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MATHEMATICS 2026. Vol. 36. Issue 2. Pp. 205–233.

MSC2020: 35R11, 35R30, 45D05, 65L10

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SOLVABILITY OF AN INVERSE PROBLEM FOR A TIME-FRACTIONAL INTEGRO-DIFFERENTIAL DIFFUSION EQUATION WITH INITIAL-BOUNDARY AND INTEGRAL OVERDETERMINATION CONDITIONS

In the article, we investigate the inverse problem for the integro-differential time-fractional diffusion equation with initial-boundary and integral overdetermination conditions on a rectangular domain. Initially, we give a definition to the classical solution for the original problem. Subsequently, the direct problem is transformed into an equivalent integral equation using the Fourier method. We prove the existence and uniqueness of the solution to this equivalent problem by applying estimates of the Mittag-Leffler function and the successive approximation method. In the second part, we address the inverse problem, which is reduced to an equivalent auxiliary problem. We utilize the contraction mapping principle for proving local existence and uniqueness of the inverse problem solution. Additionally, we provide a practical overview of numerical solutions to the original problem using the finite difference method. The results of a numerical experiment on finer grids and at smaller time steps are demonstrated. The developed algorithm allows one to simultaneously determine the time-dependent kernel in the integral term and the solution to the problem. In conclusion, a test example is given to illustrate the effectiveness of the proposed numerical algorithm.

Keywords: time-fractional diffusion equation, inverse problem, integral equation, existence, uniqueness, forward difference method.

DOI: 10.35634/vm260202

Introduction

Nowadays, the fractional integro-differential equations have been of great interest because of the many applications. These are often used to model phenomena in heat mass transfer [1], medicine [2], viscoelastic materials [3], porous media [4–6], mathematical biology and mathematical finance and economics — see [7].

The inverse coefficient problems for fractional differential equations have been extensively investigated in the past few decades (see [8–10]). For the diffusion equation, Fujishiro and Kian [9] studied the inverse problems of determining the time-dependent source term or potential coefficient in time-fractional diffusion equation with a single-term classical Caputo derivative. In the article [10], the unique solvability of an inverse coefficient problem of determining the coefficient in the lower-order term of a fractional diffusion equation was studied. The theorems of existence and uniqueness of inverse problem solution were obtained. Furthermore, they proposed a numerical algorithm based on a finite-difference scheme to accurately compute the inverse problem of simultaneously determining a time-dependent coefficient in a fractional diffusion equation, together with its solution.

Identification of missing memory kernels in evolutionary integro-differential fractional equations is relatively new in inverse problems. We are aware of only a few papers dealing with this topics, namely [11–14]. In [12], an inverse problem of determining the kernel of a fractional diffusion equation is considered. The backward problem is formulated as an initial-boundary value problem for this equation, subject to nonlocal initial and homogeneous Dirichlet boundary conditions. To identify the kernel, an additional integral-type overdetermination condition is imposed on the solution of the backward problem. In [14], an inverse problem for a fractional integro-differential equation is studied, aiming to simultaneously determine two time-dependent coefficients: a kernel function and a source function, based on an additional integral overdetermination condition. By applying a fixed point theorem in an appropriate Sobolev space, global existence and uniqueness results for the solution of the inverse problem are established. There is no description of constructive algorithms how to find a solution. Furthermore, we applied a method for solving the nonlinear inverse problem, which was introduced and studied in [15–18].

In this paper, we focus on an inverse time-dependent problem for an integro-differential diffusion equation with integral data over the domain. We first investigate the well-posedness of the direct problem using Fourier's method. Then, we apply the Banach fixed-point theorem to establish the existence and uniqueness of the inverse problem. One of the main contributions of the paper is a practical overview of numerical solutions to the original problem obtained by the finite difference method. The results of computations on finer meshes and smaller time steps are presented. This algorithm enables the simultaneous determination of the time-dependent kernel in the integral term, together with its solution. Finally, we provide a test example that demonstrates the effectiveness of the developed numerical algorithm.

The remainder of the paper is organized as follows. Section 1 presents the formulation of the problem. In Section 2, we introduce some notations and discuss the well-posedness of the direct problem. Section 3 is devoted to the classical solution of the direct problem. In Section 4, we formulate an auxiliary inverse problem and investigate its solvability. Section 5 focuses on the numerical solution of the inverse problem using the finite difference method combined with the rectangular rule for evaluating integrals. Section 6 presents computational results that validate the analysis and demonstrate the efficiency of the proposed algorithm. Finally, some comments and conclusions complete the paper.

§ 1. Formulation of problem

In this paper, we consider the following initial-boundary value problem for the time-fractional integro-differential diffusion equation

$$\partial_t^\alpha u - u_{xx} = \int_0^t k(\tau)u(x,t-\tau)\,d\tau + f(x,t), \quad (x,t)\in D_T, \tag{1.1}$$

$$u(x,0)=\varphi(x), \quad x\in[0,l], \tag{1.2}$$

$$u(0,t)=0, \quad u(l,t)=0, \quad \varphi(0)=\varphi(l)=0, \quad t\in[0,T], \tag{1.3}$$

where $D_T := \{(x,t)\colon 0<x<l,\ 0<t\le T\}$ and T is a fixed positive constant, ∂_t^α is the Caputo fractional derivative of order $0<\alpha\le 1$ with respect to time variable (see Definition 2.1), $k(t)$ is the kernel, $\varphi(x)$ is the initial temperature, $f(x,t)$ is a given smooth function.

The problem of determining a function $u(x,t)$, $(x,t)\in D_T$, that satisfies (1.1)–(1.3) with known functions $k(t)$, $f(x,t)$ and $\varphi(x)$ is referred to as the direct problem.

In the inverse problem, the goal is to determine the kernel $k(t)$, $t\in[0,T]$, in (1.1) using overdetermination condition based on solution of the direct problem (1.1)–(1.3):

$$\int_0^l \omega(x)u(x,t)\,dx = h(t), \quad t\in[0,T], \tag{1.4}$$

where $\omega(x)$, $h(t)$ are given functions. Besides, $h(t)$ represents measurement data corresponding the average temperature over a small part of $(0,l)$, since the weight function $\omega(x)$ is usually chosen to satisfy $\operatorname{supp}(\omega)\subset(0,l)$ in applied sciences [14].

The integral term on the right-hand side of equation (1.1) represents memory effects, which play a significant role in various physical processes, including those occurring in viscoelastic media (see [19]) and in the theory of reactive contaminant transport [20, Chapter 15, pp. 517–535].

Moreover, Gafiychuk et al. [21] emphasized that the inclusion of a nonlinear source term, which, in particular cases may be linear, as in equation (1.1), enables the mathematical description of a wide variety of complex phenomena such as oscillations, wave propagation of different types, excitability, diversity of stationary and spatio-temporal dissipative patterns, and bistability. This formulation offers a high degree of flexibility for modeling the diversity of self-organization phenomena and for revealing novel nonlinear effects that depend on the order of the time–space fractional derivative. The solvability of forward fractional diffusion equations have been studied, e. g., in [22–24].

In what follows, we define the functional spaces that will be used throughout the paper. $C[0,T]$ $(0<T<\infty)$ is the space of continuous functions f on $[0,T]$ with the norm

$$\|f\| = \max_{t\in[0,T]} |f(t)|.$$

We denote by $C^m[0,T]$, $m\in\mathbb{N}_0=\{0,1,\dots\}$, a space of functions f which are m times continuously differentiable on $[0,T]$ with the norm

$$\|f\|_{C^m} = \sum_{k=0}^{m} \|f^{(k)}\| = \sum_{k=0}^{m} \max_{t\in[0,T]} |f^k(t)|, \quad m\in\mathbb{N}_0.$$

We introduce the weighted space $C_\gamma[0,T]$, $\gamma\in(0,1)$, of functions f given on $(0,T]$ such that the function $t^\gamma f(t)\in C[0,T]$, with the norm

$$\|f\|_\gamma = \|t^\gamma f(t)\|.$$

For $n\in\mathbb{N}$, we denote by $C^n_\gamma[0,T]$ the Banach space of functions $f(t)$ which are continuously differentiable on $[0,T]$ up to order $n-1$ and have the derivative $f^{(n)}(t)$ of order n on $(0,T]$ such that $f^{(n)}(t)\in C_\gamma[0,T]$:

$$C^n_\gamma[0,T] = \{f\colon \|f\|_{C^n_\gamma[0,T]} = \sum_{k=0}^{n-1} \|f^{(k)}\| + \|f^{(n)}\|_\gamma\}, \qquad C^0_\gamma[0,T]=C_\gamma[0,T].$$

Let $C^{m,\alpha}(D_T)$ be the class of functions that are m times continuously differentiable with respect to x in D_T, for which a continuous derivative ∂_t^α exists.

Definition 1.1. A function $u(x,t)$ is called a *classical solution* of the initial-boundary value problem (1.1)–(1.3) if the following conditions are satisfied:

(1) $u(x,t)\in C(\overline{D_T})$;

(2) for each $t\in(0,T]$, the function $u(\cdot,t)\in C^2(0,l)$;

(3) for each $x\in(0,l)$, the fractional derivative $\partial_t^\alpha u(x,t)\in C(0,T]$;

(4) the equation (1.1) is satisfied pointwise for all $(x,t)\in D_T$, and the conditions (1.2), (1.3) are satisfied in the classical sense.

We define the Banach space X_T by

$$X_T := \left\{ u\in C(\overline{D_T})\cap C^{2,\alpha}(D_T)\colon \frac{\partial}{\partial t}(\partial_t^\alpha u),\ u_{xxt},\ u_t\in C_\gamma(\overline{D_T}),\ \gamma=1-\alpha \right\}.$$

We now provide the definition of a classical solution to (1.1)–(1.4).

Definition 1.2. A solution to the inverse problem (1.1)–(1.4) is a pair of functions

$$(u(x,t),k(t)) \in X_T \times C_\gamma[0,T], \quad \gamma = 1-\alpha,$$

that satisfies equation (1.1) and conditions (1.2)–(1.4).

Throughout this article, the functions φ, f, ω and h are assumed to satisfy the following conditions:

$(A1)$ $\varphi(x) \in C^4[0,l]$; $\varphi^{(5)}(x) \in L_2(0,l)$; $\varphi(0) = \varphi(l) = 0$; $\varphi''(0) = \varphi''(l) = 0$; $\varphi^{(4)}(0) = \varphi^{(4)}(l) = 0$;

$(A2)$ $f(x,t) \in C^{2,1}(\overline{D}_T)$; $f^{(3)}_{xxx}(.,t) \in L_2(0,l)$; $f(0,t) = f(l,t) = 0$; $f''_{xx}(0,t) = f''_{xx}(l,t) = 0$;

$(A3)$ $h(t) \in C^2[0,T]$;

$(A4)$ $\omega(x) \in C^2[0,l]$; $h(0) = \int_0^l \omega(x)\varphi(x)\,dx \neq 0$.

We will introduce the definitions of fractional derivatives and notations used in the next section.

§ 2. Preliminaries

Definition 2.1 (see [25, pp. 90–94]). The Caputo time-fractional derivative of order $0<\alpha<1$ of the one times continuously differentiable function u is defined by

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial u(x,\tau)}{\partial \tau}\,d\tau, \quad \partial_t^1 u(x,t) = \frac{\partial u(x,t)}{\partial t},$$

where $\Gamma(\cdot)$ is the Euler's Gamma function.

Definition 2.2 (see [25, pp. 69–79]). The Riemann–Liouville time-fractional derivative of order $0<\alpha<1$ of the integrable function u is defined by

$$D^\alpha_{0+,t} u(x,t) = \frac{\partial}{\partial t} I^{1-\alpha}_{0+,t} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-\tau)^{-\alpha} u(x,\tau)\,d\tau.$$

Two parameter Mittag-Leffler function [25, pp. 40–42]. The two parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by the following series:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)},$$

where $\alpha,\beta,z \in \mathbb{C}$ with $\Re(\alpha)>0$, $\Re(\alpha)$ denotes the real part of the complex number α.

Proposition 2.1 (see [26, p. 35]). *Let* $0<\alpha<2$ *and* $\beta \in \mathbb{R}$ *be arbitrary. We suppose that* κ *is such that* $\pi\alpha/2 < \kappa < \min\{\pi, \pi\alpha\}$. *Then, there exists a constant* $C = C(\alpha,\beta,\kappa) > 0$ *such that*

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|}, \quad \kappa \leq |\arg(z)| \leq \pi.$$

Note that in [27] the following estimate for the Mittag-Leffler function is proved, when $0<\alpha<1$ (not true for $\alpha \geq 1$)

$$\frac{1}{1+\Gamma(1-\alpha)z} \leq E_{\alpha,1}(-z) \leq \frac{1}{1+\Gamma(1+\alpha)^{-1}z}, \quad z>0.$$

Thus, it follows that

$$0 < E_{\alpha,1}(-z) < 1, \quad z>0.$$

Proposition 2.2. *The generalized Mittag-Leffler function* $E_{\alpha,\beta}(-t)$ *with* $t \geq 0$ *is completely monotonic if and only if* $0 < \alpha \leq 1$ *and* $\beta \geq \alpha$. *Furthermore, if* $0 < \alpha \leq 1$, $\beta \geq \alpha$, *and* $t \geq 0$, *then, for all* $n \in \mathbb{N}$,

$$(-1)^n \frac{d^n}{dt^n} E_{\alpha,\beta}(-t) \geq 0.$$

Proof. The proof for this result can be found in the references [28–31]. □

Lemma 2.1. *For* $0 < \alpha \leq 1$ *and* $\beta \geq \alpha$, *the Mittag-Leffler type function* $E_{\alpha,\beta}(-\lambda_n t^\alpha)$ *satisfies*

$$0 \leq E_{\alpha,\beta}(-\lambda_n t^\alpha) \leq E_{\alpha,\beta}(-\inf_{n\in\mathbb{N}} \lambda_n t^\alpha) \leq \frac{1}{\Gamma(\beta)}, \quad t \geq 0,$$

where $\lambda_n > 0$, $\lambda_n \to +\infty$, $n \to \infty$.

Proof. In view of Proposition 2.2, we have

$$E_{\alpha,\beta}(-z) \geq 0, \quad \forall z \geq 0,$$

and

$$\frac{d}{dz} E_{\alpha,\beta}(-z) \leq 0, \quad \forall z \geq 0.$$

These estimates mean that the function $E_{\alpha,\beta}(-z)$ is positive and non-increasing on the interval $[0,\infty)$, i. e., it satisfies the following inequality for all x, y such that $x \geq y \geq 0$

$$0 \leq E_{\alpha,\beta}(-x) \leq E_{\alpha,\beta}(-y) \leq E_{\alpha,\beta}(0) = \frac{1}{\Gamma(\beta)}.$$

These yield the desired results and complete the proof (see [32]). □

Theorem 2.1 (see [25, pp. 230–231]). *The solution* $y(t) \in AC[0,T]$ *of the linear nonhomogeneous fractional problem*

$$\begin{gathered} \partial_t^\alpha y(t) + \lambda y(t) = g(t), \quad t \in (0,T], \quad \lambda > 0, \\ y(0) = c, \end{gathered}$$

where $0 < \alpha < 1$, $g \in L(0,T)$, c *is an arbitrary real constant, is given by the integral expression*

$$y(t) = cE_{\alpha,1}(-\lambda t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda(t-\tau)^\alpha\big) g(\tau)\, d\tau.$$

Theorem 2.2 (see [25, pp. 221–224]). *The solution* $y(t) \in L(0,T)$ *of the linear nonhomogeneous fractional problem*

$$\begin{gathered} D_{0+,t}^\alpha y(t) + \lambda y(t) = g(t), \quad t \in (0,T], \quad \lambda > 0, \\ I_{0+,t}^{1-\alpha} y(0+) = c, \end{gathered}$$

where $0 < \alpha < 1$, $g \in L(0,T)$, c *is an arbitrary real constant, is given by the integral expression*

$$y(t) = ct^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda(t-\tau)^\alpha\big) g(\tau)\, d\tau.$$

We will use these facts everywhere in this article.

§ 3. Direct problem

We seek a classical solution $u(x,t)$ of problem (1.1)–(1.3) in the form

$$u(x,t)=\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}u_n(t)\sin\lambda_n x,\qquad \lambda_n=\frac{\pi n}{l},\tag{3.1}$$

where

$$u_n(t)=\sqrt{\frac{2}{l}}\int_0^l u(x,t)\sin\lambda_n x\,dx.$$

Then, applying the formal scheme of the Fourier method for determining of unknown coefficient $u_n(t)$ of function $u(x,t)$ from (1.1) and (1.2), we have

$$\partial_t^{\alpha}u_n(t)+\lambda_n^2u_n(t)=g_n(t),\tag{3.2}$$

$$u_n(t)\big|_{t=0}=\varphi_n,\qquad n=1,2,\dots,\tag{3.3}$$

where

$$\varphi_n=\sqrt{\frac{2}{l}}\int_0^l\varphi(x)\sin\lambda_n x\,dx,\qquad g_n(t)=f_n(t)+\int_0^t k(t-\tau)u_n(\tau)\,d\tau,$$

$$f_n(t)=\sqrt{\frac{2}{l}}\int_0^l f(x,t)\sin\lambda_n x\,dx.$$

According to Theorem 2.1, the solution of problem (3.2), (3.3) satisfies the following integral equation:

$$\begin{aligned}u_n(t)&=\varphi_nE_{\alpha,1}(-\lambda_n^2t^{\alpha})+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n^2(t-\tau)^{\alpha}\big)f_n(\tau)\,d\tau+\\&+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n^2(t-\tau)^{\alpha}\big)\int_0^{\tau}k(\tau-\eta)u_n(\eta)\,d\eta\,d\tau.\end{aligned}\tag{3.4}$$

We seek the solution of integral equation (3.4) by the method of successive approximations in the form

$$u_n(t)=\sum_{m=0}^{\infty}(u_n)_m(t).\tag{3.5}$$

The obtained estimates ensure that this series converges absolutely and uniformly on $[0,T]$, which justifies passing to the limit in (3.5) and shows that its sum is a solution of the integral equation.

We consider the sequence of functions

$$(u_n)_m(t)=\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n^2(t-\tau)^{\alpha}\big)\int_0^{\tau}k(\tau-\eta)(u_n)_{m-1}(\eta)\,d\eta\,d\tau,\qquad m=1,2,\dots,\tag{3.6}$$

where

$$(u_n)_0(t)=\varphi_nE_{\alpha,1}(-\lambda_n^2t^{\alpha})+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n^2(t-\tau)^{\alpha}\big)f_n(\tau)\,d\tau.$$

Using the Proposition 2.2 and Lemma 2.1, we estimate the modulus of $(u_n)_0(t)$ in the domain $[0,T]$ as

$$\begin{aligned}|(u_n)_0(t)|&\le|\varphi_nE_{\alpha,1}(-\lambda_n^2t^{\alpha})|+\left|\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^{\alpha})f_n(\tau)\,d\tau\right|\le\\&\le|\varphi_n|+\frac{\|f_n\|T^{\alpha}}{\Gamma(\alpha+1)}:=\Phi_0.\end{aligned}$$

Similarly, we estimate $(u_n)_1(t), (u_n)_2(t), (u_n)_3(t)$

$$|(u_n)_1(t)| = \left|\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n^2 (t-\tau)^\alpha\big) \int_0^\tau k(\tau-\eta)(u_n)_0(\eta)\, d\eta\, d\tau\right| \leq$$
$$\leq \frac{\|k\| \|(u_n)_0\|}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau\, d\tau \leq \frac{\Phi_0 t^{\alpha+1} \|k\|}{\Gamma(\alpha+2)},$$

where $\|k\| = \max\limits_{t\in[0,T]} |k(t)|$,

$$|(u_n)_2(t)| = \left|\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n^2 (t-\tau)^\alpha\big) \int_0^\tau k(\tau-\eta)(u_n)_1(\eta)\, d\eta\, d\tau\right| \leq$$
$$\leq \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n^2 (t-\tau)^\alpha\big) \int_0^\tau |k(\tau-\eta)| \frac{\Phi_0 \eta^{\alpha+1} \|k\|}{\Gamma(\alpha+2)}\, d\eta\, d\tau \leq \frac{\Phi_0 t^{2\alpha+2} \|k\|^2}{\Gamma(2\alpha+3)},$$
$$|(u_n)_3(t)| = \left|\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n^2 (t-\tau)^\alpha\big) \int_0^\tau k(\tau-\eta)(u_n)_2(\eta)\, d\eta\, d\tau\right| \leq$$
$$\leq \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n^2 (t-\tau)^\alpha\big) \int_0^\tau |k(\tau-\eta)| \frac{\Phi_0 \eta^{2\alpha+2} \|k\|^2}{\Gamma(2\alpha+3)}\, d\eta\, d\tau \leq \frac{\Phi_0 t^{3\alpha+3} \|k\|^3}{\Gamma(3\alpha+4)}.$$

In this way, for arbitrary $m = 4, 5, \dots,$ we have

$$|(u_n)_m(t)| \leq \frac{\Phi_0 t^{m(\alpha+1)} \|k\|^m}{\Gamma(m\alpha+m+1)}.$$

It follows from above estimates that the series (3.5) converges uniformly in $[0,T]$, since it can be majorized in $[0,T]$ by the convergent numerical series

$$\Phi_0 \sum_{m=0}^{\infty} \frac{T^{m(\alpha+1)} \|k\|^m}{\Gamma(m\alpha+m+1)}.$$

This means that the following estimate for the series (3.5) holds:

$$|u_n(t)| \leq \Phi_0 \sum_{m=0}^{\infty} \frac{T^{m(\alpha+1)} \|k\|^m}{\Gamma(m\alpha+m+1)} = \Phi_0 E_{\alpha+1,1}(T^{\alpha+1}\|k\|), \quad t \in [0,T], \tag{3.7}$$

where $E_{\alpha+1,1}(.)$ is the Mittag-Leffler function of a nonnegative real argument defined in Section 2.

Thus, under conditions $(A1)$ and $(A2)$, the series (3.7) is convergent. Therefore, in view of the above inequalities, the series (3.5) converges absolutely and uniformly on the interval $[0,T]$.

Theorem 3.1. *Let conditions $(A1)$–$(A2)$ be satisfied. Then the series*

$$u_n(t) = \sum_{m=0}^{\infty} (u_n)_m(t)$$

converges absolutely and uniformly on $[0,T]$, and its sum $u_n(t)$ is a solution of the Volterra integral equation (3.4).

Proof. Let us introduce the partial sums $S_N(t) = \sum\limits_{m=0}^{N} (u_n)_m(t)$. By construction, the sequence $\{(u_n)_m(t)\}$ is defined recursively by (3.6). Hence, the partial sums satisfy the integral relation

$$S_N(t) = (u_n)_0(t) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n^2 (t-\tau)^\alpha\big) \int_0^\tau k(\tau-\eta) S_{N-1}(\eta)\, d\eta\, d\tau.$$

From the estimates obtained above, the series

$$\sum_{m=0}^{\infty}(u_n)_m(t)$$

is absolutely and uniformly convergent on $[0,T]$. Therefore,

$$S_N(t)\to u_n(t)\ \text{ uniformly on }\ [0,T].$$

Moreover, the estimates (3.7) imply that the integrands are dominated by an integrable function independent of N. Hence, we can pass to the limit under the integral sign as $N\to\infty$ (by the dominated convergence theorem).

Passing to the limit in the above integral relation for $S_N(t)$, we obtain

$$u_n(t)=(u_n)_0(t)+\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n^2(t-\tau)^{\alpha}\big)\int_0^{\tau}k(\tau-\eta)u_n(\eta)\,d\eta\,d\tau.$$

Thus, $u_n(t)$ satisfies the Volterra integral equation (3.4). The proof is complete. ☐

Using equalities (3.2) and (3.7) we obtain the following estimates for $\partial_t^{\alpha}u_n(t)$:

$$|\partial_t^{\alpha}u_n(t)|\le(\lambda_n^2+\|k\|T)\Phi_0E_{\alpha+1,1}(T^{\alpha+1}\|k\|)+\|f_n\|.$$

Thus, we have proved the following lemma.

Lemma 3.1. *For any* $t\in[0,T]$ *the following estimates are valid:*

$$|u_n(t)|\le\left(|\varphi_n|+\frac{\|f_n\|T^{\alpha}}{\Gamma(\alpha+1)}\right)E_{\alpha+1,1}(T^{\alpha+1}\|k\|),$$
$$|\partial_t^{\alpha}u_n(t)|\le(\lambda_n^2+\|k\|T)\left(|\varphi_n|+\frac{\|f_n\|T^{\alpha}}{\Gamma(\alpha+1)}\right)E_{\alpha+1,1}(T^{\alpha+1}\|k\|)+\|f_n\|.$$

Formally, from (3.1) by term-by-term differentiation, we compose the series

$$\partial_t^{\alpha}u(x,t)=\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\partial_t^{\alpha}u_n(t)\sin\lambda_nx, \tag{3.8}$$

$$u_{xx}(x,t)=-\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\lambda_n^2u_n(t)\sin\lambda_nx. \tag{3.9}$$

Let us prove the absolute and uniform convergence of the series (3.1), (3.8), and (3.9). From (3.1) and Lemma 3.1 it follows

$$|u(x,t)|\le E_{\alpha+1,1}(T^{\alpha+1}\|k\|)\sqrt{\frac{2}{l}}\left(\sum_{n=1}^{\infty}|\varphi_n|+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\sum_{n=1}^{\infty}\|f_n\|\right). \tag{3.10}$$

Proceeding similarly for series (3.8) and (3.9), we have

$$\begin{aligned}|\partial_t^{\alpha}u(x,t)|\le E_{\alpha+1,1}(T^{\alpha+1}\|k\|)\sqrt{\frac{2}{l}}\Bigg(\sum_{n=1}^{\infty}\lambda_n^2|\varphi_n|+T\|k\|\sum_{n=1}^{\infty}\left(|\varphi_n|+\frac{\|f_n\|T^{\alpha}}{\Gamma(\alpha+1)}\right)+\\ +\frac{T^{\alpha}}{\Gamma(\alpha+1)}\sum_{n=1}^{\infty}\lambda_n^2|f_n(t)|\Bigg)+\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}|f_n(t)|,\end{aligned} \tag{3.11}$$

and

$$|u_{xx}(x,t)| \leq E_{\alpha+1,1}(T^{\alpha+1}\|k\|)\sqrt{\frac{2}{l}}\left(\sum_{n=1}^{\infty}\lambda_n^2|\varphi_n| + \frac{T^\alpha}{\Gamma(\alpha+1)}\sum_{n=1}^{\infty}\lambda_n^2|f_n(t)|\right). \tag{3.12}$$

We need the following lemma, which establishes the uniform convergence of series (3.10), (3.11) and (3.12).

Lemma 3.2. *If conditions (A1)–(A2) are valid, then there are equalities*

$$\varphi_n = -\frac{1}{\lambda_n^3}\varphi_n^{(3)}, \qquad f_n(t) = -\frac{1}{\lambda_n^3}f_n^{(3)}(t), \tag{3.13}$$

where

$$\varphi_n^{(3)} = \sqrt{\frac{2}{l}}\int_0^l \varphi^{(3)}(x)\cos\lambda_n x\,dx, \qquad f_n^{(3)}(t) = \sqrt{\frac{2}{l}}\int_0^l f_{xxx}^{(3)}(x,t)\cos\lambda_n x\,dx$$

with the following estimate:

$$\sum_{n=1}^{\infty}|\varphi_n^{(3)}|^2 \leq \|\varphi^{(3)}\|^2_{L_2(0,l)}, \qquad \sum_{n=1}^{\infty}|f_n^{(3)}(t)|^2 \leq \|f_{xxx}^{(3)}\|^2_{L_2(0,l)\times C[0,T]}. \tag{3.14}$$

Proof. We prove the corresponding statements for φ_n and $f_n(t)$ separately.

Step 1. By definition, the Fourier coefficients φ_n are given by

$$\varphi_n = \sqrt{\frac{2}{l}}\int_0^l \varphi(x)\sin(\lambda_n x)\,dx, \qquad \lambda_n = \frac{\pi n}{l}.$$

We then apply integration by parts three times.

First integration by parts yields

$$\int_0^l \varphi(x)\sin(\lambda_n x)\,dx = -\left[\varphi(x)\frac{\cos(\lambda_n x)}{\lambda_n}\right]\Bigg|_0^l + \frac{1}{\lambda_n}\int_0^l \varphi'(x)\cos(\lambda_n x)\,dx.$$

Since $\varphi(0) = \varphi(l) = 0$, the boundary term is zero.

Second integration by parts yields

$$\int_0^l \varphi'(x)\cos(\lambda_n x)\,dx = \left[\varphi'(x)\frac{\sin(\lambda_n x)}{\lambda_n}\right]\Bigg|_0^l - \frac{1}{\lambda_n}\int_0^l \varphi''(x)\sin(\lambda_n x)\,dx.$$

Since $\sin(0) = \sin(n\pi) = 0$, the boundary term is zero.

Third integration by parts yields

$$\int_0^l \varphi''(x)\sin(\lambda_n x)\,dx = -\left[\varphi''(x)\frac{\cos(\lambda_n x)}{\lambda_n}\right]\Bigg|_0^l + \frac{1}{\lambda_n}\int_0^l \varphi^{(3)}(x)\cos(\lambda_n x)\,dx.$$

Since $\varphi''(0) = \varphi''(l) = 0$, the boundary term is zero.

Summing up the above integration by parts steps, we arrive at

$$\int_0^l \varphi(x)\sin(\lambda_n x)\,dx = -\frac{1}{\lambda_n^3}\int_0^l \varphi^{(3)}(x)\cos(\lambda_n x)\,dx.$$

Thus,

$$\varphi_n = -\frac{1}{\lambda_n^3}\varphi_n^{(3)},$$

where

$$\varphi_n^{(3)} = \sqrt{\frac{2}{l}}\int_0^l \varphi^{(3)}(x)\cos(\lambda_n x)\,dx.$$

Step 2. Similarly, for

$$f_n(t)=\sqrt{\frac{2}{l}}\int_0^l f(x,t)\sin(\lambda_n x)\,dx,$$

using assumptions (*A*2) and applying integration by parts three times, we obtain

$$f_n(t)=-\frac{1}{\lambda_n^3}f_n^{(3)}(t),$$

where $f_n^{(3)}(t)$ denotes the expression

$$f_n^{(3)}(t)=\sqrt{\frac{2}{l}}\int_0^l f_{xxx}^{(3)}(x,t)\cos(\lambda_n x)\,dx.$$

Step 3. Since the system $\left\{\sqrt{\frac{2}{l}}\cos(\lambda_n x)\right\}_{n=1}^{\infty}$ is orthonormal in $L_2(0,l)$, by Bessel's inequality, we obtain

$$\sum_{n=1}^{\infty}|\varphi_n^{(3)}|^2\le\|\varphi^{(3)}\|_{L_2(0,l)}^2.$$

Similarly,

$$\sum_{n=1}^{\infty}|f_n^{(3)}(t)|^2\le\|f_{xxx}^{(3)}(\cdot,t)\|_{L_2(0,l)}^2,\quad t\in[0,T].$$

The proof of Lemma 3.2 is complete. □

If we ensure the convergence of the series $\sum\limits_{n=1}^{\infty}\lambda_n^2|\varphi_n|$, $\sum\limits_{n=1}^{\infty}\lambda_n^2\|f_n\|$, then the series (3.1), (3.8), and (3.9) will have the absolute and uniform convergence. You can see easily from (3.10), (3.11) and (3.12). We will only prove the convergence of $\sum\limits_{n=1}^{\infty}\lambda_n^2|\varphi_n|$, and for the remaining series this is proven in a similar way. By the Cauchy–Bunyakovsky inequality and conditions of Lemma 3.2, we prove convergence as follows

$$\sum_{n=1}^{\infty}\lambda_n^2|\varphi_n|=\sum_{n=1}^{\infty}\lambda_n^2\sqrt{\frac{2}{l}}\left|\int_0^l\varphi(x)\sin(\lambda_n x)\,dx\right|=\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\sqrt{\frac{2}{l}}\left|\int_0^l\varphi^{(3)}(x)\cos(\lambda_n x)\,dx\right|=$$

$$=\frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}|\varphi_n^{(3)}|\le\frac{l}{\pi}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}}\cdot\left(\sum_{n=1}^{\infty}|\varphi_n^{(3)}|^2\right)^{\frac{1}{2}}\le\frac{l}{\sqrt{6}}\|\varphi^{(3)}\|_{L^2(0,l)}.$$

If the functions $\varphi(x)$, $f(x,t)$ satisfy the conditions of Lemma 3.2, then, due to representations (3.13) and (3.14), series (3.1), (3.8) and (3.9) converge uniformly in the rectangle D_T, therefore, function $u(x,t)$ satisfies relations (1.1)–(1.3).

Using the above results, we obtain the following assertion.

Lemma 3.3. *Let $k(t)\in C[0,T]$ and (A1), (A2) be satisfied. Then, there exists a unique solution $u(x,t)\in C^{2,\alpha}(D_T)\cap C(\overline{D}_T)$ of the direct problem* (1.1)–(1.3).

In the next Section, we consider the inverse problem of reconstructing the function $k(t)$ from relations (1.1)–(1.4), employing the contraction mapping principle.

§ 4. Solvability of inverse problem

Let $u(x,t)$ and $k(t)$ be a classical solution of problem (1.1)–(1.4) (see Definition 1.2), and assume that f, φ, ω, and h are sufficiently smooth functions. The following theorem presents an equivalent auxiliary problem.

Lemma 4.1. *Problem* (1.1)–(1.4) *is equivalent to the auxiliary problem of determining the functions* $\vartheta(x,t)$, $k(t)$ *from the following equations:*

$$D^{\alpha}_{0+,t}\vartheta - \vartheta_{xx} = k(t)\varphi(x) + \int_0^t k(\tau)\vartheta(x,t-\tau)\,d\tau + f_t(x,t), \quad (x,t)\in D_T, \tag{4.1}$$

$$I^{1-\alpha}_{0+,t}\vartheta(x,t)\Big|_{t=0} = \varphi''(x) + f(x,0), \quad x\in[0,l], \tag{4.2}$$

$$\vartheta(x,t)\Big|_{x=0} = \vartheta(x,t)\Big|_{x=l} = 0, \quad t\in[0,T], \tag{4.3}$$

$$\int_0^l \omega(x)\vartheta(x,t)\,dx = h'(t), \tag{4.4}$$

where $\vartheta(x,t) := u_t(x,t), D^{\alpha}_{0+,t}\vartheta$ *is the Riemann–Liouville fractional derivative* (*see Definition* 2.2).

Proof. The auxiliary problem (4.1)–(4.4) is equivalent to problem (1.1)–(1.4). This result was proved in [11] for a similar problem. Introducing a new function $\vartheta(x,t) := u_t(x,t)$, we differentiate equation (1.1) with respect to t. For this purpose, we use the following relations:

$$\frac{\partial}{\partial t}\partial_t^{\alpha}u(x,t) = \frac{\partial}{\partial t}\left(\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^{\alpha}}\,d\tau\right) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{\vartheta(x,\tau)}{(t-\tau)^{\alpha}}\,d\tau = D^{\alpha}_{0+,t}\vartheta(x,t).$$

Hence, we obtain the following equation:

$$D^{\alpha}_{0+,t}\vartheta(x,t) - \vartheta_{xx}(x,t) = k(t)\varphi(x) + \int_0^t k(\tau)\vartheta(x,t-\tau)\,d\tau + f_t(x,t).$$

To obtain the initial condition (4.2), we use the following relation:

$$\partial_t^{\alpha}u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^{\alpha}}\,d\tau = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\vartheta(x,\tau)}{(t-\tau)^{\alpha}}\,d\tau = I^{1-\alpha}_{0+,t}\vartheta(x,t).$$

After that, we substitute $t=0$ into equation (1.1) and, using the initial condition (1.2), obtain

$$I^{1-\alpha}_{0+,t}\vartheta(x,t)\Big|_{t=0} = u_{xx}(x,t)\Big|_{t=0} + \int_0^t k(\tau)u(x,t-\tau)\,d\tau\Big|_{t=0} + f(x,0) = \varphi''(x) + f(x,0).$$

To obtain the boundary condition (4.3) and the additional condition (4.4), we differentiate both sides of equations (1.3) and (1.4) with respect to t, and using the relation $\vartheta = u_t$, we derive the required results. We have established that problem (1.1)–(1.4) leads to the auxiliary problem (4.1)–(4.4). We now prove the reverse implication, i. e., that (4.1)–(4.4) implies (1.1)–(1.4).

Step 1. Let $\vartheta(x,t)$ be a solution of the auxiliary problem (4.1)–(4.4). Define the function $u(x,t)$ by

$$u(x,t) = \eta(x) + \int_0^t \vartheta(x,\tau)\,d\tau. \tag{4.5}$$

Setting $t=0$ in (4.5), we obtain $u(x,0)=\eta(x)$. Comparing this with the initial condition (1.2), we conclude that the condition is satisfied if and only if $\eta(x)=\varphi(x)$. Thus, the initial condition (1.2) holds.

Step 2. From (4.3) we have $\vartheta(0,t)=\vartheta(l,t)=0$, $t\in[0,T]$. Using the conditions $\varphi(0)==\varphi(l)=0$ and substituting $x=0$ and $x=l$ into (4.5), we obtain $u(0,t)=\varphi(0)+\int_0^t\vartheta(0,\tau)\,d\tau==0$, $u(l,t)=\varphi(l)+\int_0^t\vartheta(l,\tau)\,d\tau=0$. Thus, the boundary conditions (1.3) are fulfilled.

Step 3. Integrating (4.4) with respect to t and using the (4.5), we get

$$\int_0^l\omega(x)u(x,t)\,dx=\int_0^l\omega(x)\varphi(x)\,dx+\int_0^t h'(\tau)\,d\tau.$$

Hence,

$$\int_0^l\omega(x)u(x,t)\,dx=h(t),$$

provided that the matching condition

$$\int_0^l\omega(x)\varphi(x)\,dx=h(0)$$

holds. Therefore, the overdetermination condition (1.4) is satisfied.

Step 4. To derive the original equation (1.1), we integrate both sides of (4.1) with respect to t over the interval $(0,t)$. Thus, we obtain

$$\begin{gathered}\int_0^t D^{\alpha}_{0+,\tau}\vartheta(x,\tau)\,d\tau-\int_0^t\vartheta_{xx}(x,\tau)\,d\tau=\\=\varphi(x)\int_0^t k(\tau)\,d\tau+\int_0^t\left(\int_0^\tau k(s)\vartheta(x,\tau-s)\,ds\right)d\tau+f(x,t)-f(x,0).\end{gathered}\tag{4.6}$$

After that, we consider each term of (4.6) separately. For the first term on the left-hand side, using the $\vartheta(x,t)=u_t(x,t)$ and (4.2), we have

$$\begin{gathered}\int_0^t D^{\alpha}_{0+,\tau}\vartheta(x,\tau)\,d\tau=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{d}{d\tau}\int_0^\tau\frac{\vartheta(x,s)}{(\tau-s)^\alpha}\,ds\,d\tau=\\=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{u_s(x,s)}{(t-s)^\alpha}\,ds-I^{1-\alpha}_{0+,t}\vartheta(x,t)\Big|_{t=0}=\partial_t^\alpha u(x,t)-\varphi''(x)-f(x,0).\end{gathered}$$

Furthermore, for the second term on the left-hand side, using (4.5), we obtain

$$\int_0^t\vartheta_{xx}(x,\tau)\,d\tau=\frac{\partial^2}{\partial x^2}\big(u(x,t)-\varphi(x)\big)=u_{xx}(x,t)-\varphi''(x).$$

Consider the convolution term on the right-hand side of (4.6). By changing the order of integration (Fubini's theorem), we obtain

$$\begin{gathered}\int_0^t\left(\int_0^\tau k(s)\vartheta(x,\tau-s)\,ds\right)d\tau=\\=\int_0^t k(s)\left(\int_0^{t-s}\vartheta(x,\xi)\,d\xi\right)ds=\int_0^t k(s)\big(u(x,t-s)-\varphi(x)\big)\,ds.\end{gathered}$$

By substituting all the obtained equalities into (4.6), we simplify the expression to arrive at (1.1). Lemma 4.1 is proved. ☐

So, we will study direct and inverse problems for the auxiliary problem (4.1)–(4.4).

Lemma 4.2. *Let $k(t) \in C_\gamma[0,T]$ and (A1), (A2) be satisfied. Then, there exists a unique solution $\vartheta(x,t) \in C_\gamma^{2,\alpha}(\overline{D}_T)$ of the direct problem* (4.1)–(4.3).

P r o o f. We will seek a solution to the problem (4.1)–(4.3) in the form

$$\vartheta(x,t) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \vartheta_n(t) \sin(\lambda_n x), \quad \lambda_n = \frac{\pi n}{l}, \tag{4.7}$$

where

$$\vartheta_n(t) = \sqrt{\frac{2}{l}} \int_0^l \vartheta(x,t) \sin(\lambda_n x)\, dx. \tag{4.8}$$

We obtain the following Cauchy problem:

$$\big(D_{0+,t}^\alpha \vartheta_n\big)(t) + \lambda_n^2 \vartheta_n(t) = F_n(t;\vartheta,k,\varphi,f), \quad n = 1,2,\dots, \tag{4.9}$$

$$I_{0+,t}^{1-\alpha} \vartheta_n(t)\Big|_{t=0} = -\lambda_n^2 \varphi_n + f_n(0), \tag{4.10}$$

where

$$F_n(t;\vartheta,k,\varphi,f) := k(t)\varphi_n + \int_0^t k(\tau)\vartheta_n(t-\tau)\, d\tau + f_n'(t).$$

Using Theorem 2.2, it is easy to derive the following Volterra integral equation from (4.9), (4.10)

$$\begin{aligned}\vartheta_n(t) = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2 t^\alpha)\big(-\lambda_n^2\varphi_n + f_n(0)\big) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n^2 (t-\tau)^\alpha\big) k(\tau)\varphi_n\, d\tau + \\ + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n^2 (t-\tau)^\alpha\big) \int_0^\tau k(s)\vartheta_n(\tau-s)\, ds\, d\tau + \\ + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n^2 (t-\tau)^\alpha\big) f_n'(\tau)\, d\tau.\end{aligned} \tag{4.11}$$

We prove the existence of a solution to (4.11) in the space $C_\gamma[0,T]$, $\gamma = 1-\alpha$, using the method of successive approximations. For this equation, we consider the sequence of functions

$$t^\gamma(\vartheta_n)_m(t) = t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n^2(t-\tau)^\alpha\big) \int_0^\tau k(s)(\vartheta_n)_{m-1}(\tau-s)\, ds\, d\tau, \quad m = 1,2,\dots,$$

where

$$\begin{aligned}t^\gamma(\vartheta_n)_0(t) = E_{\alpha,\alpha}(-\lambda_n^2 t^\alpha)\big(-\lambda_n^2\varphi_n + f_n(0)\big) + t^{1-\alpha}\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n^2(t-\tau)^\alpha\big) k(\tau)\varphi_n\, d\tau + \\ + t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n^2(t-\tau)^\alpha\big) f_n'(\tau)\, d\tau.\end{aligned}$$

Using Propositions 2.1, 2.2 and Lemma 2.1, we obtain

$$|t^\gamma(\vartheta_n)_0(t)| \le \frac{1}{\Gamma(\alpha)}\big(\lambda_n^2|\varphi_n| + |f_n(0)|\big) + \frac{\Gamma(\alpha)T^\alpha}{\Gamma(2\alpha)}|\varphi_n|\|k\|_\gamma + \frac{T}{\Gamma(\alpha+1)}\|f_n\|_{C^1[0,T]} =: F_n^0,$$

where $\|k\|_\gamma = \max\limits_{t\in[0,T]} |t^{1-\alpha}k(t)|$.

In the same way, for $m = 1,2,\dots$ we derive the following estimates

$$\begin{aligned}|t^\gamma(\vartheta_n)_1(t)| = t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n^2(t-\tau)^\alpha\big) \times \\ \times \int_0^\tau \big|s^{1-\alpha}k(s)\big|\big|(\tau-s)^{1-\alpha}(\vartheta_n)_0(\tau-s)\big| s^{\alpha-1}(\tau-s)^{\alpha-1}\, ds\, d\tau \le \frac{\Gamma^2(\alpha)\, t^{2\alpha}}{\Gamma(3\alpha)}\, \|k\|_\gamma\, F_n^0,\end{aligned}$$

$$|t^\gamma(\vartheta_n)_2(t)| = t^{1-\alpha}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n^2(t-\tau)^\alpha\big) \times$$
$$\times \int_0^\tau \big|s^{1-\alpha}k(s)\big|\big|(\tau-s)^{1-\alpha}(\vartheta_n)_1(\tau-s)\big|s^{\alpha-1}(\tau-s)^{\alpha-1}\,ds\,d\tau \le \frac{\Gamma^3(\alpha)\,t^{4\alpha}}{\Gamma(5\alpha)}\,\|k\|_\gamma^2\,F_n^0,$$

$$\ldots,$$

$$|t^\gamma(\vartheta_n)_m(t)| \le t^{1-\alpha}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_n^2(t-\tau)^\alpha\big) \times$$
$$\times \int_0^\tau \big|s^{1-\alpha}k(s)\big|\big|(\tau-s)^{1-\alpha}(\vartheta_n)_{m-1}(\tau-s)\big|s^{\alpha-1}(\tau-s)^{\alpha-1}\,ds\,d\tau \le \frac{\Gamma(\alpha)\,\Gamma^m(\alpha)\,(t^{2\alpha})^m}{\Gamma(2\alpha m+\alpha)}\,\|k\|_\gamma^m\,F_n^0.$$

It follows from the last estimates that the series

$$t^\gamma\vartheta_n(t) = \sum_{m=1}^\infty t^\gamma(\vartheta_n)_m(t)$$

converges uniformly in $[0,T]$, because it can be majorized in $[0,T]$ by the convergent numerical series

$$\Gamma(\alpha)\sum_{m=1}^\infty \frac{\big(\Gamma(\alpha)\|k\|_\gamma T^{2\alpha}\big)^m}{\Gamma(2\alpha m+\alpha)}F_n^0.$$

This means the following estimate for the solution of the integral equation (4.11) takes place:

$$|t^\gamma\vartheta_n(t)| \le \Gamma(\alpha)F_n^0E_{2\alpha,\alpha}(T^{2\alpha}\Gamma(\alpha)\|k\|_\gamma) \le \tilde{C}_1\big(\lambda_n^2|\varphi_n| + \|f_n\|_{C^1[0,T]}\big), \tag{4.12}$$

where the constant $\tilde{C}_1$ depends on T, α, $\|k\|_\gamma$.

Further, using equality (4.9), we obtain an estimate for $t^\gamma(D^\alpha_{0+,t}\vartheta_n)(t)$:

$$|t^\gamma(D^\alpha_{0+,t}\vartheta_n)(t)| \le \tilde{C}_2\big(\lambda_n^4|\varphi_n| + \lambda_n^2\|f_n\|_{C^1[0,T]}\big), \tag{4.13}$$

where the constant $\tilde{C}_2$ depend on T, α, $\|k\|_\gamma$.

Formally, from (4.7), by term-by-term differentiation, we obtain

$$t^\gamma D^\alpha_{0+,t}\vartheta(x,t) = \sqrt{\frac{2}{l}}\sum_{n=1}^\infty t^\gamma D^\alpha_{0+,t}\vartheta_n(t)\sin(\lambda_n x), \quad \gamma = 1-\alpha, \tag{4.14}$$

$$t^\gamma\vartheta_{xx}(x,t) = -\sqrt{\frac{2}{l}}\sum_{n=1}^\infty \lambda_n^2 t^\gamma\vartheta_n(t)\sin(\lambda_n x). \tag{4.15}$$

Let us prove the convergence of the series (4.7), (4.14), and (4.15). From (4.12) and (4.13), it follows that

$$|t^\gamma\vartheta(x,t)| \le \widetilde{C}_3\left(\sum_{n=1}^\infty \lambda_n^2|\varphi_n| + \sum_{n=1}^\infty \|f_n\|_{C^1[0,T]}\right), \tag{4.16}$$

$$|t^\gamma D^\alpha_{0+,t}\vartheta(x,t)| \le \widetilde{C}_4\left(\sum_{n=1}^\infty \lambda_n^4|\varphi_n| + \sum_{n=1}^\infty \lambda_n^2\|f_n\|_{C^1[0,T]}\right), \tag{4.17}$$

$$|t^\gamma\vartheta_{xx}(x,t)| \le \widetilde{C}_5\left(\sum_{n=1}^\infty \lambda_n^4|\varphi_n| + \sum_{n=1}^\infty \lambda_n^2\|f_n\|_{C^1[0,T]}\right), \tag{4.18}$$

where the constants $\widetilde{C}_i$, $i = 3, 4, 5$, depend only on T, l, α and $\|k\|_\gamma$.

If we prove the convergence of the series

$$\sum_{n=1}^{\infty} \lambda_n^4 |\varphi_n|, \qquad \sum_{n=1}^{\infty} \lambda_n^2 \|f_n\|_{C^1[0,T]},$$

then the series in (4.16), (4.17), (4.18) are convergent. In general, if the conditions of $(A1)$–$(A2)$ are satisfied, then the series (4.7), (4.14), and (4.15) converge absolutely and uniformly in $\overline{D}_T$. Thus, the function $\vartheta(x,t)$ defined by the series (4.7) is a solution to the problem (4.1)–(4.3) in $\overline{D}_T$.

Let us now prove the uniqueness of the solution. For $\varphi(x) \equiv 0$ and $f(x,t) \equiv 0$, we obtain $\varphi_n \equiv 0$, $f_n(t) \equiv 0$. Then, from formula (4.11), it follows that $\vartheta_n \equiv 0$, since ϑ_n is a solution of the homogeneous equation

$$\vartheta_n(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n (t-s)^\alpha\big) \int_0^s k(s-\tau)\vartheta_n(\tau)\, d\tau\, ds.$$

Substituting $\vartheta_n \equiv 0$ into equation (4.8), we obtain

$$\sqrt{\frac{2}{l}} \int_0^l \vartheta(x,t) \sin \lambda_n x\, dx = 0.$$

Since the system $\left\{\sqrt{\frac{2}{l}} \sin \lambda_n x\right\}$ is complete in $L_2(0,l)$, it follows that $\vartheta(x,t) = 0$ almost everywhere in $(0,l)$, $\forall t \in [0,T]$. Since $\vartheta(x,t) \in C(\overline{D}_T)$, we conclude that $\vartheta(x,t) \equiv 0$ in $\overline{D}_T$. $\square$

Lemma 4.3. *Let $\vartheta_n^1(t)$ and $\vartheta_n^2(t)$ be solutions of (4.11), corresponding to the functions $k^1(t)$ and $k^2(t)$, respectively. Then, for $t \in [0,T]$ and fixed $n \in \mathbb{N}$, the following estimate holds:*

$$|t^\gamma(\vartheta_n^1(t) - \vartheta_n^2(t))| \le \|k^1 - k^2\|_\gamma \left[\frac{\Gamma(\alpha) T^\alpha |\varphi_n|}{\Gamma(2\alpha)} + \|\vartheta_n^1\|_\gamma \frac{T^{2\alpha}\Gamma^2(\alpha)}{\Gamma(3\alpha)}\right] \times \\ \times \Gamma(\alpha)\, E_{2\alpha,\alpha}\big(T^{2\alpha}\Gamma(\alpha)\, \|k^2\|_\gamma\big). \tag{4.19}$$

Here,

$$\|\vartheta_n^1\|_\gamma \le \tilde{C}_1 \big(\lambda_n^2 |\varphi_n| + \|f_n\|_{C^1[0,T]}\big).$$

Proof. To prove estimate (4.19) for $t \in [0,T]$, we write

$$t^\gamma\big(\vartheta_n^1(t) - \vartheta_n^2(t)\big) = t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n^2 (t-\tau)^\alpha\big) \Big[\big[k^1(\tau) - k^2(\tau)\big]\varphi_n + \\ + \int_0^\tau \Big(\big[k^1(\tau-s) - k^2(\tau-s)\big]\vartheta_n^1(s) + k^2(\tau-s)\big[\vartheta_n^1(s) - \vartheta_n^2(s)\big]\Big)\, ds\Big]\, d\tau. \tag{4.20}$$

Further, applying the method of successive approximations to the integral equation (4.20), and taking into account estimate (4.12) with k^1, we obtain (4.19). $\square$

Multiplying (4.1) by $\omega(x)$ and integrating over $x \in (0,l)$, and taking into account (4.3)–(4.4), we obtain

$$D_{0+,t}^\alpha h'(t) - \int_0^l \omega(x)\vartheta_{xx}(x,t)\, dx = \\ = k(t) \int_0^l \omega(x)\varphi(x)\, dx + \int_0^t k(\tau) h'(t-\tau)\, d\tau + \int_0^l \omega(x) f_t(x,t)\, dx, \quad 0 < t \le T.$$

Thus, we arrive at the following of integral equation with respect to unknown function $k(t)$:

$$k(t) = k_0(t) - \frac{1}{h(0)}\left[\sum_{n=1}^{\infty}\lambda_n^2\vartheta_n(t;k)\omega_n + \int_0^t k(\tau)h'(t-\tau)\,d\tau\right], \quad 0 < t \le T, \tag{4.21}$$

where

$$k_0(t) = \frac{1}{h(0)}\left[D_{0+,t}^{\alpha}h'(t) - \sum_{n=1}^{\infty}\lambda_n^2 f_n'(t)\omega_n\right], \qquad h(0) = \int_0^l \omega(x)\varphi(x)\,dx.$$

Equation (4.21) can be rewritten as the operator equation

$$k = A[k], \tag{4.22}$$

where the operator A is defined by

$$A[k] = k_0(t) - \frac{1}{h(0)}\left[\sum_{n=1}^{\infty}\lambda_n^2\vartheta_n(t;k)\omega_n + \int_0^t k(\tau)h'(t-\tau)\,d\tau\right].$$

Here, $\vartheta_n(t;k)$ denotes the solution of the integral equation (4.11), which depends on the function $k(t)$, i. e., $\vartheta_n = \vartheta_n(t;k)$.

Consider the functional space of function $k \in C_\gamma(\overline{D}_T)$ with the norm given by the relation

$$\|k\|_\gamma = \max_{t\in[0,T]}|t^\gamma k(t)|, \quad \gamma = 1-\alpha.$$

Consider the set $R_T(k_0,\rho) := \big\{k \in C_\gamma[0,T]\colon \|k-k_0\|_\gamma \le \rho\big\}$, where $\rho > 0$ denotes the radius, which will be chosen later. For any $k \in R_T(k_0,\rho)$, the following inequality holds:

$$\|k\|_\gamma \le \rho + \|k_0\|_\gamma := R_0.$$

The main result of this work is presented as follows.

Theorem 4.1. *Let (A1)–(A4) be satisfied. Then, there exist a number $T > 0$ and a radius ρ such that for any $T \in (0,T^*)$, the operator equation* (4.22) *has a unique solution in the ball R_T. Consequently, there exists a unique classical solution to the problem* (4.1)–(4.4).

Proof. Let us show that for a sufficiently small $T > 0$, the operator A is a contraction mapping on $R_T(k_0,\rho)$. Let $k \in R_T(k_0,\rho)$. Then, we obtain the following estimate:

$$\begin{aligned}
\|A[k]-k_0\|_\gamma &= \max_{t\in[0,T]}\big|t^\gamma(A[k]-k_0)\big| \le \frac{1}{|h(0)|}\left[\sum_{n=1}^{\infty}\lambda_n^2|t^{1-\alpha}\vartheta_n(t;k)||\omega_n| + \right.\\
&\left. + t^{1-\alpha}\int_0^t \tau^{\alpha-1}|\tau^{1-\alpha}k(\tau)||h'(t-\tau)|\,d\tau\right] \le \frac{\Gamma(\alpha)}{|h(0)|}E_{2\alpha,\alpha}\big(T^{2\alpha}\Gamma(\alpha)\|k\|_\gamma\big)\times\\
&\times\left[\sum_{n=1}^{\infty}\left(\frac{\lambda_n^2}{\Gamma(\alpha)} + \frac{\Gamma(\alpha)\|k\|_\gamma T^\alpha}{\Gamma(2\alpha)}\right)\lambda_n^2|\varphi_n||\omega_n| + \frac{1}{\Gamma(\alpha)}\sum_{n=1}^{\infty}\lambda_n^2\|f_n\||\omega_n| + \right.\\
&\left. + \frac{T}{\Gamma(\alpha+1)}\sum_{n=1}^{\infty}\lambda_n^2\|f_n\|_{C^1[0,T]}|\omega_n|\right] + \frac{T\|k\|_\gamma}{|h(0)|}\|h\|_{C^1[0,T]} \le\\
&\le \frac{\Gamma(\alpha)}{|h(0)|}E_{2\alpha,\alpha}\big(T^{2\alpha}\Gamma(\alpha)R_0\big)\left[\sum_{n=1}^{\infty}\left(\frac{\lambda_n^2}{\Gamma(\alpha)} + \frac{\Gamma(\alpha)R_0T^\alpha}{\Gamma(2\alpha)}\right)\lambda_n^2|\varphi_n||\omega_n| + \right.\\
&\left. + \frac{1}{\Gamma(\alpha)}\sum_{n=1}^{\infty}\lambda_n^2\|f_n\||\omega_n| + \frac{T}{\Gamma(\alpha+1)}\sum_{n=1}^{\infty}\lambda_n^2\|f_n\|_{C^1[0,T]}|\omega_n|\right] +\\
&+ \frac{TR_0}{|h(0)|}\|h\|_{C^1[0,T]} =: m_1(T).
\end{aligned} \tag{4.23}$$

Note that the function $m_1(T)$ is monotonically increasing in T, and the following relation holds:

$$m_1(0) = \frac{1}{|h(0)|}\left[\sum_{n=1}^{\infty} \lambda_n^4 |\varphi_n||\omega_n| + \sum_{n=1}^{\infty} \lambda_n^2 \|f_n\||\omega_n|\right].$$

Let $\rho > m_1(0)$ be fixed. Then, by the monotonicity and continuity of $m_1(T)$, the equation $m_1(T) = \rho$ admits a unique solution, which we denote by T_1. Consequently, for any $T < T_1$, the operator A maps the ball $R_T(k_0, \rho)$ into itself, i. e., $A[k] \in R_T(k_0, \rho)$.

Now we verify the second condition of contraction operator. Let $k^j \in R_T(k_0, \rho)$, $j = 1, 2$. Using the (4.19), we obtain

$$\begin{aligned}
&\|A[k]^1 - A[k]^2\|_\gamma = \max_{t\in[0,T]} |t^\gamma (A[k]^1 - A[k]^2)| \le \\
&\le \frac{1}{|h(0)|}\Bigg[\sum_{n=1}^{\infty} \lambda_n^2 t^{1-\alpha} |\vartheta_n^1(t; k^1, k^2) - \vartheta_n^2(t; k^1, k^2)||\omega_n| + \\
&\quad + t^{1-\alpha} \int_0^t |k^1(\tau) - k^2(\tau)||h'(t-\tau)|\, d\tau\Bigg] \le \\
&\le \frac{1}{|h(0)|}\Bigg[\Gamma(\alpha)\, E_{2\alpha,\alpha}\big(T^{2\alpha}\Gamma(\alpha) R_0\big) \sum_{n=1}^{\infty}\Bigg[\frac{\Gamma(\alpha) T^\alpha \lambda_n^2 |\varphi_m|}{\Gamma(2\alpha)} + \\
&+ \tilde{C}_1\big(\lambda_n^4 |\varphi_m| + \lambda_n^2 \|f_n\|_{C^1[0,T]}\big)\frac{T^{2\alpha}\Gamma^2(\alpha)}{\Gamma(3\alpha)}\Bigg]|\omega_n| + T\|h\|_{C^1[0,T]}\Bigg]\|k^1 - k^2\|_\gamma =: \\
&=: m_2(T)\|k^1 - k^2\|_\gamma.
\end{aligned} \tag{4.24}$$

If the conditions $(A1)$–$(A4)$ are met, then all series in (4.23), (4.24) are convergent. Note that the expressions on the right sides of this inequality is monotonically increasing function $m_2(T)$ of T and $m_2(0) = 0$. It follows that the equation $m_2(T) = 1$ has a unique positive root, which we denote this by T_2.

If the radius ρ satisfies the inequality $\rho > \max\{m_1(0), m_2(0)\}$ and we set $T^* < \min\{T_1, T_2\}$, then for all $T \le T^*$ the operator A is a contraction in the ball $R_T(k_0, \rho)$. Then, according to the Banach principle [33, pp. 84–88], there exists a unique solution of equation (4.22) in the space $R_T(k_0, \rho)$. ☐

Remark 4.1. The double of functions $\vartheta(x, t)$, $k(t)$ constructed above, is the classical solution of the problem (4.1)–(4.4) in the domain D_T, the double of functions $u(x, t)$, $k(t)$ are the classical solution to the inverse problem (1.1)–(1.4), where

$$u(x, t) = \varphi(x) + \int_0^t \vartheta(x, \tau)\, d\tau.$$

§ 5. Numerical solution of the inverse problem

In this Section, we use the forward Euler difference formula for the Caputo time-fractional derivative $\partial_t^\alpha u(x, t)$ in Equation (1.1) to construct an explicit scheme, which we refer to as the fractional forward-time centered-space method.

Let N_x be the total number of discretization points. We define $\Delta x = \frac{l}{N_x}$ as the spatial step size and denote the discretized points as, $x_i = i\Delta x$, where $0 \le i \le N_x$ is a positive integer. Let u_i^n be an approximation of $u(x_i, t_n)$, where $t_n = n\Delta t$ and $\Delta t = \frac{T}{N_t}$ is the temporal step size, and N_t is the total number of time steps. For $0 < \alpha < 1$, the Caputo derivative can be discretized using the following method [34]

$$\partial_t^\alpha u(x,t) \approx \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{n} b_j \left[u_i^{n+1-j} - u_i^{n-j}\right],$$

where $\Gamma(\cdot)$ is the Euler's Gamma function, $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$.

At a grid point (x_i, t^n) with $n > 0$, using the forward finite difference approximation for $\partial_t^\alpha u(x,t)$ and the central finite difference approximation for u_{xx}, we obtain

$$\frac{1}{\Delta t^\alpha \Gamma(2-\alpha)} \sum_{j=0}^{n} b_j \left(u_i^{n-j+1} - u_i^{n-j}\right) - \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = \Delta t \sum_{j=0}^{n-1} k^j u_i^{n-j} + f_i^n,$$

where the dots denote higher-order terms, so the discretization is $O\big(\Delta t^\alpha + \Delta x^2\big)$. When $n = 0$, u_i^0 is the initial condition at the grid point $(x_i, 0)$ and from the values u_i^n at the time level n the solution of the finite difference equation at the next time level $n+1$ is

$$\begin{aligned} u_i^{n+1} = u_i^n + \frac{1}{b_0}\big(\Delta t^\alpha \Gamma(2-\alpha)\big)\left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + \Delta t \sum_{j=0}^{n-1} k^j u_i^{n-j} + f_i^n\right) - \\ - \sum_{j=1}^{n} b_j \left(u_i^{n-j+1} - u_i^{n-j}\right). \end{aligned} \tag{5.1}$$

The unknown function k^n can be determined by discretizing the integral equation (4.21) using the forward finite difference method, while the integrals are approximated by the rectangle and trapezoidal rules. First, substituting inverse transform $\vartheta = u_t$, we rewrite the integral equation (4.21) as follows:

$$k^n = \frac{1}{h^0}\left[H^\alpha - \Delta x \sum_{i=1}^{N_x-1} \omega_i \frac{A_i^n - A_i^{n-1}}{\Delta t} - F^n - \Delta t \sum_{j=0}^{n} k^j H^{n-j}\right], \tag{5.2}$$

where $h^0 = h(0)$, $H^\alpha = D_{0+,t}^\alpha h'(t)$, $H^n = h'(t)$, $F^n = \int_0^l \omega(x) f_t(x,t)\,dx$, $A_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$.

After transforming (5.2), we obtain the following scheme for k^n

$$k^n = \frac{1}{h^0 + \Delta t H^0}\left[H^\alpha - \Delta x \sum_{i=1}^{N_x-1} \omega_i \frac{A_i^n - A_i^{n-1}}{\Delta t} - F^n - \Delta t \sum_{j=0}^{n-1} k^j H^{n-j}\right]. \tag{5.3}$$

Yu and Chen [35] proposed high-order finite difference and spectral schemes for fractional diffusion equations governed by the Caputo–Fabrizio fractional derivative, demonstrating high accuracy and computational efficiency. In particular, [35, Theorem 3.4] establishes the stability condition for an explicit difference scheme applied to a one-dimensional time-space Caputo–Riesz fractional diffusion equation as follows

$$0 < \frac{\Delta t^\alpha}{(\Delta x)^\gamma} \le \frac{\Gamma(4-\gamma)(1-2^{-\alpha})}{4|\kappa_\gamma| C_{\max} \Gamma(2-\alpha)(1-2^{1-\gamma})},$$

where $0 < \alpha \le 1$, $1 < \gamma \le 2$, $\kappa_\gamma = \dfrac{1}{2\cos(\gamma\pi/2)}$, $C_{\max} = \max\limits_{0 \le i \le N_x,\ 0 \le n \le N_t} c(x_i, t_n)$. When $c(x,t) = 1$, $\gamma = 2$, the equation reduces to the time-fractional diffusion equation. Using this Theorem, we obtain the following conditional stability restriction

$$0 < \frac{\Delta t^\alpha}{(\Delta x)^2} \le \frac{1 - 2^{-\alpha}}{\Gamma(2-\alpha)}. \tag{5.4}$$

Hence, for a fixed spatial step size Δx, the stability condition (5.4) determines the maximal admissible time step

$$\Delta t_{\max} = \left((\Delta x)^2 \frac{1 - 2^{-\alpha}}{\Gamma(2-\alpha)} \right)^{\frac{1}{\alpha}}. \tag{5.5}$$

Therefore, in the numerical implementation, after choosing the spatial mesh size Δx, the time step Δt must satisfy $0 < \Delta t \le \Delta t_{\max}$. Consequently, the number of time levels is selected as $N_t = \left[\frac{T}{\Delta t}\right]$, where T denotes the final computational time. This condition provides a practical criterion for selecting the discretization parameters $(\Delta x, \Delta t, N_t)$ in order to guarantee the conditional stability of the explicit finite difference scheme.

Based on these equations, Algorithm 1 has been constructed for the simultaneous determination of $u(x,t)$ and the unknown function $k(t)$, and it has been implemented in open-source Python.

Algorithm 1: Finite difference scheme for the evolution of $u(x,t)$ and $k(t)$

```
Set parameters: $\alpha,\ T,\ N_x,\ l = 2\pi$
Compute the spatial step size: $\Delta x = \frac{l}{N_x}$
Determine the stability condition: $0 < \frac{\Delta t^\alpha}{\Delta x^2} \le \frac{1-2^{-\alpha}}{\Gamma(2-\alpha)}$
Compute the maximal admissible time step: $\Delta t_{\max} = \left( \Delta x^2 \frac{1-2^{-\alpha}}{\Gamma(2-\alpha)} \right)^{1/\alpha}$
Choose a safety factor: $0 < \theta < 1$
Define the stable time step: $\Delta t = \theta \Delta t_{\max}$
Compute the number of time levels: $N_t = \left[\frac{T}{\Delta t}\right]$,
Correct the time step: $\Delta t = \frac{T}{N_t}$
Verify the stability quantity: $S = \frac{\Delta t^\alpha}{\Delta x^2}$
if $S > \frac{1-2^{-\alpha}}{\Gamma(2-\alpha)}$ then
    Stop computation and reduce $\Delta t$
end if
Generate the grids: $x_i = i\Delta x, \quad i = 0,1,\dots,N_x, \quad t_n = n\Delta t, \quad n = 0,1,\dots,N_t$
Given initial and boundary data:
    $u_i^0 = \varphi(x_i), \qquad u_0^n = 0, \qquad u_{N_x}^n = 0$
Given terms $f_i^n$, $\omega_i$, and $h^n$
for $n = 0,1,\dots,N_t - 1$ do
    for $i = 1,2,\dots,N_x - 1$ do
        $A_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$
        for $j = 0,1,\dots,n$ do
            $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$
        end for
        if $n = 0$ then
            $B_i^0 = 0$
        else
```

```
            $B_i^n = \sum_{j=1}^{n} b_j\left(u_i^{n-j+1} - u_i^{n-j}\right)$
        end if
        Evaluate $k^n$ using Equation (5.3)
        $u_i^{n+1} = u_i^n + \frac{\Delta t^{\alpha}\Gamma(2-\alpha)}{b_0}\left(A_i^n + f_i^n + \Delta t \sum_{j=0}^{n-1} k^j u_i^{n-j} - B_i^n\right)$
    end for
    Note that $b_0 = 1$
    Apply boundary conditions: $u_0^{n+1} = 0, \quad u_{N_x}^{n+1} = 0$
    Compute the error for $k(t)$: $\rho(k^n) = |k^n - k(t_n)|$
    Compute the discrete $L^\infty$-error for $u(x,t)$: $\rho(u^{n+1}) = \max_{0 \le i \le N_x} |u_i^{n+1} - u(x_i, t_{n+1})|$
    Store or display the numerical solutions and errors
end for
```

§ 6. Numerical example

In this Section, we present the numerical results obtained using Algorithm 1 for the test example (5.1), (5.3). The results are shown for three different values of $\alpha = 0{,}5; 0{,}6; 0{,}75; 1$. In this example, we take $l = 2\pi$ and $T = 1$. The computational details are provided in Section 5. The results have been analyzed by calculating the absolute error the between the exact and numerical solutions, defined as,

$$\rho(u^{n+1}) = \max_{0 \le i \le N_x} |u_i^{n+1} - u(x_i, t_{n+1})|, \quad \rho(k^n) = |k^n - k(t_n)|.$$

We solve the fractional inverse problem (1.1)–(1.4) with the following input data

$$\varphi(x) = \sin(x), \quad f(x,t) = \left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 1 - \frac{t^3}{3} - \frac{t^5}{30} + t^2\right)\sin(x),$$
$$\omega(x) = \sin(x), \quad h(t) = \pi(t^2 + 1)$$

for $x \in (0, l = 2\pi)$ and $t \in (0, T = 1)$.

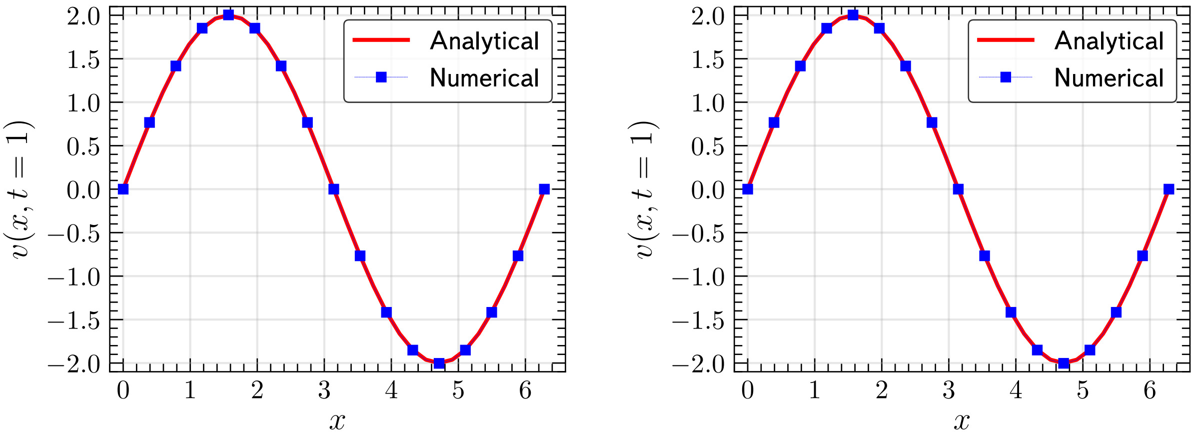
The exact solution is given by

$$u(x,t) = (1 + t^2)\sin x, \quad k(t) = t^2.$$

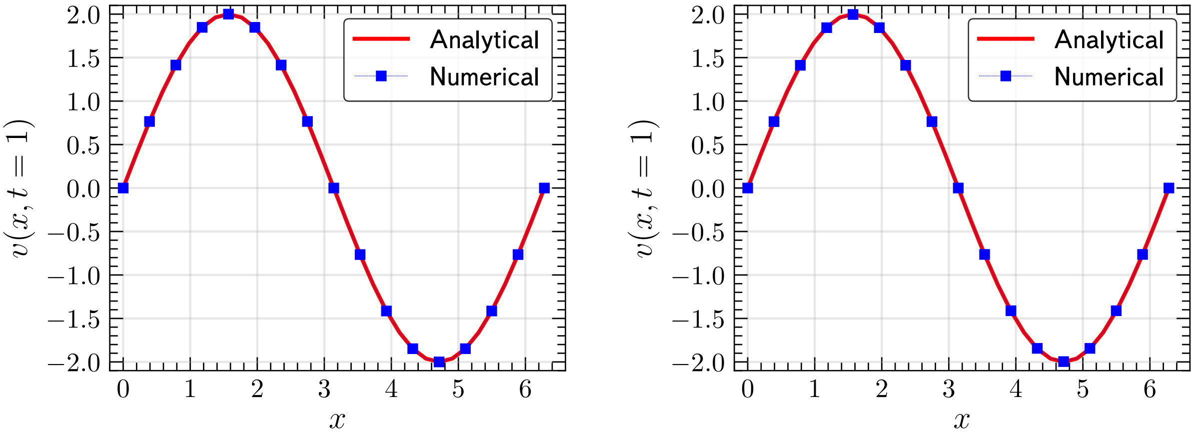
The one-dimensional domain was discretized using $N_x = 32$ grid points with spatial step size $\Delta x = \frac{l}{N_x}$, $l = 2\pi$.

The time step Δt was selected according to the conditional stability restriction of the explicit finite difference scheme, (5.4) which imposes a stability-dependent upper bound on the admissible time increment. In particular, for a prescribed spatial discretization Δx, the maximal allowable time step is given by (5.5). Accordingly, the value of Δt was chosen such that $0 < \Delta t \le \Delta t_{\max}$, thereby ensuring the conditional stability of the numerical scheme. Moreover, smaller values of Δt and finer spatial discretizations (larger N_x) may further improve the accuracy of the reconstructed coefficient $k(t)$ and the numerical solution $u(x,t)$, at the expense of increased computational cost.

In Figure 1, 2, we depict the exact and reconstructed functions $u(x,t)$ and $k(t)$ for $\alpha = 0{,}5$, $\alpha = 0{,}6$, $\alpha = 0{,}75$ and $\alpha = 1$. We observe that for all three values of the fractional order, the accuracy of the restored function is very similar.



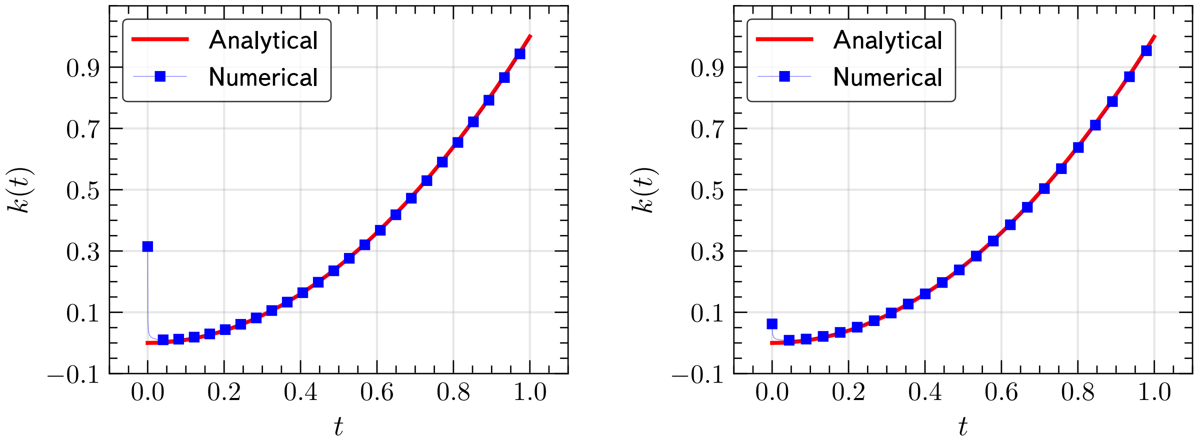
(a) $\alpha = 0.5$ (b) $\alpha = 0.6$



(b) $\alpha = 0.75$ (d) $\alpha = 1.0$

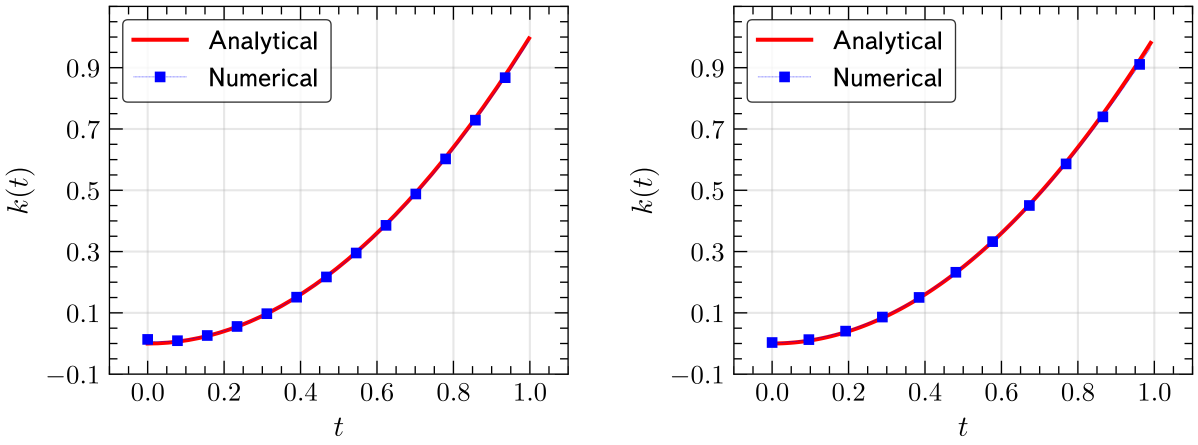
Fig. 1. (a), (b), (c) (d): comparison between the numerical and analytical solutions of $u(x, t)$

The values of the absolute errors $\rho(u^{n+1})$ and $\rho(k^n)$ between the exact and numerical solutions of $u(x, t)$ and $k(t)$, respectively, are shown in Tables 1 and 2. The computations were performed using $N_x = 32, l = 2\pi, T = 1$.



(a) $\alpha = 0.5$

(b) $\alpha = 0.6$



(c) $\alpha = 0.75$

(d) $\alpha = 1.0$

Fig. 2. (a), (b), (c), (d): comparison between the numerical and analytical solutions of $k(t)$

Table 1. The absolute error $\rho(u)$ between the exact and numerical solutions of $u(x, t)$

t	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.75$	$\alpha = 1.0$
0.0	0.00000	0.00000	0.00000	0.00000
0.1	1.13074e-03	8.23561e-04	2.63448e-04	5.77187e-04
0.2	1.59305e-03	1.21536e-03	3.39790e-04	1.11916e-03
0.3	1.94721e-03	1.52449e-03	3.78074e-04	1.53761e-03
0.4	2.24770e-03	1.78819e-03	3.96697e-04	1.92963e-03
0.5	2.51707e-03	2.02533e-03	4.00610e-04	2.23816e-03
0.6	2.76845e-03	2.24390e-03	3.90209e-04	2.51766e-03
0.7	3.01129e-03	2.44917e-03	3.62258e-04	2.81509e-03
0.8	3.25345e-03	2.64766e-03	3.11904e-04	3.10245e-03
0.9	3.50215e-03	2.84153e-03	2.32317e-04	3.47317e-03
1.0	3.76444e-03	3.03605e-03	1.14786e-04	3.89785e-03

Table 2. The absolute error $\rho(k)$ between the exact and numerical solutions of $k(t)$

t	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.75$	$\alpha = 1.0$
0.0	3.14612e-01	6.21643e-02	1.33447e-02	3.19834e-03
0.1	4.87237e-03	4.33933e-03	2.61352e-03	3.41166e-03
0.2	2.32395e-03	2.24725e-03	1.26278e-03	3.38707e-03
0.3	6.65906e-04	7.77824e-04	1.82402e-04	3.04575e-03
0.4	6.92886e-04	4.79524e-04	8.65110e-04	2.22179e-03
0.5	1.87812e-03	1.63035e-03	1.94756e-03	9.79391e-04
0.6	2.92186e-03	2.69354e-03	3.06921e-03	8.12339e-04
0.7	3.82716e-03	3.66748e-03	4.29890e-03	3.50510e-03
0.8	4.58660e-03	4.55507e-03	5.62466e-03	6.68321e-03
0.9	5.18917e-03	5.33964e-03	7.05633e-03	1.10582e-02
1.0	5.62313e-03	6.01599e-03	8.56630e-03	1.53712e-02

Table 3. Stability parameters and maximum admissible time step $\Delta t_{\max}$ for different values of α

α	Δx	$\Delta t_{\max}$	Δt (used)	N_t
0.5	1.96350e-01	1.62349e-04	8.11688e-05	12320
0.6	1.96350e-01	8.905675e-04	4.452360e-04	2246
0.75	1.96350e-01	4.45477e-03	2.22717e-03	449
1.0	1.96350e-01	1.92766e-02	9.61539e-03	104

It is important to note that in Figures 1, *a*, *b* and 2, *a*, *b* a relatively larger discrepancy between the numerical and exact solutions is observed near $t = 0$. This error can be attributed to the limited accuracy of the numerical approximation at early time steps, where the fractional derivative has a singularity point at $t = 0$. Moreover, since we chose a modest number of spatial grid points $N_x = 32$, the spatial accuracy is also slightly reduced. Despite this, as t increases, the numerical solutions rapidly improve and closely match the exact solutions. This convergence is ensured under the condition $T \leq T^*$ which arises from the stability and contraction estimates of the underlying operator. More precisely, for T satisfying this restriction, the corresponding solution operator remains contractive in the chosen functional space, guaranteeing both stability and convergence of the iterative scheme. This behavior is consistent across all fractional orders $\alpha = 0{,}5,\ 0{,}6,\ 0{,}75,\ 1$, as supported by the absolute error values presented in Tables 1 and 2. Table 3 presents the stability-related discretization parameters for different values of the fractional order α. The time step Δt is chosen as $\Delta t = \theta \Delta t_{\max}$, $\theta = 0{,}5$, where θ is the safety factor. This choice guarantees that the conditional stability restriction of the explicit finite difference scheme is satisfied, thereby ensuring numerical stability of the proposed method.

Conclusion

In this paper, we investigated an inverse problem for a time-fractional integro-differential diffusion equation subject to initial-boundary conditions and an integral overdetermination condition in a rectangular domain. Firstly, the direct problem is transformed into an equivalent integral equation using the Fourier method. We proved the existence and uniqueness of the solution to this equivalent problem by applying of estimates of the Mittag-Leffler function and the method of successive approximations. In the second part, we studied the inverse problem, which was reduced to an equivalent auxiliary problem. Using the contraction mapping principle, we proved local existence and uniqueness of the solution to the inverse problem. In addition, we developed an approach for numerically recovering the time-dependent kernel using observations at a finite number of time and space points. We have experimentally demonstrated the efficacy of the proposed method. The order of the fractional derivative does not significantly affect the results. The numerical tests with perturbed data demonstrated accurate recovery of the kernel with a satisfactory level of error.

Acknowledgment

The authors are grateful for the anonymous referees for their constructive comments.

Funding. The research was executed at North–Caucasus Center for Mathematical Research of the Vladikavkaz Scientific Centre of the Russian Academy of Sciences, with the support of the Ministry of Science and Higher Education of the Russian Federation, agreement No. 075-02-2026-738.

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Received 27.05.2025
Accepted 10.06.2026

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Citation: J. J. Jumaev, D. K. Durdiev. Solvability of an inverse problem for a time-fractional integro-differential diffusion equation with initial-boundary and integral overdetermination conditions, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, 2026, vol. 36, issue 2, pp. 205–233.

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Разрешимость обратной задачи для интегро-дифференциального уравнения диффузии с дробной производной по времени, начально-граничными условиями и интегральным условием переопределения

Ключевые слова: уравнение диффузии с дробной производной по времени, обратная задача, интегральное уравнение, существование, единственность, метод конечных разностей.

УДК 517.968.74

DOI: 10.35634/vm260202

В данной статье рассматривается обратная задача для интегро-дифференциального уравнения диффузии с дробной производной по времени с начальными, граничными и интегральными переопределяющими условиями в прямоугольной области. Вначале дается определение классического решения исходной задачи. Затем прямая задача преобразуется в эквивалентное интегральное уравнение с использованием метода Фурье. Доказываются существование и единственность решения этой эквивалентной задачи на основе оценок функции Миттаг-Лефлера и метода последовательных приближений. Во второй части статьи рассматривается обратная задача, которая сводится к эквивалентной вспомогательной задаче. Для доказательства локального существования и единственности решения обратной задачи используется принцип сжимающего отображения. Кроме того, дается практический обзор численного решения исходной задачи с использованием метода конечных разностей. Демонстрируются результаты численного эксперимента на более мелких сетках и при меньших шагах по времени. Разработанный алгоритм позволяет одновременно определять зависящее от времени ядро в интегральном члене и решение задачи. В заключение приводится тестовый пример, иллюстрирующий эффективность предложенного численного алгоритма.

Финансирование. Исследование выполнено в Северо-Кавказском центре математических исследований Владикавказского научного центра Российской академии наук при поддержке Министерства науки и высшего образования Российской Федерации (соглашение № 075-02-2026-738).

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Поступила в редакцию 27.05.2025
Принята к публикации 10.06.2026

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Цитирование: Ж. Ж. Жумаев, Д. К. Дурдиев. Разрешимость обратной задачи для интегро-дифференциального уравнения диффузии с дробной производной по времени, начально-граничными условиями и интегральным условием переопределения // Вестник Удмуртского университета. Математика. Механика. Компьютерные науки. 2026. Т. 36. Вып. 2. С. 205–233.