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CONSTRUCTION OF THREE-DIMENSIONAL STABILITY REGIONS FOR TWO LINEAR THIRD-ORDER DIFFERENCE EQUATIONS WITH DELAY

For linear third-order difference equations with delay

$$x(n+3)+ax(n+2)+bx(n+1)+dx(n-1)=0,\quad n\in\mathbb{Z},\quad x\in\mathbb{R},$$

$$x(n+3)+bx(n+1)+cx(n)+dx(n-1)=0,\quad n\in\mathbb{Z},\quad x\in\mathbb{R},$$

stability regions are studied and constructed in three-dimensional spaces $\{(a,b,d)\}\subset\mathbb{R}^3$ and $\{(b,c,d)\}\subset$ $\subset\mathbb{R}^3$, respectively.

Keywords: linear difference equation, Schur stability, stability region.

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Introduction

Consider a linear time-invariant difference equation

$$x(n+k)+p_1x(n+k-1)+p_2x(n+k-2)+\ldots+p_kx(n)=0,\quad n\in\mathbb{Z},\tag{0.1}$$

here $x\in\mathbb{R}$ is a state, $p_1,\ldots,p_k\in\mathbb{R}$, and $k\in\mathbb{N}$. We study the problem of asymptotic stability of equation (0.1). Let us construct the *characteristic polynomial* corresponding to the equation (0.1):

$$p(\lambda)=\lambda^k+p_1\lambda^{k-1}+\ldots+p_k.\tag{0.2}$$

Let λ_i, $i=\overline{1,k}$, be the roots of (0.2); they are called *eigenvalues* of the equation (0.1). The zero solution of (0.1) is asymptotically stable (or, equivalently, equation (0.1) is asymptotically of exponentially stable) [12, Section 5.1] iff

$$\forall\, i=\overline{1,k}\quad |\lambda_i|<1.\tag{0.3}$$

In this case, the polynomial (0.2) is called *Schur stable*. There are various criteria for checking the condition (0.3), for example, the Schur–Cohn criterion [12, Section 5.1], [16, Sections 3.4, 3.5], the Jury criterion [16, Sections 3.9, 3.10], the criterion using reduction to the Routh–Hurwitz conditions [16, Section 3.3], and others.

By $\mathcal{S}_k=\{(p_1,\ldots,p_k)\}\subset\mathbb{R}^k$ denote the set of values of the coefficients $p_1,\ldots,p_k$ of equation (0.1), for which the equation is asymptotically stable. This set is called *the stability region* in the space $\mathbb{R}^k$. The problem of constructing a stability region has been studied in the works of many authors, see, e. g., [3, 5, 6, 10, 14, 19]. Some sufficient conditions for the asymptotic stability of linear autonomous difference systems, including those with delays, have been obtained in the papers [7, 11, 18, 20]. The issues of constructing stability regions of difference and differential equations, including equations with delays, were studied, in particular, in the papers [8, 9, 22–25]. The papers [1, 2, 4, 15] are devoted to the study and construction of stability regions for linear difference equations with complex coefficients.

In this paper, we consider equation (0.1) of the fourth order. For this equation, we consider two cases in which one of the coefficients is zero. For these cases, we investigate and construct stability regions in the three-dimensional space of the remaining parameters.

§ 1. The set $\mathcal{S}_4$

Let $k = 4$ in (0.1). Let us consider a fourth-order linear difference equation

$$x(n+4) + ax(n+3) + bx(n+2) + cx(n+1) + dx(n) = 0, \quad n \in \mathbb{Z}, \tag{1.1}$$

where $a, b, c, d \in \mathbb{R}$ are constant. The necessary condition for the Schur stability of the characteristic polynomial $p(\lambda) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d$ is the following (see [21]):

$$-4 < a < 4, \quad -6 < b < 6, \quad -4 < c < 4, \quad -1 < d < 1.$$

Consider the stability region $\mathcal{S}_4 = \{(a, b, c, d)\} \subset \mathbb{R}^4$ for the equation (1.1).

According to [13, Theorem 1], the convex hull of the set $\mathcal{S}_4$ is a polyhedron in four-dimensional space with vertices at the points

$$N_1(4, 6, 4, 1), \quad N_2(2, 0, -2, -1), \quad N_3(0, -2, 0, 1), \quad N_4(-2, 0, 2, -1), \quad N_5(-4, 6, -4, 1).$$

(see, e. g., [28, (30)]).

According to [16, (3.53)–(3.57)], the stability region $\mathcal{S}_4$ is determined by the system of inequalities

$$\begin{cases} 1 + a + b + c + d > 0, & (1.2) \\ 1 - a + b - c + d > 0, & (1.3) \\ -a^2 d + acd - bd^2 + d^3 + ac + 2bd - c^2 - d^2 - b - d + 1 > 0, & (1.4) \\ d^2 - 1 + ad - c < 0, & (1.5) \\ d^2 - 1 - ad + c < 0. & (1.6) \end{cases}$$

Consider the hyperplane $\mathcal{H}(d_0) = \{(a, b, c, d) \colon d = d_0\} \subset \mathbb{R}^4$, $d_0 \in (-1, 1)$, and consider the set $\mathcal{HS}_4(d_0) := \mathcal{S}_4 \cap \mathcal{H}(d_0)$. This is a section of the set $\mathcal{S}_4$ by the hyperplane $\mathcal{H}(d_0)$. The set $\mathcal{HS}_4(d_0) = \{(a, b, c)\}$ is contained in the three-dimensional space $\mathbb{R}^3$. The set $\mathcal{HS}_4(0)$ coincides with the set $\mathcal{S}_3$. The form of the set $\mathcal{S}_3$ is well known; see, for example, [13, 26, 27].

In the paper [28], the structure of the set $\mathcal{HS}_4(d_0)$ was studied for all $d_0 \in (-1, 1)$ and an image of this set in three-dimensional space was constructed.

Let us now consider the hyperplanes $\mathcal{G}(c_0) = \{(a, b, c, d) \colon c = c_0\} \subset \mathbb{R}^4$, $c_0 \in (-4, 4)$, and $\mathcal{F}(a_0) = \{(a, b, c, d) \colon a = a_0\} \subset \mathbb{R}^4$, $a_0 \in (-4, 4)$, and consider the sections $\mathcal{GS}_4(c_0) := \mathcal{S}_4 \cap \mathcal{G}(c_0)$ and $\mathcal{FS}_4(a_0) := \mathcal{S}_4 \cap \mathcal{F}(a_0)$ of the set $\mathcal{S}_4$ by these hyperplanes. These sets are contained in three-dimensional spaces: $\mathcal{GS}_4(c_0) = \{(a, b, d)\} \subset \mathbb{R}^3$, $\mathcal{FS}_4(a_0) = \{(b, c, d)\} \subset \mathbb{R}^3$. We denote $\mathcal{C} := \mathcal{GS}_4(0)$ and $\mathcal{A} := \mathcal{FS}_4(0)$. In this paper, we investigate the structure of the sets $\mathcal{C}$ and $\mathcal{A}$ and construct an image of these sets in three-dimensional space.

§ 2. Construction of the set $\mathcal{C}$

Let us consider the set $\mathcal{C} = \mathcal{GS}_4(0)$. This set is the stability domain of the fourth-order equation

$$x(n+4) + ax(n+3) + bx(n+2) + dx(n) = 0, \quad n \in \mathbb{Z},$$

or the third-order equation with delay

$$x(n+3) + ax(n+2) + bx(n+1) + dx(n-1) = 0, \quad n \in \mathbb{Z}, \tag{2.1}$$

in the parameter space $\{(a,b,d) \in \mathbb{R}^3\}$. According to (1.2), (1.3), (1.4), (1.5), (1.6), the stability domain $\mathcal{C}$ is defined by the system of inequalities

$$\begin{cases} b > g_1(a,d) := -a-(1+d), & (2.2)\\ b > g_2(a,d) := a-(1+d), & (2.3)\\ b < g_3(a,d) := \dfrac{-a^2d}{(1-d)^2}+(1+d), & (2.4)\\ d^2-1+ad<0, & (2.5)\\ d^2-1-ad<0. & (2.6)\end{cases}$$

The stability domain $\mathcal{C}$ is a region in the three-dimensional space $Odab$ that is bounded by hypersurfaces

$$\gamma_1: \quad b=-a-(1+d), \tag{2.7}$$
$$\gamma_2: \quad b=a-(1+d), \tag{2.8}$$
$$\gamma_3: \quad b=\frac{-a^2d}{(1-d)^2}+(1+d), \tag{2.9}$$

as (d,a) run through the set $\Omega \subset \{(d,a) \in \mathbb{R}^2\}$ defined by inequalites (2.5), (2.6).

1. Consider the curves

$$\omega_1: \quad d^2-1+ad=0, \tag{2.10}$$
$$\omega_2: \quad d^2-1-ad=0. \tag{2.11}$$

They bound the set Ω. From (2.10) it follows

$$\omega_1: \quad a=\frac{1}{d}-d. \tag{2.12}$$

The graph of the function $a=a(d)$ (2.12) is a hyperbola with asymptotes $d=0$, $a=-d$; the function $a=a(d)$ is decreasing on the interval $d \in (-\infty,0)$ and on the interval $d \in (0,+\infty)$; the graph of the function $a=a(d)$ (2.12) is shown in Fig. 1. The hyperbola ω_1 consists of two branches: $\omega_1=\omega_1^+ \cup \omega_1^-$; here ω_1^+ is a branch of the hyperbola ω_1 for $d>0$; ω_1^- is a branch of the hyperbola ω_1 for $d<0$.

From (2.11) it follows

$$\omega_2: \quad a=d-\frac{1}{d}. \tag{2.13}$$

The graph of the function $a=a(d)$ (2.13) is a hyperbola with asymptotes $d=0$, $a=d$; the function $a=a(d)$ is increasing on the interval $d \in (-\infty,0)$ and on the interval $d \in (0,+\infty)$; the graph of the function $a=a(d)$ (2.13) is shown in Fig. 2. The hyperbola ω_2 consists of two branches: $\omega_2=\omega_2^+ \cup \omega_2^-$; here ω_2^+ is a branch of the hyperbola ω_2 for $d>0$; ω_2^- is a branch of the hyperbola ω_2 for $d<0$.

Thus, it is easy to see that the domain Ω is the region enclosed between the hyperbolas ω_1 and ω_2, containing zero within itself (see Fig. 3).

Denote by A_+ the point $(1,0) \in \{(d,a) \in \mathbb{R}^2\}$, and by A_- the point $(-1,0) \in \{(d,a) \in \mathbb{R}^2\}$.

2. The surfaces γ_1 and γ_2 are planes. The surface γ_3 is a cubic surface. Let's find curves of intersection of the surfaces γ_1 and γ_2.

Let

$$l_{12} := \gamma_1 \cap \gamma_2: \quad \begin{cases} b=-a-(1+d),\\ b=a-(1+d).\end{cases}$$

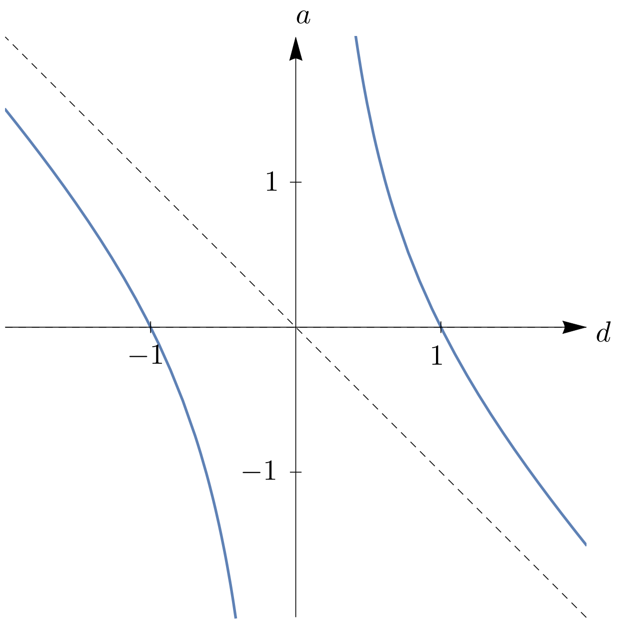

Fig. 1. The graph of the hyperbola ω_1 (2.12)

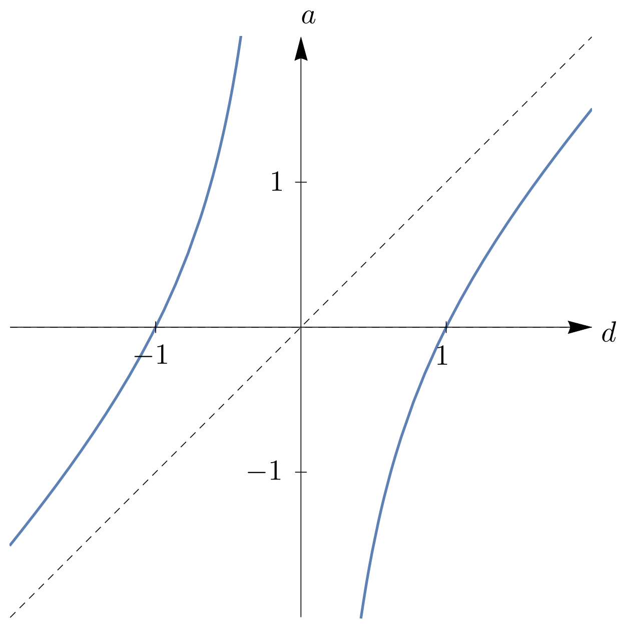

Fig. 2. The graph of the hyperbola ω_2 (2.13)

We get a straight line

$$l_{12}\colon \begin{cases} a = 0, \\ b = -(1+d). \end{cases}$$

If $a < 0$, then $g_1(a, d) > g_2(a, d)$. Therefore, by (2.2), (2.3), and (2.4), the set $\mathcal{C}$ is bounded by the surfaces γ_1 and γ_3.

If $a > 0$, then $g_2(a, d) > g_1(a, d)$. Therefore, by (2.2), (2.3), and (2.4), the set $\mathcal{C}$ is bounded by the surfaces γ_2 and γ_3.

Let us represent the domain Ω as follows: $\Omega = \Omega_+ \cup \Omega_- \cup \Omega_0$. Here $\Omega_+ = \{(d, a) \in \Omega\colon a > 0\}$, $\Omega_- = \{(d, a) \in \Omega\colon a < 0\}$, $\Omega_0 = \{(d, a) \in \Omega\colon a = 0\} = (A_-, A_+)$.

3. Let $(d, a) \in \Omega_-$. Then, the stability domain $\mathcal{C}$ is bounded by the surfaces γ_1 and γ_3. Let us find curves of intersection of surfaces (2.7) and (2.9). Let

$$l_{13} := \gamma_1 \cap \gamma_3\colon \quad \begin{cases} b = -a - (1+d), \\ b = \dfrac{-a^2 d}{(1-d)^2} + (1+d). \end{cases} \tag{2.14}$$

From the system (2.14), we obtain a quadratic equation

$$\frac{-a^2 d}{(1-d)^2} + (1+d) = -a - (1+d).$$

After transformation, we get a quadratic equation

$$-a^2 d + a(d-1)^2 + 2(1+d)(d-1)^2 = 0. \tag{2.15}$$

Let us calculate the discriminant $\mathfrak{D}$ of the quadratic equation (2.15). We obtain that

$$\mathfrak{D} = (3d+1)^2(d-1)^2.$$

The quadratic equation (2.15) has a solution

$$\left[\begin{array}{l} a = -\dfrac{d^2-1}{d}, \\ a = 2d - 2. \end{array}\right.$$

From here, taking into account (2.14), we obtain that the curve l_{13} consists of two curves: $l_{13} = l_{131} \cup l_{132}$,

$$l_{131}: \begin{cases} a = -\dfrac{d^2-1}{d}, \\ b = -1 - \dfrac{1}{d}, \end{cases} \qquad l_{132}: \begin{cases} a = 2d-2, \\ b = -3d+1. \end{cases}$$

The projection of the curve l_{131} onto the plane $Oda = \{b = 0\}$ coincides with the curve

$$\omega_1: \quad a = -d + \frac{1}{d}.$$

The projection of the line l_{132} onto the plane $Oda = \{b = 0\}$ is a line

$$\psi_1: \quad a = 2d - 2. \tag{2.16}$$

Let's find points of intersection of the curves ψ_1 and ω_1:

$$\begin{cases} a = 2d-2, \\ a = -\dfrac{d^2-1}{d}. \end{cases}$$

We obtain two points: $(1,0)$ and $\left(-\frac{1}{3}, -\frac{8}{3}\right)$. We denote the point $\left(-\frac{1}{3}, -\frac{8}{3}\right)$ by B_1.

Let's find the intersection point of the curves ψ_1 and ω_2:

$$\begin{cases} a = 2d-2, \\ a = d - \dfrac{1}{d}. \end{cases}$$

We obtain the point $(1,0)$. This point coincides with the point A_+.

The line (2.16) divides the domain Ω_- into two parts: in the lower part Ω_-^1 we have $a < 2d-2$, in the upper part Ω_-^2 we have $a > 2d-2$ (see Fig. 4). The following properties are easy to verify:

(i) if $(d,a) \in \Omega_-^1$, then $g_3(a,d) - g_1(a,d) < 0$; therefore, $g_3(a,d) < g_1(a,d)$;

(ii) if $(d,a) \in \psi_1$, then $g_3(a,d) - g_1(a,d) = 0$; therefore, $g_3(a,d) = g_1(a,d)$;

(iii) if $(d,a) \in \Omega_-^2$, then $g_3(a,d) - g_1(a,d) > 0$; therefore, $g_3(a,d) > g_1(a,d)$.

From this, it follows that if $(d,a) \in \Omega_-^2 = \operatorname{int}(A_-A_+B_1)$, then the region $\mathcal{C}$ is bounded from below by the plane γ_1 and bounded from above by the surface γ_3 (we suppose that the b-axis is directed vertically); and if $(d,a) \in \Omega_- \setminus \Omega_-^2$, then system (2.2)–(2.6) has no solution, and for these values of (d,a) the set $\mathcal{C}$ is empty.

When the parameters (d,a) of the plane $Oda = \{b = 0\}$ are located at the vertex points $A_-(-1,0)$, $A_+(1,0)$, $B_1(-1/3,-8/3)$ of the set Ω_-^2, one can find the coordinates b of the corresponding points $V_-, V_+, V_1 \in \{(d,a,b) \in \mathbb{R}^3\}$ of intersection of the surfaces γ_1 and γ_3. To do this, we substitute $A_-(-1,0)$ into l_{131}, $A_+(1,0)$ into l_{132}, and $B_1(-1/3,-8/3)$ into l_{131} (or into l_{132}). We get

$$V_-(-1,0,0), \tag{2.17}$$
$$V_+(1,0,-2), \tag{2.18}$$
$$V_1(-1/3,-8/3,2).$$

4. Let $(d,a) \in \Omega_+$. Then, the stability domain $\mathcal{C}$ is bounded by the surfaces γ_2 and γ_3. Let us find curves of intersection of surfaces (2.8) and (2.9). Let

$$l_{23} := \gamma_2 \cap \gamma_3: \quad \begin{cases} b = a - (1+d), \\ b = \dfrac{-a^2 d}{(1-d)^2} + (1+d). \end{cases} \tag{2.19}$$

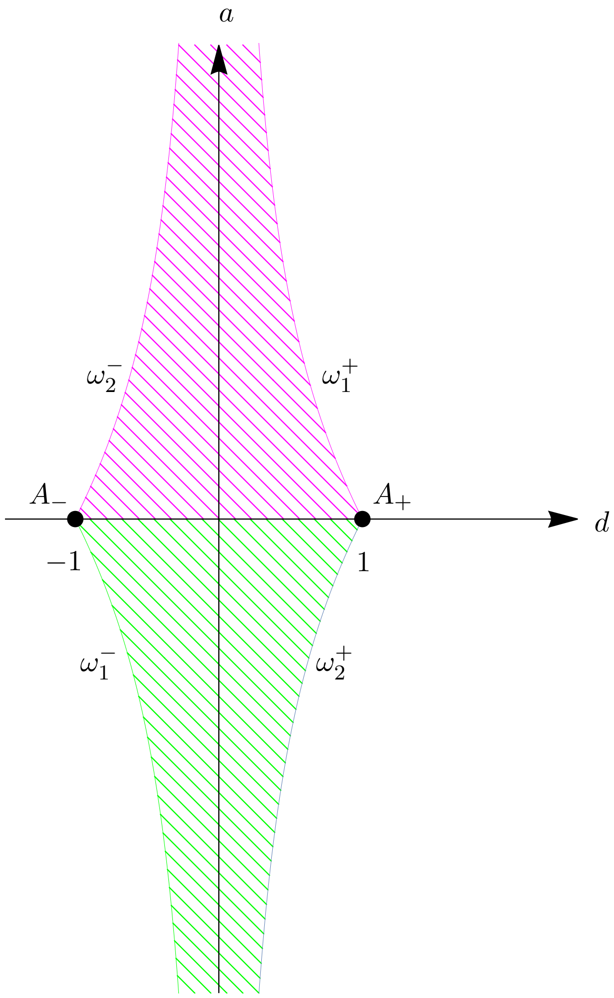

Fig. 3. The set Ω

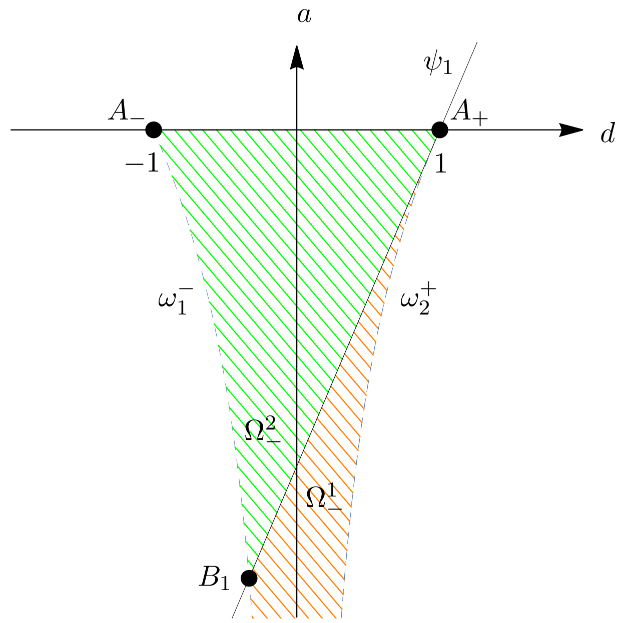

Fig. 4. Parts Ω_-^1 and Ω_-^2 of Ω_-

From the system (2.19), we obtain a quadratic equation

$$\frac{-a^2 d}{(1-d)^2} + (1+d) = a - (1+d).$$

After transformation, we get a quadratic equation

$$-a^2 d - a(d-1)^2 + 2(1+d)(d-1)^2 = 0. \tag{2.20}$$

Let us calculate the discriminant $\mathfrak{D}$ of the quadratic equation (2.20). We obtain that

$$\mathfrak{D} = (3d+1)^2(d-1)^2.$$

The quadratic equation (2.20) has a solution

$$\left[\begin{aligned} a &= \frac{d^2-1}{d}, \\ a &= 2 - 2d. \end{aligned}\right.$$

From here, taking into account (2.19), we obtain that the curve l_{23} consists of two curves: $l_{23} = l_{231} \cup l_{232}$,

$$l_{231}\colon \begin{cases} a = \dfrac{d^2-1}{d}, \\ b = -1 - \dfrac{1}{d}, \end{cases} \qquad l_{232}\colon \begin{cases} a = 2 - 2d, \\ b = -3d + 1. \end{cases}$$

The projection of the curve l_{231} onto the plane $Oda = \{b = 0\}$ coincides with the curve

$$\omega_2: \quad a = d - \frac{1}{d}.$$

The projection of the line l_{232} onto the plane $Oda = \{b = 0\}$ is a line

$$\psi_2: \quad a = 2 - 2d. \tag{2.21}$$

Let's find points of intersection of the curves ψ_2 and ω_2:

$$\begin{cases} a = 2 - 2d, \\ a = \dfrac{d^2 - 1}{d}. \end{cases}$$

We obtain two points: $(1, 0)$ and $\left(-\frac{1}{3}, \frac{8}{3}\right)$. We denote the point $\left(-\frac{1}{3}, \frac{8}{3}\right)$ by B_2.

Let's find points of intersection of the curves ψ_2 and ω_1:

$$\begin{cases} a = 2 - 2d, \\ a = -d + \dfrac{1}{d}. \end{cases}$$

We obtain the point $(1, 0)$. This point coincides with the point A_+.

The line (2.21) divides the domain Ω_+ into two parts: in the lower part Ω_+^1 we have $a < 2-2d$, in the upper part Ω_+^2 we have $a > 2-2d$ (see Fig. 5). The following properties are easy to verify:

(i) if $(d, a) \in \Omega_+^1$, then $g_3(a, d) - g_2(a, d) > 0$; therefore, $g_3(a, d) > g_2(a, d)$;

(ii) if $(d, a) \in \psi_2$, then $g_3(a, d) - g_2(a, d) = 0$; therefore, $g_3(a, d) = g_2(a, d)$;

(iii) if $(d, a) \in \Omega_+^2$, then $g_3(a, d) - g_2(a, d) < 0$; hence $g_3(a, d) < g_2(a, d)$.

From this, it follows that if $(d, a) \in \Omega_+^1 = \operatorname{int}(A_- A_+ B_2)$, then the region $\mathcal{C}$ is bounded from below by the plane γ_2 and bounded from above by the surface γ_3 (we suppose that the b-axis is directed vertically); and if $(d, a) \in \Omega_+ \setminus \Omega_+^1$, then system (2.2)–(2.6) has no solution, and for these values of (d, a) the set $\mathcal{C}$ is empty.

Substituting $A_-(-1, 0)$ into l_{231}, $A_+(1, 0)$ into l_{232}, and $B_2(-1/3, 8/3)$ into l_{231} (or into l_{232}), we find the coordinates b of the points of intersection of the surfaces γ_2 and γ_3. We obtain the points $V_-(-1, 0, 0)$ (2.17), $V_+(1, 0, -2)$ (2.18), and $V_2(-1/3, 8/3, 2)$.

5. Let $(d, a) \in \Omega_0 = (A_-, A_+)$. Then, $a = 0$. Substituting $a = 0$ into $g_1(a, d)$ (or into $g_2(a, d)$) and into $g_3(a, d)$, respectively, we obtain that the stability domain $\mathcal{C}$ is bounded from below by the line $l_{12} = \{a = 0,\ b = -1 - d\}$, and bounded from above by the line $l_4 := \{a = 0,\ b = 1 + d\}$. Substituting the point $A_-(-1, 0)$ into l_{12} and into l_4, we obtain the point $V_-(-1, 0, 0)$ (2.17). Substituting the point $A_+(1, 0)$ into l_{12} and l_4, we obtain the point $V_+(1, 0, -2)$ and the point $\widetilde{V}(1, 0, 2)$. The region $\mathcal{C}$ is constructed in Fig. 7.

So, we have the following theorem.

Theorem 1. *For a linear time-invariant difference equation with delay* (2.1)*, the stability region $\mathcal{C} \subset \mathbb{R}^3$ is defined by the system of inequalities* (2.2)*,* (2.3)*,* (2.4)*,* (2.5)*,* (2.6)*. This set is a connected open set. This set has the following properties.*

(a) *The convex hull of the set $\mathcal{C} = \{(d, a, b)\} \subset \mathbb{R}^3$ is the polyhedron $V_- V_+ V_1 V_2 \widetilde{V}$ with the vertices $V_-(-1, 0, 0)$, $V_+(1, 0, -2)$, $V_1(-1/3, -8/3, 2)$, $V_2(-1/3, 8/3, 2)$, $\widetilde{V}(1, 0, 2)$.*

(b) *The projection $\mathcal{P}(\mathcal{C})$ of the set $\mathcal{C}$ onto the plane $Oda = \{b = 0\}$ forms the interior of the region $A_-B_1A_+B_2$ with vertices $A_-(-1,0)$, $B_1(-1/3,-8/3)$, $A_+(1,0)$, $B_2(-1/3,8/3)$ (see Fig. 6); this region is bounded by the curves*

$$\begin{aligned}
\omega_1^- = A_-B_1: &\quad a = \frac{1-d^2}{d}, \quad d \in [-1,-1/3],\\
\omega_2^- = A_-B_2: &\quad a = \frac{d^2-1}{d}, \quad d \in [-1,-1/3],\\
\psi_2 = B_2A_+: &\quad a = 2-2d, \quad d \in [-1/3,1],\\
\psi_1 = B_1A_+: &\quad a = 2d-2, \quad d \in [-1/3,1].
\end{aligned}$$

(c) *The edge curves of the domain $\mathcal{C}$ are straight lines*

$$\begin{aligned}
l_{12} = V_-V_+: &\quad a = 0, \ \ b = -(1+d), \ \ d \in [-1,1],\\
l_{132} = V_1V_+: &\quad a = 2d-2, \ \ b = -3d+1, \ \ d \in [-1/3,1],\\
l_{232} = V_2V_+: &\quad a = 2-2d, \ \ b = -3d+1, \ \ d \in [-1/3,1],\\
V_+\widetilde{V}: &\quad d = 1, \ \ a = 0, \ \ b \in [-2,2],
\end{aligned}$$

and curved lines

$$\begin{aligned}
l_{131} = V_-V_1: &\quad a = -\frac{d^2-1}{d}, \ \ b = -1-\frac{1}{d}, \ \ d \in [-1,-1/3],\\
l_{231} = V_-V_2: &\quad a = \frac{d^2-1}{d}, \ \ b = -1-\frac{1}{d}, \ \ d \in [-1,-1/3].
\end{aligned}$$

(d) *In the coordinate system (d,a,b):*
if $a \leq 0$, then the set $\mathcal{C}$ is bounded from below by the plane

$$\gamma_1: \quad b = -a-(1+d)$$

(the green front plane in Fig. 7) and bounded from above by the surface

$$\gamma_3: \quad b = \frac{-a^2d}{(1-d)^2} + (1+d)$$

(the blue top surface in Fig. 7);
if $a > 0$, then the set $\mathcal{C}$ is bounded from below by the plane

$$\gamma_2: \quad b = a-(1+d)$$

(the magenta back plane in Fig. 7) and bounded from above by the surface

$$\gamma_3: \quad b = \frac{-a^2d}{(1-d)^2} + (1+d)$$

(the blue top surface in Fig. 7).

§ 3. Construction of the set $\mathcal{A}$

Let us consider the set $\mathcal{A} = \mathcal{FS}_4(0)$. This set is the stability domain of the fourth-order equation

$$x(n+4) + bx(n+2) + cx(n+1) + dx(n) = 0, \quad n \in \mathbb{Z},$$

or the third-order equation with delay

$$x(n+3) + bx(n+1) + cx(n) + dx(n-1) = 0, \quad n \in \mathbb{Z}, \tag{3.1}$$

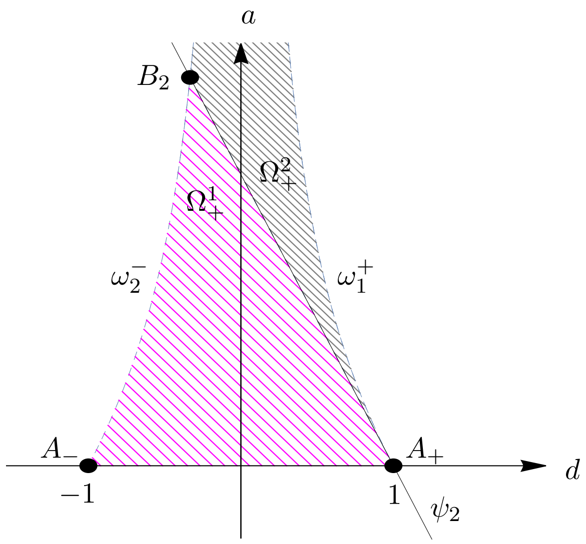

Fig. 5. Parts Ω_+^1 and Ω_+^2 of Ω_+

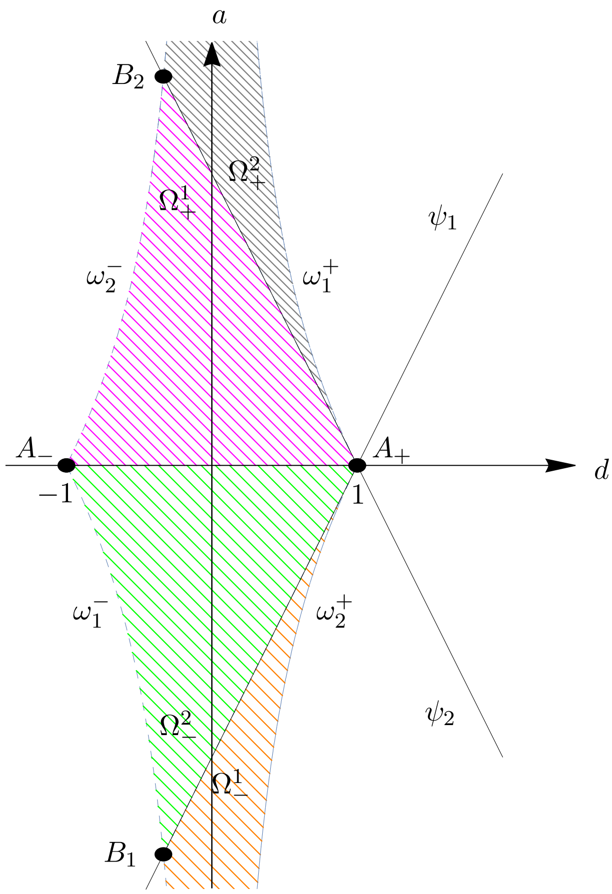

Fig. 6. The projection $\mathcal{P}(\mathcal{C})$ of the region $\mathcal{C}$

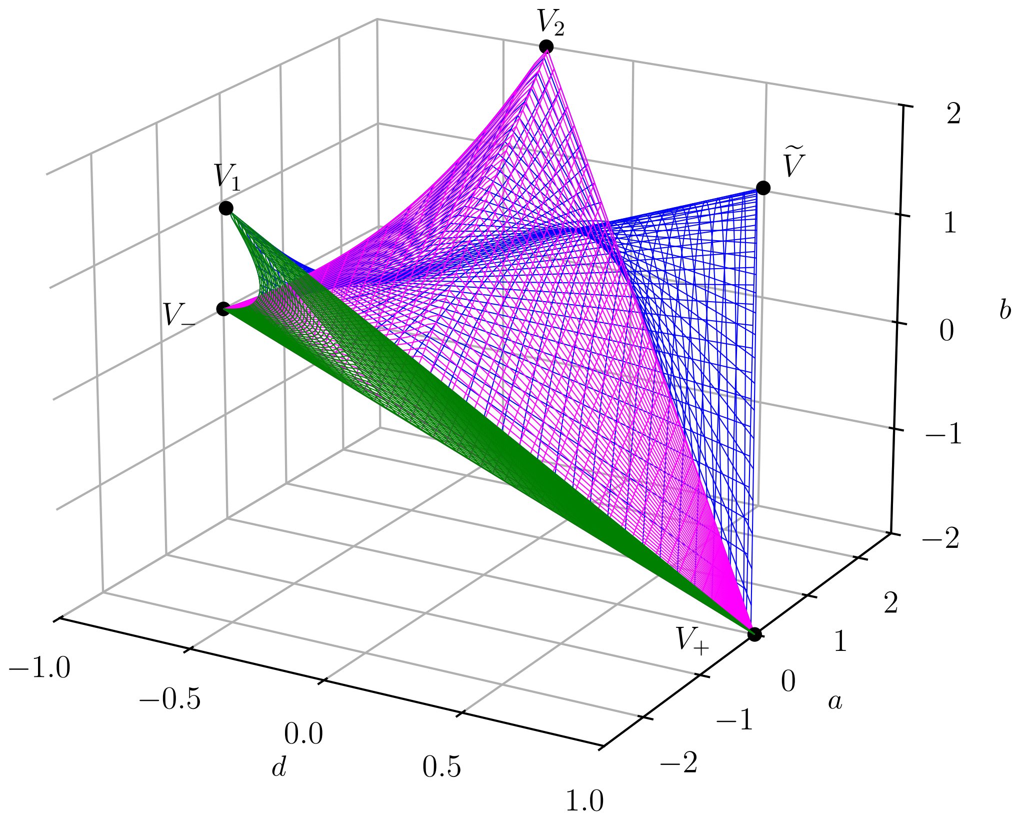

Fig. 7. The region $\mathcal{C}$

in the parameter space $\{(b,c,d)\in\mathbb{R}^3\}$. According to (1.2), (1.3), (1.4), (1.5), (1.6), the stability domain $\mathcal{A}$ is defined by the system of inequalities

$$\begin{cases} b > f_1(c,d) := -c-1-d, & (3.2)\\ b > f_2(c,d) := c-1-d, & (3.3)\\ b < f_3(c,d) := (1+d) - \dfrac{c^2}{(1-d)^2}, & (3.4)\\ d^2-1-c<0, & (3.5)\\ d^2-1+c<0. & (3.6)\end{cases}$$

The stability domain $\mathcal{A}$ is a region in the three-dimensional space $Odcb$ that is bounded by hypersurfaces

$$\beta_1:\quad b=-c-(1+d), \tag{3.7}$$
$$\beta_2:\quad b=c-(1+d), \tag{3.8}$$
$$\beta_3:\quad b=(1+d)-\frac{c^2}{(1-d)^2}, \tag{3.9}$$

as (d,c) run through the set $\Sigma\subset\{(d,c)\in\mathbb{R}^2\}$ defined by inequalites (3.5), (3.6).

1. Consider the curves

$$\sigma_1:\quad d^2-1-c=0, \tag{3.10}$$
$$\sigma_2:\quad d^2-1+c=0. \tag{3.11}$$

From (3.10) and (3.11) it follows

$$\sigma_1:\quad c=d^2-1, \tag{3.12}$$
$$\sigma_2:\quad c=1-d^2. \tag{3.13}$$

The graphs of the functions (3.12) and (3.13) are parabolas. They intersect at the point C_- with coordinates $d=-1$, $c=0$ and at the point C_+ with coordinates $d=1$, $c=0$.

The region Σ bounded by the curves (3.10) and (3.11) is the region enclosed between the parabolas σ_1 and σ_2, containing zero within itself (see Fig. 8).

2. The surfaces β_1 and β_2 are planes. The surface β_3 is a cubic surface. Let's find curves of intersection of the surfaces β_1 and β_2.

Let

$$m_{12} := \beta_1\cap\beta_2 \colon\quad \begin{cases} b=-c-(1+d),\\ b=c-(1+d).\end{cases}$$

We get a straight line

$$m_{12}\colon\ \begin{cases} c=0,\\ b=-(1+d).\end{cases}$$

If $c<0$, then $f_1(c,d)>f_2(c,d)$. Therefore, by (3.2), (3.3), and (3.4), the set $\mathcal{A}$ is bounded by the surfaces β_1 and β_3.

If $c>0$, then $f_2(c,d)>f_1(c,d)$. Therefore, by (3.2), (3.3), and (3.4), the set $\mathcal{A}$ is bounded by the surfaces β_2 and β_3.

Let us represent the domain Σ as follows: $\Sigma=\Sigma_+\cup\Sigma_-\cup\Sigma_0$ (see Fig. 8). Here $\Sigma_+ = \{(d,c)\in\Sigma\colon c>0\}$, $\Sigma_-=\{(d,c)\in\Sigma\colon c<0\}$, $\Sigma_0=\{(d,c)\in\Sigma\colon c=0\}=(C_-,C_+)$.

3. Let $(d, c) \in \Sigma_-$. Then, the stability domain $\mathcal{A}$ is bounded by the surfaces β_1 and β_3. Let's find curves of intersection of surfaces (3.7) and (3.9). Let

$$m_{13} := \beta_1 \cap \beta_3 \colon \quad \begin{cases} b = -c - (1+d), \\ b = (1+d) - \dfrac{c^2}{(1-d)^2}. \end{cases} \tag{3.14}$$

From the system (3.14), we obtain a quadratic equation

$$(1+d) - \frac{c^2}{(1-d)^2} = -c - (1+d).$$

After transformation, we get a quadratic equation

$$c^2 - c(1-d)^2 - 2(1+d)(d-1)^2 = 0. \tag{3.15}$$

Let us calculate the discriminant $\mathfrak{D}$ of the quadratic equation (3.15). We obtain that

$$\mathfrak{D} = (d+3)^2(d-1)^2.$$

The quadratic equation (3.15) has a solution

$$\left[\begin{aligned} & c = d^2 - 1, \\ & c = 2 - 2d. \end{aligned}\right.$$

From here, taking into account (3.14), we obtain that the curve m_{13} consists of two curves: $m_{13} = m_{131} \cup m_{132}$,

$$m_{131} \colon \begin{cases} c = d^2 - 1, \\ b = -d^2 - d, \end{cases} \qquad m_{132} \colon \begin{cases} c = 2 - 2d, \\ b = d - 3. \end{cases}$$

The projection of the curve m_{131} onto the plane $Odc = \{b = 0\}$ coincides with the curve σ_1. The projection of the line m_{132} onto the plane $Odc = \{b = 0\}$ is a line

$$\chi_1 \colon \quad c = 2 - 2d. \tag{3.16}$$

Let's find points of intersection of the curves χ_1 and σ_1: $2 - 2d = d^2 - 1$. We obtain the point $C_+(1, 0)$ and the point X_1 with coordinates $d = -3$, $c = 8$.

Let's find points of the intersection of the curves χ_1 and σ_2: $2 - 2d = 1 - d^2$. We obtain the point $C_+(1, 0)$.

Thus, the line (3.16) does not intersect the region Σ_- (see Fig. 9).

From this, it follows that the function $f_3(c, d) - f_1(c, d)$ preserves its sign in the domain Σ_-. It is easy to verify that if $(d, c) \in \Sigma_-$, then $f_3(c, d) - f_1(c, d) > 0$; therefore, $f_3(c, d) > f_1(c, d)$.

From this, it follows that **for all** $(d, c) \in \Sigma_-$, the stability domain $\mathcal{A}$ is bounded from below by the plane β_1 and bounded from above by the surface β_3 (we assume that the b-axis is directed vertically).

When the parameters (d, c) of the plane $Odc = \{b = 0\}$ are located at the vertices C_-, C_+, we can find the coordinates b of the corresponding points $W_-, W_+ \in \{(d, c, b) \in \mathbb{R}^3\}$ of intersection of the surfaces β_1 and β_3. To do this, we should substitute the coordinates of the points C_- into m_{131} and C_+ into m_{131} (or into m_{132}). We obtain

$$W_-(-1, 0, 0), \tag{3.17}$$

$$W_+(1, 0, -2), \tag{3.18}$$

4. Let $(d, c) \in \Sigma_+$. Then, the stability domain $\mathcal{A}$ is bounded by the surfaces β_2 and β_3. Let's find curves of intersection of surfaces (3.8) and (3.9). Let

$$m_{23} := \beta_2 \cap \beta_3 \colon \quad \begin{cases} b = c - (1+d), \\ b = (1+d) - \dfrac{c^2}{(1-d)^2}. \end{cases} \tag{3.19}$$

From the system (3.19), we obtain a quadratic equation

$$(1+d) - \frac{c^2}{(1-d)^2} = c - (1+d).$$

After transformation, we get a quadratic equation

$$c^2 + c(1-d)^2 - 2(1+d)(d-1)^2 = 0. \tag{3.20}$$

Let us calculate the discriminant $\mathfrak{D}$ of the quadratic equation (3.20). We obtain that

$$\mathfrak{D} = (d+3)^2(d-1)^2.$$

The quadratic equation (3.20) has a solution

$$\left[\begin{array}{l} c = 1 - d^2, \\ c = 2d - 2. \end{array}\right.$$

From here, taking into account (3.19), we obtain that the curve m_{23} consists of two curves: $m_{23} = m_{231} \cup m_{232}$,

$$m_{231} \colon \begin{cases} c = 1 - d^2, \\ b = -d^2 - d, \end{cases} \qquad m_{232} \colon \begin{cases} c = 2d - 2, \\ b = d - 3. \end{cases}$$

The projection of the curve m_{231} onto the plane $Odc = \{b = 0\}$ coincides with the curve σ_2. The projection of the line m_{232} onto the plane $Odc = \{b = 0\}$ is a line

$$\chi_2 \colon \quad c = 2d - 2. \tag{3.21}$$

Let's find points of intersection of the curves χ_2 and σ_2: $2d - 2 = 1 - d^2$. We obtain the point $C_+(1, 0)$ and the point X_2 with coordinates $d = -3$, $c = -8$.

Let's find points of the intersection of the curves χ_2 and σ_1: $2d - 2 = d^2 - 1$. We obtain the point $C_+(1, 0)$.

Thus, the line (3.21) does not intersect the region Σ_- (see Fig. 9).

From this, it follows that the function $f_3(c, d) - f_2(c, d)$ preserves its sign in the domain Σ_+. It is easy to verify that if $(d, c) \in \Sigma_+$, then $f_3(c, d) - f_2(c, d) > 0$; therefore, $f_3(c, d) > f_2(c, d)$.

From this, it follows that **for all** $(d, c) \in \Sigma_+$, the domain $\mathcal{A}$ is bounded from below by the plane β_2 and bounded from above by the surface β_3.

Substituting C_- into m_{231} and C_+ into m_{231} (or into m_{232}), we obtain the points $W_-(-1, 0, 0)$ (3.17), $W_+(1, 0, -2)$ (3.18) of intersection of the surfaces β_2 and β_3.

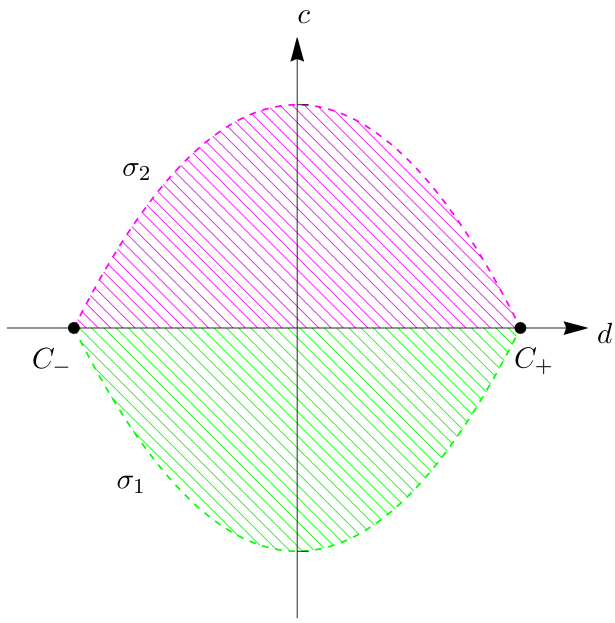

Fig. 8. The set Σ

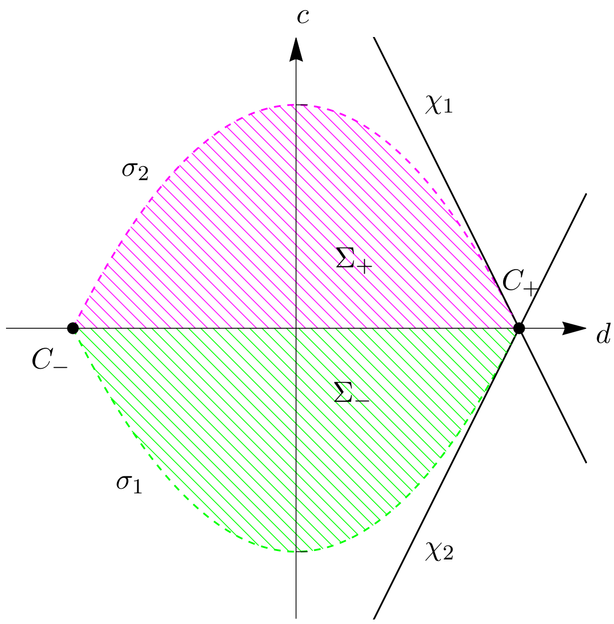

Fig. 9. The projection $\mathcal{P}(\mathcal{A})$ of the region $\mathcal{A}$

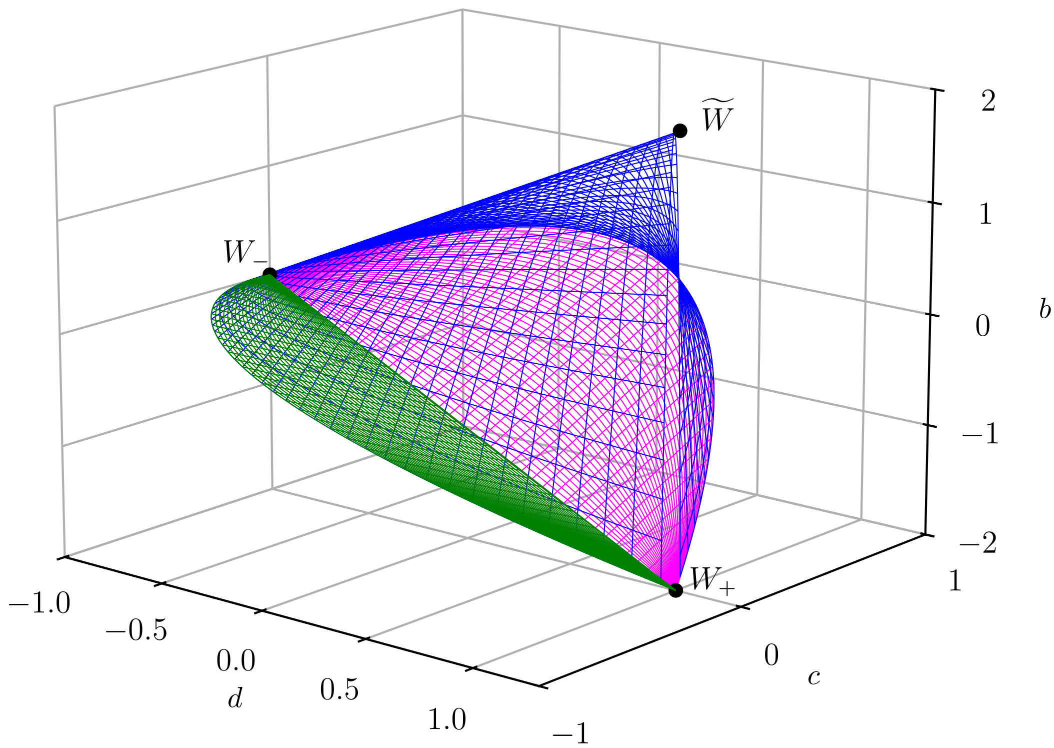

Fig. 10. The region $\mathcal{A}$

5. Let $(d, c) \in \Sigma_0 = (C_-, C_+)$. Then $c = 0$. Substituting $c = 0$ into $f_1(c, d)$ (or into $f_2(c, d)$) and into $f_3(c, d)$, respectively, we obtain that the stability domain $\mathcal{A}$ is bounded from below by the line $m_{12} = \{c = 0,\ b = -1 - d\}$, and bounded from above by the line $m_4 := \{c = 0,\ b = 1 + d\}$. Substituting the point $C_-(-1, 0)$ into m_{12} and into m_4, we obtain the point $W_-(-1, 0, 0)$ (3.17). Substituting the point $C_+(1, 0)$ into m_{12} and m_4, we obtain the point $W_+(1, 0, -2)$ and the point $\widetilde{W}(1, 0, 2)$. The region $\mathcal{A}$ is constructed in Figures 10, 11, 12 from different viewing angles.

So, we have the following theorem.

Theorem 2. *For an autonomous linear difference equation with delay* (3.1)*, the stability region* $\mathcal{A} \subset \mathbb{R}^3$ *is defined by the system of inequalities* (3.2)*,* (3.3)*,* (3.4)*,* (3.5)*,* (3.6)*. This set is a connected open set. This set has the following properties.*

(a) *The projection* $\mathcal{P}(\mathcal{A})$ *of the set* $\mathcal{A}$ *onto the plane* $Odc = \{b = 0\}$ *forms the interior of the region bounded by parabolas*

$$\sigma_1 = C_-C_+ : \quad c = d^2 - 1, \quad d \in [-1, 1],$$
$$\sigma_2 = C_-C_+ : \quad c = 1 - d^2, \quad d \in [-1, 1];$$

(*see Fig.* 9).

(b) *The edge curves of* $\mathcal{A}$ *are straight lines*

$$m_{12} = W_-W_+ : \quad c = 0, \ \ b = -(1 + d), \ \ d \in [-1, 1],$$
$$W_+\widetilde{W} : \quad d = 1, \ \ a = 0, \ \ b \in [-2, 2],$$

and the curved lines m_{131} (*green*) *and* m_{231} (*magenta*)

$$m_{131} = W_-W_+ : \quad a = d^2 - 1, \ \ b = -d^2 - d, \ \ d \in [-1, 1],$$
$$m_{231} = W_-W_+ : \quad a = 1 - d^2, \ \ b = -d^2 - d, \ \ d \in [-1, 1].$$

(c) *In the coordinate system* (d, c, b)*:*
if $c \leq 0$*, then the set* $\mathcal{A}$ *is bounded from below by the plane*

$$\beta_1 : \quad b = -c - (1 + d)$$

(*the green front plane in Fig.* 10) *and bounded from above by the surface*

$$\beta_3 : \quad b = (1 + d) - \frac{c^2}{(1 - d)^2}$$

(*the blue top surface in Fig.* 10);
if $c > 0$*, then the set* $\mathcal{A}$ *is bounded from below by the plane*

$$\beta_2 : \quad c - (1 + d)$$

(*the magenta back plane in Fig.* 10) *and bounded from above by the surface*

$$\beta_3 : \quad b = (1 + d) - \frac{c^2}{(1 - d)^2}$$

(*the blue top surface in Fig.* 10).

(d) *The upper blue surface has the shape of a concave conical surface* (*a surface with concave generators*) *whose directrix curves are* m_{131} *and* m_{132}*; the convex hull of* $\mathcal{A}$ *is the region bounded from below by the planes* β_1 *and* β_2*, and bounded from above by the conical surface* β_4 *with vertex at* $\widetilde{W}$*, with straight line generators and with directrix curves* m_{131} *and* m_{132}.

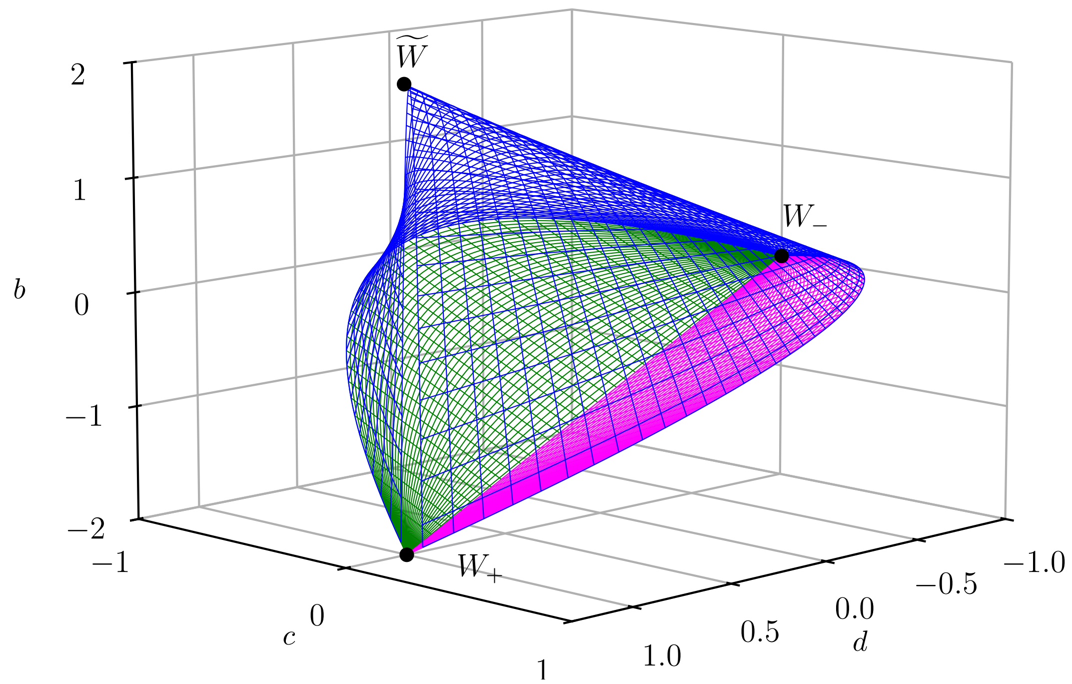

Fig. 11. The region $\mathcal{A}$

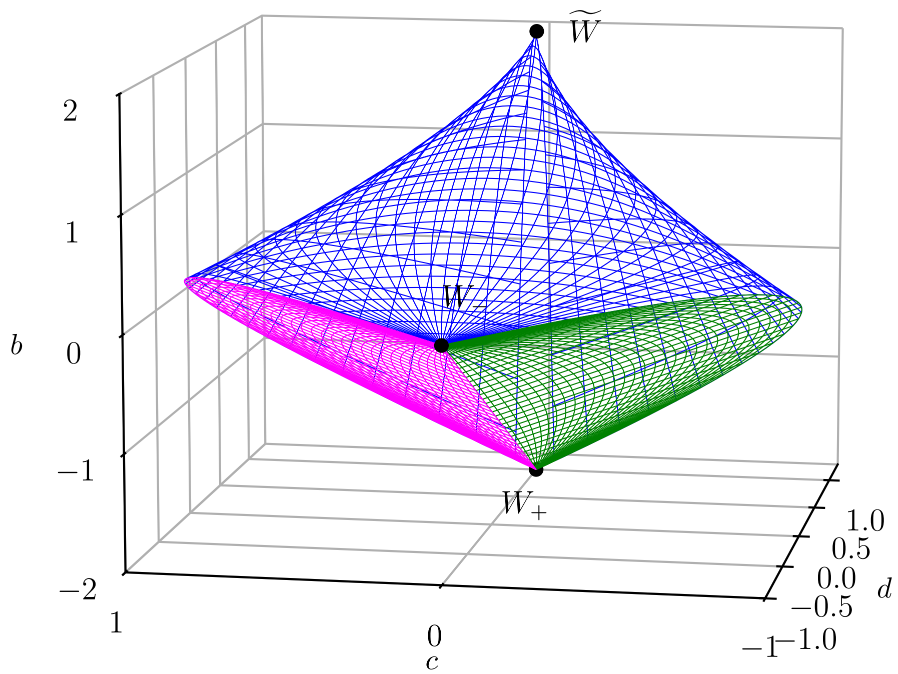

Fig. 12. The region $\mathcal{A}$

§ 4. Lined-structure property

Following [17] we say that a region has a lined-structure property if its boundary is a union of straight line segments, straight half-lines, and straight lines. In [17], sketches of the $\mathcal{HS}_4(0)$ and $\mathcal{FS}_4(0)$ sets were constructed. They were shown to have a lined-structure property. The study was conducted using the D-partitioning method. The following proposition follows from [17, Theorem 3].

Proposition 1. *The set* $\mathcal{HS}_4(0)$ *has a lined-structure property.*

Let us show that this result can be obtained from the results of [28]. In fact, the region $\mathcal{HS}_4(0) = \mathcal{S}_3$ is bounded by the hypersurfaces (22), (23), (24) of [28]. The surfaces (22) and (23) are planes. The surface (24) $b = 1 + ac - c^2$ is a hyperbolic paraboloid. The sections of (24) by planes $c = \text{const}$ are straight lines.

Moreover, from [28] one can obtain a more general statement.

Proposition 2. *For any $d_0 \in (-1, 1)$, the set $\mathcal{HS}_4(d_0)$ has a lined-structure property.*

P r o o f. Let the set $\mathcal{HS}_4(d_0) = \{(a, b, c)\} \subset \mathbb{R}^3$ be given where $d_0 \in (-1, 1)$. Let us make a linear change of variables.

$$a = \widetilde{a} + d_0\widetilde{c}, \qquad c = d_0\widetilde{a} + \widetilde{c}, \qquad b = \widetilde{b}(1 + d_0). \tag{4.1}$$

From the proof of Theorem 1 of [28], it is easy to see that the linear change of variables (4.1) maps the set $\mathcal{HS}_4(d_0) = \{(a, b, c)\} \subset \mathbb{R}^3$ to the set $\mathcal{HS}_4(0) = \{(\widetilde{a}, \widetilde{b}, \widetilde{c})\} \subset \mathbb{R}^3$. Now the assertion of Proposition 2 follows from Proposition 1 and the fact that the change (4.1) is linear. □

Further, the following proposition follows from [17, Theorem 5].

Proposition 3. *The set $\mathcal{A} = \mathcal{FS}_4(0)$ has a lined-structure property.*

Let us show that this result can be obtained from the results of Section 3 here. In fact, the region $\mathcal{A} = \mathcal{FS}_4(0)$ is bounded by the hypersurfaces (3.7), (3.8), (3.9). The surfaces (3.7) and (3.8) are planes. The sections of the surface (3.9) by planes $c = k(d-1)$ (where $k \in (-2, 2)$) are segments of the straight lines $\{c = k(d-1),\ b = (1+d) - k^2\}$.

Let us show that the same statement holds for the set $\mathcal{C}$.

Proposition 4. *The set $\mathcal{C} = \mathcal{GS}_4(0)$ has a lined-structure property.*

In fact, the region $\mathcal{C} = \mathcal{GS}_4(0)$ is bounded by the hypersurfaces (2.7), (2.8), (2.9). The surfaces (2.7) and (2.8) are planes. The sections of the surface (2.9) by planes $a = k(d-1)$ (where $k \in (-2, 2)$) are segments of the straight lines $\{a = k(d-1),\ b = (1+d) - k^2 d\}$.

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Построение трехмерных областей устойчивости для двух линейных разностных уравнений третьего порядка с запаздыванием

Ключевые слова: линейное разностное уравнение, устойчивость по Шуру, область устойчивости.

УДК 517.925.51

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Для линейных разностных уравнений третьего порядка с запаздыванием

$$x(n+3)+ax(n+2)+bx(n+1)+dx(n-1)=0, \quad n\in\mathbb{Z}, \quad x\in\mathbb{R},$$

$$x(n+3)+bx(n+1)+cx(n)+dx(n-1)=0, \quad n\in\mathbb{Z}, \quad x\in\mathbb{R},$$

исследованы и построены области устойчивости в трехмерных пространствах $\{(a,b,d)\}\subset\mathbb{R}^3$ и $\{(b,c,d)\}\subset\mathbb{R}^3$ соответственно.

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