathrm{I}_2$, it must be that $A^{6k} = \mathrm{I}_2$ for every $A \in U(R)$ and $k \geq 1$. Thus, $(\mathrm{I}_2 - A^{3k})^2 = 0$. As $n = 3k$, we infer that R is an n-∇U ring.

Now consider $n = 6$ and the ring $T = \mathbb{Z}_{36} \times M_2(\mathbb{Z}_2)$. Then, T is a 6-∇U ring. Choose $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $e = (9, A) \in T$. One can check that $e^2 = e$, and so the corner ring

$$eTe = \{(0,0),\ (9,0),\ (18,0),\ (27,0),\ (0,A),\ (9,A),\ (18,A),\ (27,A)\}$$

is a 6-∇U ring.

We provide examples of matrix rings that are not n-∇U. In the general case, we do not yet have an answer.

Example 2.11. For any ring $R \neq 0$ and any integer $n \geq 3$, $\mathrm{M}_n(R)$ is *not* a 3-∇U ring.

P r o o f. Based on Proposition 2.7, it is sufficient to show that $\mathrm{M}_3(R)$ is not a 3-∇U ring. Assume, in a way of contradiction, that $\mathrm{M}_3(R)$ is a 3-∇U ring. Since $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \in \mathrm{GL}_3(R)$ and $\mathrm{M}_3(R)$ is a 3-∇U ring, we have $\begin{pmatrix} 5 & 4 & 3 \\ 3 & 1 & 1 \\ 4 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}^3 - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \nabla(\mathrm{M}_3(R))$. Let $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{M}_3(R)$. Then, $e^2 = e$ and so $R \cong e\mathrm{M}_3(R)e$ is a 3-∇U ring by Proposition 2.7. From Proposition 2.2, it infers that $2 \in \Delta(R)$. We have that $\Delta(R)$ is a subring of R, and so, $9 \in U(R)$, whence one calculates that

$$\begin{pmatrix} 5 & 4 & 3 \\ 3 & 1 & 1 \\ 4 & 3 & 1 \end{pmatrix}^{-1} = \frac{1}{9}\begin{pmatrix} -2 & 5 & 1 \\ 1 & -7 & 4 \\ 5 & 1 & -7 \end{pmatrix},$$

which is a contradiction. This completes the proof. □

In the above example, we cannot assume $n \geq 2$; indeed, this is because, according to Example 2.10, the ring $\mathrm{M}_2(\mathbb{Z}_2)$ is 3-∇U .

Proposition 2.8. *For any ring $R \neq 0$ and any integer $n \geq 2$, $\mathrm{M}_n(R)$ is not a k-∇U ring with $k = 5, 7, 11, 13$.*

P r o o f. Since it is well known that $\mathrm{M}_2(R)$ is isomorphic to a corner ring of $\mathrm{M}_n(R)$ for $n \geq 2$, it suffices to demonstrate that $\mathrm{M}_2(R)$ is not a k-∇U ring by Proposition 2.7.

Assume the contrary that $\mathrm{M}_2(R)$ is a 5-∇U ring. Since

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(R),$$

we extract

$$\begin{pmatrix}-1 & 1\\ 1 & 0\end{pmatrix}^5 - \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}-9 & 5\\ 5 & -4\end{pmatrix} \in \nabla(\mathrm{M}_2(R)).$$

Let $e = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \in \mathrm{M}_2(R))$. Then, $e^2 = e$ and so $R \cong e\mathrm{M}_2(R))e$ is a 5-∇U ring by Proposition 2.7. From Proposition 2.2, it infers that $2 \in \Delta(R)$, and so $11 \in U(R)$. Then, one verifies that

$$\begin{pmatrix}-9 & 5\\ 5 & -4\end{pmatrix}^{-1} = \frac{1}{11}\begin{pmatrix}-4 & -5\\ -5 & -9\end{pmatrix},$$

a contradiction. Similarly, we have $A^{11} - I = \begin{pmatrix}-145 & 89\\ 89 & -56\end{pmatrix}$ and $199 \in U(R)$, and so $(A^{11} - I)^{-1} = \frac{1}{199}\begin{pmatrix}-56 & -89\\ -89 & -145\end{pmatrix}$. It follows that $\mathrm{M}_n(R)$ is not a 11-∇U ring.

Let us consider the matrix:

$$B = \begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix} \in \mathrm{GL}_2(R).$$

Then, we have $C := B^7 - \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}$. One can check that

$$C^{-1} = \begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}.$$

It follows that $B^7 - I \notin \nabla(\mathrm{M}_2(R))$. Thus, $\mathrm{M}_2(R)$ is not 7-∇U .

Take $U = \begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}$. Then, $U \in U(R)$, and

$$\det(U^{13} - I) = 1202500001 \in U(R), \text{ and so } U^{13} - I \in U(\mathrm{M}_2(R)).$$

Thus, $\mathrm{M}_2(R)$ is not 13-∇U. This concludes the proof. □

2.5. Study of n-∇U rings and classical rings

In this subsection, we study certain classes of n-∇U rings together with some classical classes of rings such as semi-simple, Dedekind-finite, and regular rings.

Let us recall that a set $\{e_{ij} : 1 \le i, j \le n\}$ of non-zero elements in a ring R is called a system of n^2 *matrix units* if it satisfies the multiplication rule $e_{ij}e_{st} = \delta_{js}e_{it}$, where $\delta_{js} = 1$ if $j = s$ and $\delta_{js} = 0$ otherwise. In this case, the element $e := \sum_{i=1}^n e_{ii}$ is an idempotent in R, and the corner ring eRe is isomorphic to the full matrix ring $\mathrm{M}_n(S)$, where

$$S = \{r \in eRe \mid re_{ij} = e_{ij}r \text{ for all } 1 \le i, j \le n\}.$$

Furthermore, a ring R is said to be *Dedekind-finite* if, for any $a, b \in R$, the equality $ab = 1$ implies $ba = 1$. That is, every one-sided inverse in R is automatically two-sided.

We are now prepared to establish the following.

Proposition 2.9. *Every $(2k-1)$-∇U ring is Dedekind-finite, provided $k \in \{3, 4, 6, 7\}$.*

P r o o f. Suppose, to the contrary, that R is *not* a Dedekind-finite ring. Then there exist elements $a, b \in R$ such that $ab = 1$ but $ba \neq 1$. Define $e_{ij} := a^i(1 - ba)b^j$ and let $e := \sum_{i=1}^{n} e_{ii}$. Under this construction, there exists a non-zero ring S such that $eRe \cong \mathrm{M}_n(S)$. However, by Proposition 2.7, the ring eRe is $(2k-1)$-∇U, and hence $\mathrm{M}_n(S)$ must also be $(2k-1)$-∇U. This contradicts Proposition 2.8, as expected. □

Example 2.12. Let $R = \prod_{i=1}^{m} \mathbb{F}_{q_i}$ where each $(q_i - 1) \mid (2k-1)$ for a fixed $k \in \{3, 4, 6, 7\}$. By the direct product property, R is $(2k-1)$-∇U, and so R is Dedekind-finite.

Concrete choices:

- $k = 4$: $R = \mathbb{F}_8 \times \mathbb{F}_2$ is 7-∇U and Dedekind-finite;
- $k = 3$: $R = \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$ is 5-∇U and Dedekind-finite.

Proposition 2.10. *Let R be a ring and $n \in \{3, 4, 6, 7\}$. Then, the following two conditions are equivalent.*

(1) *$R/J(R)$ is a semi-simple $(2n-1)$-∇U ring.*

(2) *$R/J(R) \cong \prod_{i=1}^{m} \mathbb{F}_{p^{k_i}}$, where $(p^{k_i} - 1) | n$ and $\mathbb{F}_{p^{k_i}}$ is a field with p^{k_i} elements.*

P r o o f. (1) $\Longrightarrow$ (2). Since the quotient $R/J(R)$ is semi-simple, hence,

$$R/J(R) \cong \prod_{i=1}^{m} \mathrm{M}_{n_i}(D_i),$$

where each D_i is a division ring. By Proposition 2.8, it follows that $R/J(R) \cong \prod_{i=1}^{m} D_i$. Moreover, invoking Proposition 2.4, we deduce that $D_i \cong \mathbb{F}_{p^{k_i}}$, where $p^{k_i} - 1$ divides n, as claimed.

(2) $\Longrightarrow$ (1). By Proposition 2.4, each field $\mathbb{F}_{p^{k_i}}$ is $(2n-1)$-∇U. Then, applying Proposition 2.1, we conclude that the product $\prod_{i=1}^{m} \mathbb{F}_{p^{k_i}}$ is also $(2n-1)$-∇U. Therefore, $R/J(R)$ is a semi-simple $(2n-1)$-∇U ring, as asserted. □

Example 2.13. Take $n = 6$, so that $2n - 1 = 11$.

A correct semi-simple example is

$$R/J(R) \;\cong\; \mathbb{F}_7 \times \mathbb{F}_3 \times \mathbb{F}_2,$$

since

$$|\mathbb{F}_7| - 1 = 6, \quad |\mathbb{F}_3| - 1 = 2, \quad |\mathbb{F}_2| - 1 = 1,$$

and each of $6, 2, 1$ divides $n = 6$. Therefore $R/J(R)$ is $(2n-1)$-∇U (i. e., 11-∇U).

By contrast, $R/J(R) \cong \mathbb{F}_{11} \times \mathbb{F}_2$ is not $(2n-1)$-∇U, because $|\mathbb{F}_{11}| - 1 = 10 \nmid 6$.

A ring R is called *reduced* if it has no nonzero nilpotent elements; that is, whenever $a^n = 0$ for some $a \in R$ and integer $n \geq 1$, then $a = 0$.

Lemma 2.3. *Let R be a $(2n-1)$-∇U ring with $n \in \{3, 4, 6, 7\}$. If $J(R) = \{0\}$ and every non-zero right ideal of R contains a non-zero idempotent, then R is reduced.*

P r o o f. Suppose, to the contrary, that R is not reduced. Then there exists a non-zero element $a \in R$ such that $a^2 = 0$. By [7, Theorem 2.1], there exists an idempotent $e \in RaR$ such that $eRe \cong \mathrm{M}_2(T)$ for some non-trivial ring T. However, by Proposition 2.7, the ring eRe is $(2n-1)$-∇U, and hence, $\mathrm{M}_2(T)$ must also be $(2n-1)$-∇U. This contradicts Proposition 2.8, as expected. □

A ring R is called *regular* (or von Neumann regular) if, for every $a \in R$, there exists $x \in R$ such that $a = axa$. It is called *π-regular* if for each $a \in R$ there exists $n \geq 1$ and $x \in R$ such that $a^n = a^n x a^n$. The ring R is *strongly regular* if for every $a \in R$ there exists $x \in R$ with $a = a^2 x = x a^2$. Finally, R is *strongly π-regular* if for each $a \in R$ there exists $n \geq 1$ and $x \in R$ such that $a^n = a^{n+1} x = x a^{n+1}$.

Our second main statement is the following.

Theorem 2.2. *Let R be a ring and $n \in \{3, 4, 6, 7\}$. The following three items are equivalent.*

(1) *R is a regular $(2n-1)$-∇U ring.*

(2) *R is a π-regular reduced $(2n-1)$-∇U ring.*

(3) *For each $x \in R$, $x^{2n} = x$.*

P r o o f. (1) $\Rightarrow$ (2). Since R is regular, we have $J(R) = \{0\}$, and thus every non-zero right ideal contains a non-zero idempotent. By Lemma 2.3, it follows that R is reduced. Moreover, every regular ring is known to be π-regular, so the implication follows immediately, as claimed.

(2) $\Rightarrow$ (3). Notice that reduced rings are always abelian, so R is abelian regular (i. e., R is strongly regular). Assume that $y \in \nabla(R)$. Then, we have $y = eu$ for some $u \in U(R)$ and $e^2 = e \in Z(R)$. One can check that $uy = yu$. Then, we have $e = yu^{-1}$ and so $1 - e = 1 - yu^{-1}$. Note that $yu^{-1} = u^{-1}y$ and $y \in \nabla(R)$. From this, it immediately infers that $1 - e = 1 - yu^{-1} \in$ $\in U(R)$. It follows that $e = 0$. Thus, $y = 0$, and consequently, $\nabla(R) = 0$. This implies that $\mathrm{Nil}(R) = J(R) = 0$. Then, for each $x \in R$, we write $x = eu$ for some idempotent $e \in R$ and a unit $u \in R$, and so $x = ex$. Moreover, since R is a $(2n-1)$-∇U ring, $u^{2n-1} = 1$. It follows now that

$$x^{2n-1} = (eu)^{2n-1} = u^{2n-1} e^{2n-1} = e.$$

From this, it immediately infers that $x^{2n} = x^{2n-1}.x = ex = x$.

(3) $\Rightarrow$ (1). It is trivial that R is regular. Let $u \in U(R)$. Then, we have $u^{2n} = u$ forcing that $u^{2n-1} = 1$ and thus, R is a $(2n-1)$-∇U ring, as promised. □

A ring R is called *unit-regular* if for every $a \in R$ there exists a unit $u \in U(R)$ such that $a = aua$.

We now can record the following interesting consequence.

Corollary 2.4. *The following four conditions are equivalent for a ring R and $n \in \{3, 4, 6, 7\}$.*

(1) *R is a regular $(2n-1)$-∇U ring.*

(2) *R is a strongly regular $(2n-1)$-∇U ring.*

(3) *R is a unit-regular $(2n-1)$-∇U ring.*

(4) *R has the identity $x^{2n} = x$.*

P r o o f. (1) $\Rightarrow$ (2). By Lemma 2.3, the ring R is reduced and therefore abelian. It follows that R is strongly regular.

(2) $\Rightarrow$ (3). This is pretty obvious, so we leave out the argumentation.

(3) $\Rightarrow$ (4). Let $x \in R$. Then, $x = ue$ for some $u \in U(R)$ and $e \in \mathrm{Id}(R)$. We know that every unit-regular ring is, by definition, regular, so R is regular $(2n-1)$-∇U, whence R is abelian. It follows that $\nabla(R) = \{0\}$. Therefore, for any $u \in U(R)$, we have $u^{2n-1} = 1$, which means that $x^{2n-1} = u^{2n-1} e^{2n-1} = e$. So, we detect that $x^{2n} = x$, as required.

(4) $\Rightarrow$ (1). It is clear by a direct appeal to Theorem 2.2. □

Example 2.14. Let $n \in \{3,4,6,7\}$ and take

$$R \cong \mathbb{F}_7 \times \mathbb{F}_4.$$

Then R is (von Neumann) regular and reduced, since finite products of fields have both properties. Moreover, $|\mathbb{F}_7^\times| = 6$ and $|\mathbb{F}_4^\times| = 3$, so on each factor every unit u satisfies $u^n = 1$ (because $6 \mid n$ and $3 \mid n$). Thus, R is $(2n-1)$-∇U. Finally, for each $x = (a,b) \in R$, we have

$$x^{2n-1} = (a^{2n-1},\, b^{2n-1}) = \begin{cases} (0,0), & \text{if } a = 0 \text{ or } b = 0, \\ (1,1), & \text{if } a, b \in \mathbb{F}^\times, \end{cases}$$

so x^{2n-1} is an idempotent (component-wise either 0 or 1), i. e., $x^{2n-1} \in \mathrm{Id}(R)$.

We now come to the next two pivotal assertions.

Theorem 2.3. *Let R be a $(2n-1)$-∇U ring and $n \in \{3,4,6,7\}$. Then, the following conditions are equivalent.*

(1) *R is an exchange ring.*

(2) *R is a clean ring.*

P r o o f. (2) $\Rightarrow$ (1). The claim follows immediately, as every clean ring is known to be exchange.

(1) $\Rightarrow$ (2). If R is both exchange and $(2n-1)$-∇U, then by Lemma 2.3, R is reduced, and hence, abelian. Therefore, R is an abelian exchange ring, which implies that R is clean. □

2.6. Some extensions of n-∇U rings

Finally, we examine how the n-∇U property behaves under ring extensions.

As usual, we say that B is a unital subring of a ring A if $\emptyset \neq B \subseteq A$ and, for any $x, y \in B$, the relations $x - y$, $xy \in B$, and $1_A \in B$ hold. Let A be a ring and let B a unital subring of A, we denote by $R[A,B]$ the set

$$\{(a_1, \ldots, a_n, b, b, \ldots) \colon a_i \in A,\ b \in B,\ 1 \leq i \leq n\}.$$

Then, a routine check establishes that $R[A,B]$ forms a ring under the usual component-wise addition and multiplication. The ring $R[A,B]$ is called the *tail ring extension*.

We start our considerations here with the following helpful statement.

Proposition 2.11. *$R[A,B]$ is an n-∇U ring if, and only if, both A and B are n-∇U rings.*

P r o o f. Suppose $R[A,B]$ is an n-∇U rings. Firstly, we prove that A is an n-∇U ring. Let $u \in U(A)$. Then, $\bar{u} = (u, 1, 1, \ldots) \in U(R[A,B])$. By hypothesis, we have $(u^n - 1, 0, 0, \ldots) \in \nabla(R[A,B])$. Let $v \in U(A)$ with $uv = vu$, then $(u^n - 1, 0, 0, \ldots)(v, 1, 1, \ldots) = (v, 1, 1, \ldots)(u^n - 1, 0, 0, \ldots)$ and so,

$$((1 - (u^n - 1)v, 1, 1, \ldots) = 1 - (u^n - 1, 0, 0, \ldots)(v, 1, 1, \ldots) \in U(R[A,B]).$$

Hence, $1 - (u^n - 1)v \in U(A)$, which insures that $u^n - 1 \in \nabla(A)$. Now, we show that B is an n-∇U ring. To this target, choose $v \in U(B)$. Then, $(1, \ldots, 1, 1, v, v, \ldots) \in U(R[A,B])$. By hypothesis, $\bar{v} = (0, \ldots, 0, v^n - 1, v^n - 1, \ldots) \in \nabla(R[A,B])$. Thus, for all $u \in B$, with $uv = vu$ we have

$$\bar{v}(0, \ldots, 0, u, u, \ldots) = (0, \ldots, 0, u, u, \ldots)\bar{v}.$$

It follows that $1-(0,\ldots,0,v^n-1,v^n-1,\ldots)(0,\ldots,0,u,u,\ldots)\in U(R[A,B])$, and so $1-(v^n-1)u\in U(B)$ and hence, $v^n-1\in\nabla(B)$, as required. The case of A is treated absolutely analogously, so we remove the arguments.

Conversely, assume that A and B are both n-∇U rings. Let $\bar{u}=(u_1,\ldots,u_k,v,v,\ldots)\in$ $\in U(R[A,B])$, where $u_i\in U(A)$, and $v\in U(B)\subseteq U(A)$. We must show that $\bar{u}^n-1\in$ $\in\nabla(R[A,B])$. In fact, for all $\bar{a}=(a_1,\ldots,a_m,b,b,\ldots)\in U(R[D,C])$ with $a_i\in U(D)$ and $b\in U(C)\subseteq U(D)$ such that

$$(\bar{u}^n-1)(a_1,\ldots,a_m,b,b,\ldots)=(a_1,\ldots,a_m,b,b,\ldots)(\bar{u}^n-1).$$

If $k\le m$, then we have $(u_i^2-1)a_i=a_i(u_i^2-1)$ for all $1\le i\le k$, $(v^n-1)a_{n+j}=a_{n+j}(v^n-1)$ for all $1\le j\le m-n$, and $(v^n-1)b=b(v^n-1)$. We have

$$1-(\bar{u}^2-1)\bar{a}=(1-(1-u_1^n)a_1,\ldots,1-(1-u_k^n)a_n,1-(v^2-1)a_{n+1},\ldots,$$
$$\ldots,1-(v^2-1)a_m,1-(v^2-1)b,\ldots).$$

Note that $1-(u_i^n-1)a_i\in U(D)$ for all $1\le i\le n$, $1-(v^n-1)a_{n+j}\in U(D)$ for all $1\le j\le m-n$, and $1-(v^n-1)b\in U(C)\subseteq U(D)$. It follows that $1-(\bar{u}^n-1)\bar{a}\in U(R[D,C])$. The case $n>m$ is similar.

We deduce that $\bar{u}^2-1\in\nabla(R[D,C])$ or $\bar{u}^2=n\in 1+\nabla(R[D,C])$. This shows that $R[D,C]$ is a n-∇U ring. □

Let R be a ring and M be a bi-module over R. The *trivial extension* of R and M is stated as

$$T(R,M)=\{(r,m)\colon r\in R\ \text{ and }\ m\in M\},$$

with addition defined componentwise and multiplication defined by

$$(r,m)(s,n)=(rs,rn+ms).$$

The trivial extension $T(R,M)$ is isomorphic to the subring

$$\left\{\begin{pmatrix} r & m\\ 0 & r\end{pmatrix}\colon r\in R\ \text{ and }\ m\in M\right\}$$

of the formal 2×2 matrix ring $\begin{pmatrix} R & M\\ 0 & R\end{pmatrix}$ (see [4]), and also $T(R,R)\cong R[x]/\langle x^2\rangle$. We, likewise, note that the set of units of the trivial extension $T(R,M)$ is

$$U(T(R,M))=T(U(R),M).$$

Assume that

$$\begin{pmatrix} r & m\\ 0 & r\end{pmatrix}\in T(\nabla(R),M)\ \text{ with }\ r\in\nabla(R)$$

and $\begin{pmatrix} r_1 & m_1\\ 0 & r_1\end{pmatrix}\in U(T(R,M))$ with $\begin{pmatrix} r & m\\ 0 & r\end{pmatrix}\begin{pmatrix} r_1 & m_1\\ 0 & r_1\end{pmatrix}=\begin{pmatrix} r_1 & m_1\\ 0 & r_1\end{pmatrix}\begin{pmatrix} r & m\\ 0 & r\end{pmatrix}$. Then, $r_1\in U(R)$ and $rr_1=r_1r$, and so $1-rr_1\in U(R)$. It follows that

$$1-\begin{pmatrix} r & m\\ 0 & r\end{pmatrix}\begin{pmatrix} r_1 & m_1\\ 0 & r_1\end{pmatrix}=\begin{pmatrix} 1-rr_1 & *\\ 0 & 1-rr_1\end{pmatrix}\in T(U(R),M)=U(T(R,M)).$$

Thus, $\begin{pmatrix} r & m\\ 0 & r\end{pmatrix}\in\nabla(T(R,M))$. This is shown that $T(\nabla(R),M)\subseteq\nabla(T(R,M))$.

Proposition 2.12. *Assume that $cm = mc$ for any $c \in R$ and $m \in M$. Then R is an n-∇U ring if and only if $T(R, M)$ is an n-∇U ring.*

P r o o f. Let $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \in U(T(R, M))$. Then, $r \in U(R)$. Since R is an ∇U ring, $r^n \in 1+\nabla(R)$. Then

$$\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}^n - 1 = \begin{pmatrix} r^n - 1 & * \\ 0 & r^n - 1 \end{pmatrix} \in \nabla(T(R, M)).$$

We deduce that $T(R, M)$ is a n-∇U ring.

Now, assume that $T(R, M)$ is an n-∇U ring. We show that R is an n-∇U ring. Call $r \in U(R)$. Then $\begin{pmatrix} r^n & 0 \\ 0 & r^n \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}^n \in 1 + \nabla(T(R, M))$. It follows that there is $\begin{pmatrix} q & m \\ 0 & q \end{pmatrix} \in$ $\in \nabla(T(R, M)$ such that

$$\begin{pmatrix} r^n & 0 \\ 0 & r^n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} q & m \\ 0 & q \end{pmatrix}.$$

Then, $r^n = 1 + q$. Next, we show that $q \in \nabla(R)$. Indeed, let $u \in R$ with $qu = uq$. Then, we have

$$\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} q & m \\ 0 & q \end{pmatrix} = \begin{pmatrix} uq & um \\ 0 & uq \end{pmatrix} = \begin{pmatrix} qu & mu \\ 0 & qu \end{pmatrix} = \begin{pmatrix} q & m \\ 0 & q \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}.$$

It follows that $1 - \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} q & m \\ 0 & q \end{pmatrix} \in U(T(R, M)) = T(U(R), M)$. It means that $1 - uq \in U(R)$. We deduce that $q \in \nabla(R)$. □

Corollary 2.5. *Assume that R is a commutative ring. Then R is an n-∇U ring if and only if $T(R, R)$ is an n-∇U ring.*

Conclusion

We have introduced and investigated the class of n-∇U rings and their variants. Our study established key structural properties, clarified their behavior under standard ring-theoretic constructions, and connected them with classical notions such as regularity, cleanness, exchange rings, and Dedekind-finiteness. Illustrative examples demonstrated both positive and negative cases, highlighting the richness of the theory. Overall, these results extend the understanding of unit powers in relation to radical-like subsets and provide a foundation for further exploration of algebraic structures within ring theory.

Acknowledgement

The author would like to thank the referees for the very helpful comments.

REFERENCES

1. Karabacak F., Koşan M. T., Quynh Truong Cong, Tai Dinh Duc. A generalization of UJ-rings, *Journal of Algebra and Its Applications*, 2021, vol. 20, no. 12, 2150217. https://doi.org/10.1142/S0219498821502170
2. Koşan M. T., Leroy A., Matczuk J. On UJ-rings, *Communications in Algebra*, 2018, vol. 46, issue 5, pp. 2297–2303. https://doi.org/10.1080/00927872.2017.1388814
3. Koşan M. Tamer, Quynh Truong Cong, Yıldırım Tülay, Žemlička Jan. Rings such that, for each unit u, $u - u^n$ belongs to the Jacobson radical, *Hacettepe Journal of Mathematics and Statistics*, 2020, vol. 49, no. 4, pp. 1397–1404. https://doi.org/10.15672/hujms.542574
4. Koşan M. Tamer, Quynh Truong Cong, Žemlička Jan. UNJ-Rings, *Journal of Algebra and Its Applications*, 2020, vol. 19, no. 09, 2050170. https://doi.org/10.1142/S0219498820501704

5. Lam Tsit-Yuen. *A first course in noncommutative rings*, New York: Springer, 2001. https://doi.org/10.1007/978-1-4419-8616-0
6. Lam Tsit-Yuen. *Exercises in classical ring theory*, New York: Springer, 2003. https://doi.org/10.1007/b97448
7. Levitzki J. On the structure of algebraic algebras and related rings, *Transactions of the American Mathematical Society*, 1953, vol. 74, no. 3, pp. 384–409. https://doi.org/10.2307/1990809
8. Nicholson W. K. Lifting idempotents and exchange rings, *Transactions of the American Mathematical Society*, 1977, vol. 229, pp. 269–278. https://doi.org/10.2307/1998510
9. Tien Nguyen Quoc, Koşan M. Tamer, Quynh Truong Cong. Remarks on rings with $u-1$ quasi-nilpotent for each unit u, *Journal of Algebra and Its Applications*, 2026, vol. 25, no. 02, 2550317. https://doi.org/10.1142/S0219498825503177
10. Tin Phan Hong. On 2-UNJ rings, *Asian-European Journal of Mathematics*, 2025 (Online Ready). https://doi.org/10.1142/S1793557125501050
11. Tin Phan Hong, Tien Nguyen Quoc. A note on strongly clean elements, *Asian-European Journal of Mathematics*, 2025, vol. 18, no. 11, 2550047. https://doi.org/10.1142/S1793557125500470
12. Trang Dao Thi. On Morita equivalent of indecomposable automorphism-invariant rings, *Asian-European Journal of Mathematics*, 2023, vol. 16, no. 02, 2350014. https://doi.org/10.1142/S1793557123500146

Received 09.10.2025
Accepted 25.01.2026

Dao Thi Trang, Faculty of Applied Sciences, Ho Chi Minh City University of Industry and Trade, Ho Chi Minh City, Vietnam.
ORCID: https://orcid.org/0000-0002-5733-1594
E-mail: trangdt@huit.edu.vn

Citation: D. T. Trang. Rings with ∇-nilpotent unit deviations, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, 2026, vol. 36, issue 1, pp. 82–98.

МАТЕМАТИКА 2026. Т. 36. Вып. 1. С. 82–98.

Д. Т. Транг

Кольца с ∇-нильпотентными отклонениями обратимых элементов

Ключевые слова: n-∇U кольцо, ∇-нильпотентный элемент, радикал Джекобсона, конечное по Дедекинду кольцо, чистое кольцо, обменное кольцо, регулярное кольцо, π-регулярное кольцо.

УДК 512.552, 512.54

DOI: 10.35634/vm260105

Мы вводим и исследуем новый класс ассоциативных колец, называемых n-∇U кольцами, характеризующихся условием, что для каждого обратимого элемента u в кольце элемент $u^n - 1$ принадлежит выделенному подмножеству $\nabla(R)$ ∇-нильпотентных элементов. Это подмножество состоит из элементов $x \in R$ таких, что $1 - ux$ обратим для всех обратимых элементов u, коммутирующих с x. Мы исследуем структурные свойства n-∇U и π-∇U колец, приводим иллюстративные примеры и изучаем их поведение относительно различных теоретико-кольцевых конструкций, включая прямые произведения, факторкольца и тривиальные расширения. Наши результаты устанавливают связи между условием n-∇U и классическими понятиями такими, как регулярность, чистота и конечность по Дедекинду, предлагая новые идеи о взаимодействии между степенями обратимых элементов и нильпотентностью в теории колец.

СПИСОК ЛИТЕРАТУРЫ

1. Karabacak F., Koşan M. T., Quynh Truong Cong, Tai Dinh Duc. A generalization of UJ-rings // Journal of Algebra and Its Applications. 2021. Vol. 20. No. 12. 2150217. https://doi.org/10.1142/S0219498821502170
2. Koşan M. T., Leroy A., Matczuk J. On UJ-rings // Communications in Algebra. 2018. Vol. 46. Issue 5. P. 2297–2303. https://doi.org/10.1080/00927872.2017.1388814
3. Koşan M. Tamer, Quynh Truong Cong, Yıldırım Tülay, Žemlička Jan. Rings such that, for each unit u, $u - u^n$ belongs to the Jacobson radical // Hacettepe Journal of Mathematics and Statistics. 2020. Vol. 49. No. 4. P. 1397–1404. https://doi.org/10.15672/hujms.542574
4. Koşan M. Tamer, Quynh Truong Cong, Žemlička Jan. UNJ-Rings // Journal of Algebra and Its Applications. 2020. Vol. 19. No. 09. 2050170. https://doi.org/10.1142/S0219498820501704
5. Lam Tsit-Yuen. A first course in noncommutative rings. New York: Springer, 2001. https://doi.org/10.1007/978-1-4419-8616-0
6. Lam Tsit-Yuen. Exercises in classical ring theory. New York: Springer, 2003. https://doi.org/10.1007/b97448
7. Levitzki J. On the structure of algebraic algebras and related rings // Transactions of the American Mathematical Society. 1953. Vol. 74. No. 3. P. 384–409. https://doi.org/10.2307/1990809
8. Nicholson W. K. Lifting idempotents and exchange rings // Transactions of the American Mathematical Society. 1977. Vol. 229. P. 269–278. https://doi.org/10.2307/1998510
9. Tien Nguyen Quoc, Koşan M. Tamer, Quynh Truong Cong. Remarks on rings with $u - 1$ quasinilpotent for each unit u // Journal of Algebra and Its Applications. 2026. Vol. 25. No. 02. 2550317. https://doi.org/10.1142/S0219498825503177
10. Tin Phan Hong. On 2-UNJ rings // Asian-European Journal of Mathematics. 2025 (Online Ready). https://doi.org/10.1142/S1793557125501050
11. Tin Phan Hong, Tien Nguyen Quoc. A note on strongly clean elements // Asian-European Journal of Mathematics. 2025. Vol. 18. No. 11. 2550047. https://doi.org/10.1142/S1793557125500470
12. Trang Dao Thi. On Morita equivalent of indecomposable automorphism-invariant rings // Asian-European Journal of Mathematics. 2023. Vol. 16. No. 02. 2350014. https://doi.org/10.1142/S1793557123500146

Поступила в редакцию 09.10.2025
Принята к публикации 25.01.2026

Транг Дао Тхи, факультет прикладных наук, Хошиминский университет промышленности и торговли, Вьетнам, г. Хошимин.
ORCID: https://orcid.org/0000-0002-5733-1594
E-mail: trangdt@huit.edu.vn

Цитирование: Д. Т. Транг. Кольца с ∇-нильпотентными отклонениями обратимых элементов // Вестник Удмуртского университета. Математика. Механика. Компьютерные науки. 2026. Т. 36. Вып. 1. С. 82–98.