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# $d$-MULTIPLE CAPTURE IN A LINEAR DIFFERENTIAL GAME OF PLAYERS WHOSE CONTROLS ARE SUBJECTED TO MIXED CONSTRAINTS

We study a linear pursuit differential game of multiple capture with one evader $y$ and $m$ pursuers $x_1, x_2, \dots, x_m$ in $\mathbb{R}^n$. The first $k$ coordinates of each player's control function obey geometric constraints, while the remaining coordinates satisfy integral constraints. We say that $d$-multiple capture occurs if distinct pursuers $x_{i_1}, x_{i_2}, \dots, x_{i_d}$, $i_1, i_2, \dots, i_d \in \{1, 2, \dots, m\}$, capture the evader at times $0 < \delta_1 \le \delta_2 \le \dots \le \delta_d$ satisfying $x_{i_j}(\delta_j) = y(\delta_j)$ for all $j = 1, 2, \dots, d$. We establish a sufficient condition for $d$-multiple capture and construct guaranteed pursuit strategies to achieve it.

*Keywords*: differential game, pursuer, evader, strategy, $d$-multiple capture.

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## Introduction

Multi-pursuer differential games are widely studied in the literature (see, for example, [3, 7, 9, 12–15]). In these types of games, the main problem is to find the conditions of evasion or completion of pursuit. Another important aspect is to develop strategies for the players. Such games have attracted significant attention in studies such as [4, 5, 16, 27].

Different versions of these games are examined in the literature. Some studies focus on the case of a single pursuer and a single evader [1,23], while others examine games involving multiple pursuers and multiple evaders [6, 18]. A key aspect in modeling these games is how to limit the movement capabilities of the players. Such limits arise as a result of constraints on the players' controls. The two main types of constraints are geometric (e.g., related to maximum velocity) and integral (e.g., related to limited energy resources). Scott and Leonard [24] study optimal pursuit–evasion control with one speed-limited pursuer and multiple heterogeneous evaders. They show that an optimal pursuer strategy guarantees the finite-time capture of at least one evader, while the others evade using simple reactive rules.

There are many studies, which focus on games with integral constraints on the controls of players such as [9, 10, 22, 28]. Ibragimov et al. [10], explore games with integral constraints and propose pursuit and evasion strategies for the differential game of multiple pursuers and one evader.

The idea of dividing players into groups to achieve capture was explored by Blagodatskikh and Petrov [2], who studied the simultaneous capture of coordinated evaders. They established sufficient and necessary conditions for the simultaneous multiple capture of rigidly coordinated evaders in a nonstationary differential game, where the pursuers use piecewise-programmed counterstrategies. Makkapati and Tsiotras [15] developed optimal strategies for multi-player pursuit-evasion. For multi-pursuer single-evader games, the authors proved that the time-optimal evading strategy depends only on the initial positions of the pursuers. These are the ones that eventually capture the evader simultaneously, under both constant bearing and pure pursuit strategies.

Fang et al. [5] studied cooperative pursuit in scenarios where the evader is faster than the pursuers. They developed a distributed pursuit algorithm that balances encirclement formation and evader approach. This algorithm enables the pursuers to trap a faster, freely moving evader using cooperative control with an adaptive tradeoff between surrounding and chasing actions. Pan and Yuan [17] proposed relay pursuit schemes for multiple pursuers. In particular, they introduced a region-based relay pursuit scheme, which divides the environment into safe regions, where pursuers perform interception, and an escape region, where the evader's motion is restricted. This method enables dynamic role-switching among pursuers to maintain continuous pursuit pressure. Deng et al. [4] extended the pursuit–evasion model to constant flow fields. They derived tight capture conditions, proving that evasion is impossible if the evader's initial position lies within the convex hull of the pursuers' dominance regions under flow dynamics. Explicit bounds on the capture time were also established, based on flow velocities and spatial configurations. Sun et al. [27] investigated pursuit–evasion dynamics in time-varying environments. They derived analytical capture conditions, showing that interception is guaranteed only when the evader's initial position lies within a time-dependent "capture envelope", defined by the joint reachability of the pursuers under flow disturbances.

Other relevant works include Salimi and Ferrara [22], who studied the optimal pursuit of one evader by many pursuers. They showed that capture can be achieved without the need for any individual pursuer to have more energy resources than the evader. Umar et al. [28] proved that capture is guaranteed if the pursuer's energy resource exceeds that of the evader, regardless of the initial positions. The problem of multiple capture of rigidly coordinated evaders was studied by Blagodatskikh and Petrov [2], and Petrov and Solov'eva [18, 19] studied linear recurrent differential games. In addition, optimal strategies for the players are obtained for linear differential games in these three articles.

Additionally, Ramana and Kothari [20] focused on high-speed evaders, and Sakharov [21] looked at multiple capture in Pontryagin's almost periodic example. Ibragimov [8] analyzed group pursuit with geometric constraints, and Selvakumar and Bakolas [25] proposed feedback strategies for reach-avoid games.

Shaunak and Subhash [26] investigated the $k$-capture problem in multi-agent systems, drawing parallels with the "lion and hyenas" scenario. They studied a discrete-time multiple-capture differential game and established that $k$-capture is achievable if and only if the evader's initial position lies in the interior of the $k$-Hull formed by the initial positions of the pursuers. The paper by Yan et al. [30] reviewed multiplayer reach-avoid games with simple motion models. Their work focused on recent developments in multi-agent reach-avoid games played between two adversarial teams. Wang et al. [29] studied cooperative hunting scenarios involving a faster evader. They proposed an initial state distribution strategy for multiple pursuers to successfully capture the fast evader, based on the geometric theory of Apollonius circles.

Ibragimov et al. [11] studied a simple-motion $d$-multiple capture game with mixed constraints. In this work, a linear differential game is considered, in which at least $d$ distinct pursuers capture the evader at $d$ different times. We show that if the pursuers can be grouped such that each group has enough speed and energy advantage over the evader, then $d$-multiple capture is possible.

The present paper contributes to the field by analyzing $d$-multiple capture in a linear differential game where players face both geometric and integral constraints. We provide sufficient conditions under which such capture is possible and construct the pursuers' strategies for achieving it.

## § 1. Statement of problem

The motion of the $m$ pursuers $x_1, x_2, \ldots, x_m$ and the single evader $y$ are described by the following system of linear differential equations:

$$\begin{aligned} &\dot{x}_i = -\mu x_i + u_i, \quad x_i(0) = x_i^0, \quad i = 1, 2, \ldots, m, \\ &\dot{y} = -\mu y + v, \qquad y(0) = y^0, \end{aligned} \tag{1.1}$$

where $x_i, x_i^0, y, y^0, u_i, v \in \mathbb{R}^n$, with distinct initial positions $x_i^0 \neq y^0$, $u_i = (u_{i1}, \dots, u_{in})$ and $v = (v_1, \dots, v_n)$ are the control parameters of the pursuer $x_i$ and evader $y$, respectively, $\mu$ is a given positive number.

The admissible control functions for the players are measurable functions

$$u_i(t) = (u_{i1}(t), \dots, u_{in}(t)), \quad i = 1, 2, \dots, m, \quad v(t) = (v_1(t), v_2(t), \dots, v_n(t))$$

which satisfy the following constraints:

$$\sum_{j=1}^{k} u_{ij}^2(t) \le \varrho_{i1}^2, \quad t \ge 0, \quad i = 1, 2, \dots, m, \tag{1.2}$$

$$\int_0^\infty \sum_{j=k+1}^{n} u_{ij}^2(t)\, dt \le \varrho_{i2}^2, \quad i = 1, 2, \dots, m, \tag{1.3}$$

$$\sum_{j=1}^{k} v_j^2(t) \le \sigma_1^2, \quad t \ge 0, \tag{1.4}$$

$$\int_0^\infty \sum_{j=k+1}^{n} v_j^2(t)\, dt \le \sigma_2^2, \tag{1.5}$$

where $\varrho_{i1}, \varrho_{i2}$, $i = 1, 2, \dots, m$, and $\sigma_1, \sigma_2$ are given positive numbers, $k$ is a given nonnegative integer with $k < n$.

To clarify the constraints, we divide the vector $x \in \mathbb{R}^n$ into two parts: let $x^{(1)}$ and $x^{(2)}$ denote the vectors consisting of the first $k$ and the remaining $n - k$ coordinates, respectively.

Under this notation, conditions (1.2) and (1.4) specify that the maximum speeds of the components $x_i^{(1)}$ and $y^{(1)}$ are limited by $\varrho_{i1}$ and $\sigma_1$, respectively. At the same time, conditions (1.3) and (1.5) impose restrictions on the control energy, requiring that the energy associated with $x_i^{(2)}$ and $y^{(2)}$ does not exceed $\varrho_{i2}^2$ and $\sigma_2^2$, respectively.

**Definition 1.1.** A function

$$U_i(t, x_i, y, v) \colon [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

is said to be a strategy of the pursuer $x_i$ if, for an arbitrary admissible control of the evader $v(t)$, $t \ge 0$, the initial value problem (1.1) admits a unique solution $(x_i(t), y(t))$ with $u_i = U_i(t, x_i, y, v)$ and $v = v(t)$, and this solution satisfies the following constraints:

$$\sum_{j=1}^{k} U_{ij}^2(t, x_i(t), y(t), v(t)) \le \varrho_{i1}^2, \quad t \ge 0,$$

$$\int_0^\infty \sum_{j=k+1}^{n} U_{ij}^2(t, x_i(t), y(t), v(t))\, dt \le \varrho_{i2}^2.$$

The above definition means that, in constructing the strategy of the pursuer $x_i$, the information about the values $t$, $x_i(t)$, $y(t)$, and $v(t)$ available at the current time $t$ is used.

Consider $d$ such that $1 \le d \le m$.

**Definition 1.2.** $d$-multiple capture in the game (1.1)–(1.5) is said to be possible if there exist strategies of the pursuers such that, for any admissible control of the evader,

$$x_{i_1}(\delta_1) = y(\delta_1), \quad \dots, \quad x_{i_d}(\delta_d) = y(\delta_d)$$

for some pairwise distinct numbers $i_1, \dots, i_d \in \{1, \dots, m\}$ and times $0 < \delta_1 \le \dots \le \delta_d$.

If at some time $\delta_1$ the equality $x_{i_1}(\delta_1) = y(\delta_1)$ holds, that is, the evader is captured by the pursuer $x_{i_1}$ at that moment, then the pursuer $x_{i_1}$ is destroyed at the same time. Subsequently, if the evader is captured $d-1$ more times by other distinct pursuers $x_{i_2}, \dots, x_{i_d}$, then by definition a $d$-multiple capture is said to occur.

The pursuers use strategies, while the evader uses any admissible control. The pursuers' aim is to achieve the $d$-multiple capture; on the other hand, the evader's aim is the opposite.

## § 2. Main results

In this section, we present the main results of the article and provide their proofs. Let us introduce the following notations for the first $k$ coordinates that satisfy the geometric constraint

$$\begin{aligned} x_i^{(1)} &= (x_{i1}, x_{i2}, \dots, x_{ik}), & x_{i0}^{(1)} &= (x_{i1}^0, x_{i2}^0, \dots, x_{ik}^0), \\ y^{(1)} &= (y_1, y_2, \dots, y_k), & y_0^{(1)} &= (y_1^0, y_2^0, \dots, y_k^0), \\ u_i^{(1)} &= (u_{i1}, u_{i2}, \dots, u_{ik}), & v^{(1)} &= (v_1, v_2, \dots, v_k). \end{aligned}$$

The coordinates from $k+1$ to $n$ that satisfy the integral constraint are denoted as follows:

$$\begin{aligned} x_i^{(2)} &= (x_{i,k+1}, x_{i,k+2}, \dots, x_{i,n}), & x_{i0}^{(2)} &= (x_{i,k+1}^0, x_{i,k+2}^0, \dots, x_{i,n}^0), \\ y^{(2)} &= (y_{k+1}, y_{k+2}, \dots, y_n), & y_0^{(2)} &= (y_{k+1}^0, \dots, y_n^0), \\ u_i^{(2)} &= (u_{i,k+1}, \dots, u_{i,n}), & v^{(2)} &= (v_{k+1}, \dots, v_n). \end{aligned}$$

According to these notations, $u_i^{(1)}$ and $v^{(1)}$ are the control parameters of the points $x_i^{(1)}$ and $y^{(1)}$, while $u_i^{(2)}$ and $v^{(2)}$ are, respectively, the control parameters of the points $x_i^{(2)}$ and $y^{(2)}$.

The solutions of equations (1.1) can be written as follows:

$$\begin{aligned} x_i(t) &= e^{-\mu t}\left(x_{i0} + \int_0^t e^{\mu s} u_i(s)\,ds\right) = e^{-\mu t}\bar{x}_i(t), \\ y(t) &= e^{-\mu t}\left(y_0 + \int_0^t e^{\mu s} v(s)\,ds\right) = e^{-\mu t}\bar{y}(t), \end{aligned}$$

where we define the auxiliary variables:

$$\bar{x}_i(t) = x_{i0} + \int_0^t e^{\mu s} u_i(s)\,ds, \quad \bar{y}(t) = y_0 + \int_0^t e^{\mu s} v(s)\,ds.$$

Note that the equality $x_i(t) = y(t)$ holds if and only if $\bar{x}_i(t) = \bar{y}(t)$. Therefore, the pursuit problem reduces to making $\bar{x}_i(t)$ coincide with $\bar{y}(t)$.

We prove the following statement.

**Theorem 2.1.** *If there exist pairwise disjoint subsets $H_1, H_2, \dots, H_d$ of the set*

$$H = \big\{i \mid \varrho_{i1} > \sigma_1,\ i \in \{1, \dots, m\}\big\}$$

*such that*

$$\sum_{i \in H_j} \varrho_{i2}^2 > \sigma_2^2, \quad \textit{for each} \ \ j = 1, \dots, d,$$

*then, for any initial state $(y_0, x_{10}, \dots, x_{m0})$, $d$-multiple capture is possible in game (1.1)–(1.5).*

P r o o f. To prove the theorem, we consider the following three possible cases.

**Case 1:** assume $n \geq 2$ and $1 \leq k < n$. From the construction of the sets $H_1, \ldots, H_d$, it follows that, for all $j = 1, \ldots, d$, we have

$$\varrho_{i1} > \sigma_1, \quad i \in H_j, \quad \sum_{i \in H_j} \varrho_{i2}^2 > \sigma_2^2. \tag{2.1}$$

To prove the theorem, it suffices to show that for each $j \in \{1, 2, \ldots, d\}$, there exists an index $i_j \in H_j$ and a time $\delta_j > 0$ such that $x_{i_j}(\delta_j) = y(\delta_j)$. First, we demonstrate this for $H_1$; the same reasoning applies to subsets $H_2, \ldots, H_d$.

For the set $H_1$, (2.1) becomes:

$$\varrho_{i1} > \sigma_1, \quad i \in H_1, \quad \sum_{i \in H_1} \varrho_{i2}^2 > \sigma_2^2.$$

For each $i \in \bigcup_{j=1}^{d} H_j$, we define the following numbers

$$\tau_i = \frac{1}{\mu} \ln\left(1 + \frac{\mu \big|y_0^{(1)} - x_{i0}^{(1)}\big|}{\varrho_{i1} - \sigma_1}\right). \tag{2.2}$$

For pursuers $x_i$, $i \in \bigcup_{j=1}^{d} H_j$, define the first $k$ coordinates of the strategies as follows:

$$U_i^{(1)}(t, x_{i0}, y_0, v(t)) = \begin{cases} \dfrac{\mu\big(y_0^{(1)} - x_{i0}^{(1)}\big)}{e^{\mu\tau_i} - 1} + v^{(1)}(t), & 0 \leq t \leq \tau_i, \\ v^{(1)}(t), & t > \tau_i. \end{cases} \tag{2.3}$$

If $x_{i0}^{(1)} = y_0^{(1)}$, then $\tau_i = 0$ and (2.3) is undefined, and we simply set $U_i^{(1)}(t) = v^{(1)}(t)$ for all $t \geq 0$.

We use (2.2) to verify constraint (1.2) for (2.3):

$$\begin{aligned} \big|U_i^{(1)}(t, x_{i0}, y_0, v(t))\big| &\leq \left|\frac{\mu\big(y_0^{(1)} - x_{i0}^{(1)}\big)}{e^{\mu\tau_i} - 1}\right| + \big|v^{(1)}(t)\big| \\ &\leq \frac{\mu\big|y_0^{(1)} - x_{i0}^{(1)}\big|}{\frac{\mu\left|y_0^{(1)} - x_{i0}^{(1)}\right|}{\varrho_{i1} - \sigma_1}} + \sigma_1 = \varrho_{i1} - \sigma_1 + \sigma_1 = \varrho_{i1}. \end{aligned}$$

Thus, (1.2) is satisfied.

Next, we show that the equality $x_i^{(1)}(t) = y^{(1)}(t)$ holds for all $t \geq \tau_i$. This means that, for each pursuer $x_i$ with $i \in \bigcup_{j=1}^{d} H_j$, the first $k$ coordinates of $x_i$ and those of the evader $y$ coincide for all $t \geq \tau_i$. Indeed, from (2.3) we obtain the following

$$\begin{aligned} \bar{x}_i^{(1)}(t) &= x_{i0}^{(1)} + \int_0^t e^{\mu s} u_i^{(1)}(s)\, ds \\ &= x_{i0}^{(1)} + \int_0^{\tau_i} e^{\mu s} \left(\frac{\mu\big(y_0^{(1)} - x_{i0}^{(1)}\big)}{e^{\mu\tau_i} - 1} + v^{(1)}(s)\right) ds + \int_{\tau_i}^t e^{\mu s} v^{(1)}(s)\, ds \\ &= x_{i0}^{(1)} + \frac{\mu\big(y_0^{(1)} - x_{i0}^{(1)}\big)}{e^{\mu\tau_i} - 1} \cdot \frac{e^{\mu\tau_i} - 1}{\mu} + \int_0^t e^{\mu s} v^{(1)}(s)\, ds \\ &= x_{i0}^{(1)} + y_0^{(1)} - x_{i0}^{(1)} + \int_0^t e^{\mu s} v^{(1)}(s)\, ds = \bar{y}^{(1)}(t). \end{aligned}$$

Thus, $\bar{x}_i^{(1)}(t) = \bar{y}^{(1)}(t)$ for all $t \geq \tau_i$. At the same time, we denote $\tau_0 = \max\limits_{i \in \bigcup_{j=1}^{d} H_j} \tau_i$. Therefore, we obtain,

$$\bar{x}_i^{(1)}(t) = \bar{y}^{(1)}(t) \ \text{ for all } \ t \geq \tau_0 \ \text{ and } \ i \in \bigcup_{j=1}^{d} H_j, \tag{2.4}$$

equivalent to $x_i^{(1)}(t) = y^{(1)}(t)$. In other words, all the points $x_i^{(1)}$, $i \in H_1 \cup H_2 \cup \ldots \cup H_d$, capture the point $y^{(1)}$ at the time $\tau_0$ and keep the equations $x_i^{(1)}(t) = y^{(1)}(t)$ for all $t \geq \tau_0$.

For $0 \leq t \leq \tau_0$, define the strategies of the points $x_i^{(2)}$, $i \in H_1 \cup H_2 \cup \ldots \cup H_d$, and strategies of the points $x_i$, $i \in \{1, 2, \ldots, m\} \setminus \bigcup\limits_{j=1}^{d} H_j$, as follows:

$$U_i^{(2)}(t, x_i(t), y(t), v(t)) = 0, \quad 0 \leq t < \tau_0, \quad i \in \bigcup_{j=1}^{d} H_j,$$

$$U_i(t, x_i(t), y(t), v(t)) = 0, \quad t \geq 0, \quad i \in \{1, 2, \ldots, m\} \setminus \bigcup_{j=1}^{d} H_j.$$

Thus, it follows from the former equation, the points $x_i^{(2)}$, $i \in H_1 \cup \ldots \cup H_d$, do not join the pursuit on $0 \leq t < \tau_0$, so $x_i^{(2)}(\tau_0) = x_{i0}^{(2)}$ for these points. From the latter equation, all pursuers with $i \in \{1, 2, \ldots, m\} \setminus \bigcup\limits_{j=1}^{d} H_j$ stay at rest during the game. Thus, only pursuers $x_i$, $i \in \bigcup\limits_{j=1}^{d} H_j$, can realize the $d$-multiple capture.

Next, for the construction of $U_i^{(2)}(t, x_i, y, v)$ when $t \geq \tau_0$, $i \in H_1$, we denote

$$\varrho_1 = \left( \sum_{i \in H_1} \varrho_{i2}^2 \right)^{1/2}, \quad \sigma_{i2} = \frac{\varrho_{i2}}{\varrho_1} \sigma_2, \quad i \in H_1.$$

Without loss of generality, we assume that $H_1 = \{1, 2, \ldots, l\}$, where $l \leq m$.

Let

$$\beta_1 = \frac{1}{2\mu} \ln \left( 1 + \frac{2\mu \left| \bar{y}^{(2)}(\tau_0) - \bar{x}_{10}^{(2)} \right|^2}{e^{2\mu\tau_0} (\varrho_{12} - \sigma_{12})^2} \right) \tag{2.5}$$

and

$$U_1^{(2)}(t) = \begin{cases} \dfrac{2\mu \left( \bar{y}^{(2)}(\tau_0) - \bar{x}_{10}^{(2)} \right) e^{\mu t}}{e^{2\mu\tau_0} \left( e^{2\mu\beta_1} - 1 \right)} + v^{(2)}(t), & \tau_0 \leq t < \tau_0 + \beta_1, \\ 0, & t \geq \tau_0 + \beta_1, \end{cases} \tag{2.6}$$
$$U_i^{(2)}(t) = 0, \quad \tau_0 \leq t \leq \tau_0 + \beta_1, \quad i \in \bigcup_{j=1}^{d} H_j \setminus \{1\}.$$

If the following inequality

$$\int_{\tau_0}^{\tau_0 + \beta_1} \left| v^{(2)}(t) \right|^2 dt \leq \sigma_{12}^2 \tag{2.7}$$

is satisfied, then, according to (2.6), we obtain

$$\left(\int_{\tau_0}^{\tau_0+\beta_1}\left|U_1^{(2)}(t)\right|^2 dt\right)^{1/2} \le \left(\int_{\tau_0}^{\tau_0+\beta_1}\left|\frac{2\mu\big(\bar y^{(2)}(\tau_0)-\bar x_{10}^{(2)}\big)e^{\mu t}}{e^{2\mu\tau_0}(e^{2\mu\beta_1}-1)}+v^{(2)}(t)\right|^2 dt\right)^{1/2}$$
$$\le \frac{2\mu\big|\bar y^{(2)}(\tau_0)-\bar x_{10}^{(2)}\big|}{e^{2\mu\tau_0}(e^{2\mu\beta_1}-1)}\left(\int_{\tau_0}^{\tau_0+\beta_1} e^{2\mu t}\,dt\right)^{1/2}+\sigma_{12}$$
$$= \frac{2\mu\big|\bar y^{(2)}(\tau_0)-\bar x_{10}^{(2)}\big|}{e^{2\mu\tau_0}(e^{2\mu\beta_1}-1)}\cdot\frac{1}{\sqrt{2\mu}}\big(e^{2\mu(\tau_0+\beta_1)}-e^{2\mu\tau_0}\big)^{1/2}+\sigma_{12}$$
$$= \big|\bar y^{(2)}(\tau_0)-\bar x_{10}^{(2)}\big|\sqrt{\frac{2\mu}{e^{2\mu\tau_0}(e^{2\mu\beta_1}-1)}}+\sigma_{12}.$$

From (2.5) we obtain

$$e^{2\mu\beta_1}-1=\frac{2\mu\big|\bar y^{(2)}(\tau_0)-\bar x_{10}^{(2)}\big|^2}{e^{2\mu\tau_0}(\varrho_{12}-\sigma_{12})^2},$$

so

$$\sqrt{\frac{2\mu}{e^{2\mu\tau_0}(e^{2\mu\beta_1}-1)}}=\frac{\varrho_{12}-\sigma_{12}}{\big|\bar y^{(2)}(\tau_0)-\bar x_{10}^{(2)}\big|}.$$

Substituting this into the inequality above yields

$$\left(\int_{\tau_0}^{\tau_0+\beta_1}\left|U_1^{(2)}(t)\right|^2 dt\right)^{1/2} \le \big|\bar y^{(2)}(\tau_0)-\bar x_{10}^{(2)}\big|\cdot\frac{\varrho_{12}-\sigma_{12}}{\big|\bar y^{(2)}(\tau_0)-\bar x_{10}^{(2)}\big|}+\sigma_{12}$$
$$=\varrho_{12}-\sigma_{12}+\sigma_{12}=\varrho_{12}.$$

In other words, provided that condition (2.7) is satisfied, strategy (2.6) is admissible. Furthermore, from (2.6) we obtain:

$$\bar x_1^{(2)}(\tau_0+\beta_1)-\bar x_{10}^{(2)}=\int_0^{\tau_0+\beta_1} e^{\mu s}u_1^{(2)}(s)\,ds$$
$$=\int_{\tau_0}^{\tau_0+\beta_1} e^{\mu s}\left(\frac{2\mu\big(\bar y^{(2)}(\tau_0)-\bar x_{10}^{(2)}\big)e^{\mu s}}{e^{2\mu\tau_0}(e^{2\mu\beta_1}-1)}+v^{(2)}(s)\right)ds$$
$$=\frac{2\mu\big(\bar y^{(2)}(\tau_0)-\bar x_{10}^{(2)}\big)}{e^{2\mu\tau_0}(e^{2\mu\beta_1}-1)}\int_{\tau_0}^{\tau_0+\beta_1} e^{2\mu s}\,ds+\int_{\tau_0}^{\tau_0+\beta_1} e^{\mu s}v^{(2)}(s)\,ds$$
$$=\frac{2\mu\big(\bar y^{(2)}(\tau_0)-\bar x_{10}^{(2)}\big)}{e^{2\mu\tau_0}(e^{2\mu\beta_1}-1)}\cdot\frac{e^{2\mu(\tau_0+\beta_1)}-e^{2\mu\tau_0}}{2\mu}+\int_{\tau_0}^{\tau_0+\beta_1} e^{\mu s}v^{(2)}(s)\,ds$$
$$=\bar y^{(2)}(\tau_0)-\bar x_{10}^{(2)}+\int_{\tau_0}^{\tau_0+\beta_1} e^{\mu s}v^{(2)}(s)\,ds.$$

Hence,

$$\bar x_1^{(2)}(\tau_0+\beta_1)=\bar x_{10}^{(2)}+\bar y^{(2)}(\tau_0)-\bar x_{10}^{(2)}+\int_{\tau_0}^{\tau_0+\beta_1} e^{\mu s}v^{(2)}(s)\,ds$$
$$=\bar y^{(2)}(\tau_0)+\int_{\tau_0}^{\tau_0+\beta_1} e^{\mu s}v^{(2)}(s)\,ds=\bar y^{(2)}(\tau_0+\beta_1).$$

Therefore, if condition (2.7) holds, the evader is captured at time $\tau_0+\beta_1$, since, by equation (2.4) and the one above, we have $x_1(\tau_0+\beta_1)=y(\tau_0+\beta_1)$.

Let us now assume that (2.7) is not satisfied, i. e.,

$$\int_{\tau_0}^{\tau_0+\beta_1} \left|v^{(2)}(t)\right|^2 dt > \sigma_{12}^2.$$

Then, the strategy given by (2.6) may not satisfy the condition.

$$\int_{\tau_0}^{\tau_0+\beta_1} \left|U_1^{(2)}(t)\right|^2 dt \le \varrho_{12}^2.$$

In other words, the control resource $\varrho_{12}$ of the pursuer $x_1$ may be spent before the time $\tau_0 + \beta_1$, or before the point $x_1^{(2)}$ captures the point $y^{(2)}$. Consequently, the pursuer $x_1$ may not be able to move further. Next, we define the number

$$\beta_2 = \frac{1}{2\mu} \ln\left(1 + \frac{2\mu\left|\bar{y}^{(2)}(\tau_0+\beta_1) - \bar{x}_{20}^{(2)}\right|^2}{e^{2\mu(\tau_0+\beta_1)}(\varrho_{22}-\sigma_{22})^2}\right).$$

We then construct the strategies for pursuers $x_2, x_3, \ldots, x_r$ inductively. It is assumed that the numbers $\tau_0 + \beta_1, \tau_0 + \beta_1 + \beta_2, \ldots, \tau_0 + \beta_1 + \cdots + \beta_r$ are defined and let

$$\int_{\tau_0+\beta_1+\cdots+\beta_{i-1}}^{\tau_0+\beta_1+\cdots+\beta_i} \left|v^{(2)}(t)\right|^2 dt > \sigma_{i2}^2, \quad i = 1, 2, \ldots, r, \quad 1 \le r < l, \quad \beta_0 = 0. \tag{2.8}$$

Subsequently, we define the number

$$\beta_{r+1} = \frac{1}{2\mu} \ln\left(1 + \frac{2\mu\left|\bar{y}^{(2)}(\tau_0+\beta_1+\beta_2+\ldots+\beta_r) - \bar{x}_{r+1,0}^{(2)}\right|^2}{e^{2\mu(\tau_0+\beta_1+\ldots+\beta_r)}(\varrho_{r+1,2}-\sigma_{r+1,2})^2}\right),$$

and construct strategies for the pursuer $x_{r+1}$ and other pursuers $x_i$, $i \in \bigcup_{j=1}^{d} H_j \setminus \{r+1\}$, as follows:

$$U_{r+1}^{(2)}(t) = \begin{cases} \dfrac{2\mu\left(\bar{y}^{(2)}(\tau_0+\beta_1+\ldots+\beta_r) - \bar{x}_{r+1,0}^{(2)}\right)e^{\mu t}}{e^{2\mu(\tau_0+\beta_1+\ldots+\beta_r)}\left(e^{2\mu\beta_{r+1}}-1\right)} + v^{(2)}(t), \\ \qquad \tau_0+\beta_1+\ldots+\beta_r < t \le \tau_0+\beta_1+\cdots+\beta_{r+1}, \\ 0, \qquad t \ge \tau_0+\beta_1+\ldots+\beta_{r+1}. \end{cases} \tag{2.9}$$

$$U_i^{(2)}(t) = 0, \qquad i \in \bigcup_{j=1}^{d} H_j \setminus \{r+1\}.$$

From the discussion above, we conclude that if

$$\int_{\tau_0+\beta_1+\ldots+\beta_r}^{\tau_0+\beta_1+\ldots+\beta_{r+1}} \left|v^{(2)}(t)\right|^2 dt \le \sigma_{r+1,2}^2,$$

then strategy (2.9) guarantees the capture of the evader:

$$x_{r+1}(\tau_0+\beta_1+\ldots+\beta_{r+1}) = y(\tau_0+\beta_1+\cdots+\beta_{r+1}).$$

However, it is impossible for the inequalities (2.8) to be satisfied for all $i = 1, 2, \ldots, l$, because this would result in the following contradiction:

$$\int_0^\infty \left|v^{(2)}(t)\right|^2 dt \ge \sum_{i=1}^{l} \int_{\tau_0+\beta_1+\ldots+\beta_{i-1}}^{\tau_0+\beta_1+\cdots+\beta_i} \left|v^{(2)}(t)\right|^2 dt$$
$$> \sum_{i=1}^{l} \sigma_{i2}^2 = \sum_{i=1}^{l} \frac{\varrho_{i2}^2}{\varrho_1^2}\sigma_2^2 = \sigma_2^2,$$

meaning that $v^{(2)}(t)$ is not admissible. Thus, for some $i_1 \in H_1$, the inequality

$$\int_{\tau_0+\beta_1+\ldots+\beta_{i_1-1}}^{\tau_0+\beta_1+\ldots+\beta_{i_1}} \left|v^{(2)}(t)\right|^2 dt \le \sigma_{i_1 2}^2$$

is satisfied, so the evader is captured by the pursuer $x_{i_1}$.

Similarly, for each $r = 2, 3, \ldots, d$, it can be shown that a pursuer $x_{i_r}$, with $i_r \in H_r$, captures the evader.

**Case 2:** $k = 0$, $n = 1$. Since $k = 0$, only the integral constraints (1.3), (1.5) remain. The sets $H_1, \ldots, H_d$ satisfy the theorem's condition. We set $\tau_0 = 0$ and use the scalar auxiliary variables $\bar{x}_i(t) = e^{\mu t} x_i(t)$, $\bar{y}(t) = e^{\mu t} y(t)$.

Consider a group, for instance $H_1 = \{1, \ldots, l\}$, and set

$$\varrho_1 = \left(\sum_{i=1}^{l} \varrho_{i2}^2\right)^{1/2}, \qquad \sigma_{i2} = \frac{\varrho_{i2}}{\varrho_1}\,\sigma_2.$$

Define

$$\beta_1 = \frac{1}{2\mu} \ln\left(1 + \frac{2\mu\left|\bar{y}(0) - \bar{x}_{10}\right|^2}{(\varrho_{12} - \sigma_{12})^2}\right),$$

and the strategy for pursuer $x_1$ as

$$U_1(t) = \begin{cases} \dfrac{2\mu(\bar{y}(0) - \bar{x}_{10})\, e^{\mu t}}{e^{2\mu\beta_1} - 1} + v(t), & 0 \le t < \beta_1, \\ 0, & t \ge \beta_1, \end{cases}$$

while $U_i(t) = 0$ for all $i \in H_1 \setminus \{1\}$ on $[0, \beta_1]$. If $\int_0^{\beta_1} v^2(t)\, dt \le \sigma_{12}^2$, the same estimate as in Case 1 yields $\int_0^{\beta_1} U_1^2(t)\, dt \le \varrho_{12}^2$ and $\bar{x}_1(\beta_1) = \bar{y}(\beta_1)$ capture occurs.

Otherwise we construct $\beta_2, \beta_3, \ldots$ and the corresponding strategies $U_2, U_3, \ldots$ by the same induction as in the Case 1. If capture did not occur within any of these intervals, summing the inequalities over $i = 1, \ldots, l$ gives

$$\int_0^{\infty} v^2(t)\, dt > \sum_{i=1}^{l} \sigma_{i2}^2 = \sigma_2^2,$$

contradicting the admissibility condition (1.5). Hence some pursuer in $H_1$ captures the evader. Applying the same reasoning to $H_2, \ldots, H_d$ guarantees $d$-multiple capture for $n = 1$.

**Case 3:** $k = 0$, $n \ge 2$. Setting $\tau_0 = 0$ reduces the construction to the integral part of the main proof, with the same definitions of $\beta_1, \beta_2, \ldots$ and the strategies $U_i^{(2)}$. The contradiction argument from Case 1 guarantees a capture in each group $H_j$.

Thus, in all possible cases, $d$-multiple capture is guaranteed. □

## § 3. Conclusions

This work studies the problem of $d$-multiple capture (capturing an evader $d$ distinct times by different pursuers) in a linear differential game with $m$ pursuers and one evader, subject to mixed constraints on players' controls. The first $k$ coordinates of each player's control function obey geometric constraints (speed limits), while the remaining $n - k$ coordinates satisfy integral constraints (energy/resource limits).

We have proposed pursuit strategies to achieve $d$-multiple capture. Unlike the strategies constructed in [11], our developed strategies depend on the parameters of linear differential equations. Also, they depend on the times $\tau_i$ and $\beta_i$, for which we have derived new formulas.

The cooperative strategies constructed for the pursuers consist of coordination in the first $k$ coordinates. The pursuers satisfying the condition $\varrho_{i1} > \sigma_1$ align with the evader during the time interval $\tau_0$, followed by pursuit in the remaining $n-k$ coordinates. If each group $H_j$ satisfies $\sum\limits_{i\in H_j} \varrho_{i2}^2 > \sigma_2^2$ then at least one pursuer in $H_j$ captures the evader. An important observation is that the pursuers should be divided into $d$ pairwise disjoint subsets $H_1, H_2, \dots, H_d$ such that $\bigcup\limits_{j=1}^{d} H_j \subset H$, where $H = \{i \mid \varrho_{i1} > \sigma_1\}$.

The main theorem can be extended by considering the boundary case $\varrho_{i1} = \sigma_1$, $i \in H_j$, for some of the pursuer groups $H_j$, $j \in \{1, 2, \dots, d\}$. For a finite set of points $X \subset \mathbb{R}^k$, denote by $\mathrm{Hull}_q(X)$ its $q$-hull, that is, the set of all points $z \in \mathbb{R}^k$ such that every hyperplane through $z$ leaves at least $q$ points of $X$ in each of the two closed half-spaces determined by that hyperplane. Let $N_j = |H_j|$. Suppose that for some integer $1 \le q_j \le N_j$ the following two conditions hold:

$$y_0^{(1)} \in \operatorname{int} \mathrm{Hull}_{q_j}\{x_{i0}^{(1)} \mid i \in H_j\}, \tag{3.1}$$

and

$$\sum_{i\in I} \varrho_{i2}^2 > \sigma_2^2 \quad \text{for every} \quad I \subseteq H_j, \quad |I| = q_j. \tag{3.2}$$

If the integral resources in $H_j$ are arranged increasingly as

$$\varrho_{j(1),2} \le \varrho_{j(2),2} \le \cdots \le \varrho_{j(N_j),2},$$

then condition (3.2) is equivalent to

$$\sum_{\ell=1}^{q_j} \varrho_{j(\ell),2}^2 > \sigma_2^2.$$

We next show that at least one pursuer in $H_j$ can capture the evader. The condition (3.1) ensures that $q_j$-multiple capture of the evader $y^{(1)}$ by the pursuers $x_i^{(1)}$, $i \in H_j$, is possible. This means that there exist $q_j$ distinct pursuers with indices $i_{j1}, \dots, i_{jq_j} \in H_j$ and times $\tau_{j1} \le \tau_{j2} \le \cdots \le \tau_{jq_j}$ such that

$$x_{i_{jr}}^{(1)}(\tau_{jr}) = y^{(1)}(\tau_{jr}), \qquad r = 1, \dots, q_j.$$

After the first-coordinate meeting of the pursuer $x_{i_{jr}}$ with the evader, set

$$u_{i_{jr}}^{(1)}(t) = v^{(1)}(t), \qquad t \ge \tau_{jr}.$$

Since $\varrho_{i_{jr},1} = \sigma_1$, this control is admissible and preserves

$$x_{i_{jr}}^{(1)}(t) = y^{(1)}(t), \qquad t \ge \tau_{jr}.$$

Let $T_j = \max\limits_{1\le r\le q_j} \tau_{jr}$, and define $I_j^* = \{i_{j1}, \dots, i_{jq_j}\}$. Then, for every $i \in I_j^*$,

$$x_i^{(1)}(t) = y^{(1)}(t), \qquad t \ge T_j.$$

The set $I_j^*$ depends on the evader's control and is not known in advance. This is precisely why the integral-resource assumption is imposed for every $q_j$-element subset of $H_j$. In particular,

$$\sum_{i\in I_j^*} \varrho_{i2}^2 > \sigma_2^2. \tag{3.3}$$

Choose the integral-coordinate controls of the pursuers in $H_j$ to be zero before time $T_j$. Thus, their integral resources remain unspent up to time $T_j$.

Starting from $T_j$, apply to the pursuers in $I_j^*$ the sequential integral-coordinate pursuit construction used in the paper. By (3.3) the corresponding contradiction argument shows that at least one pursuer $i_j \in I_j^*$ must achieve $x_{i_j}^{(2)}(\delta_j) = y^{(2)}(\delta_j)$ at some finite time $\delta_j \geq T_j$. At the same time, $x_{i_j}^{(1)}(\delta_j) = y^{(1)}(\delta_j)$. Hence,

$$x_{i_j}(\delta_j) = y(\delta_j).$$

Therefore, at least one pursuer in $H_j$ can capture the evader.

Although the proposed model is developed for a single evader, it may be generalized to multi-evader systems. In such cases, the study of group-based $d$-multiple capture strategies remains a relevant and important direction.

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*Г. И. Ибрагимов, С. О. Гуломов, Х. Ш. Кушаков*

**$d$-кратная поимка в линейной дифференциальной игре игроков, управления которых подчинены смешанным ограничениям**

*Ключевые слова*: дифференциальная игра, преследователь, убегающий, стратегия, $d$-кратная поимка.

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Мы исследуем линейную дифференциальную игру преследования с множественной поимкой, в которой один убегающий $y$ и $m$ преследователей $x_1, x_2, \ldots, x_m$ движутся в пространстве $\mathbb{R}^n$. Первые $k$ координат управляющих функций каждого игрока подчиняются геометрическим ограничениям, а остальные координаты удовлетворяют интегральным ограничениям. Мы говорим, что происходит $d$-кратная поимка, если различные преследователи $x_{i_1}, x_{i_2}, \ldots, x_{i_d}$, $i_1, i_2, \ldots, i_d \in \{1, 2, \ldots, m\}$, захватывают убегающего в моменты времени $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_d$, удовлетворяющие условиям $x_{i_j}(\delta_j) = y(\delta_j)$ для всех $j = 1, 2, \ldots, d$. Мы устанавливаем достаточное условие для $d$-кратной поимки и строим гарантированные стратегии преследования, обеспечивающие его достижение.

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