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WELL-POSEDNESS OF AN INITIAL-BOUNDARY VALUE PROBLEM FOR A DEGENERATE PARTIAL DIFFERENTIAL EQUATION OF HIGH EVEN ORDER

In the present paper, an initial-boundary problem for a degenerate partial differential equation of high even order in a rectangle is formulated and investigated. Using the method of separation of variables, a spectral problem for an ordinary differential equation is obtained. By constructing the Green's function, the spectral problem is equivalently reduced to the second kind Fredholm integral equation with a symmetric kernel. Based on the theory of integral equations with symmetric kernel, the existence of eigenvalues and a system of eigenfunctions of the spectral problem were established. Moreover, with the help of the derived integral equation and Mercer's theorem, the uniform convergence of some bilinear series involving the obtained eigenfunctions has been proved. The solution to the problem under consideration is presented as the sum of a Fourier series with respect to the system of eigenfunctions of the spectral problem. The uniqueness of the solution is proved using the energy integral method. To demonstrate the continuous dependence of the solution on the given data, appropriate estimates are derived.

Keywords: degenerate equation, initial-boundary value problem, variable separation method, spectral problem, Green function method, integral equation, Fourier series.

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Introduction

Due to numerous applications in science and technology, the theory of partial differential equations is developing rapidly. In particular, the mathematical modeling of many problems — such as the oscillations of rods, plates, shells, beams, etc. — leads to high even-order equations and corresponding boundary value problems for such equations [1–3].

To date, numerous scientific papers have been devoted to the formulation and study of initial-boundary value problems for higher-order equations. For instance, the work [4] focuses on problems involving ultra-hyperbolic equations of higher even order with respect to spatial variables and second order with respect to the time variable in a multidimensional domain. For results concerning the well-posedness of boundary value problems for higher even-order equations, we refer to the works [5, 6] and the references therein.

In addition to equations involving integer-order derivatives, researchers have also paid considerable attention to equations with fractional derivatives. For example, in [7], the initial-boundary value problem is studied for the following equation in a multidimensional domain with a sufficiently smooth boundary:

$$\partial_t^\rho u(x,t) + A(x,D)u(x,t) = f(x,t),$$

where ∂_t^ρ is the Riemann–Liouville differential operator of fractional order $\rho \in (0,1]$, and $A(x,D)$ is an arbitrary positive formally self-adjoint elliptic differential operator of even order.

In the aforementioned works, the authors consider, in a certain sense, general equations with sufficiently smooth coefficients, and the unique solvability of the associated problems is established under more general conditions on the coefficients.

In recent years, equations with specific coefficients, such as degenerate equations, have attracted increasing attention due to their significant theoretical implications and practical applications, including in climate science [8], bending vibrations [9, 10], and other fields. Degenerate equations have been extensively studied by many researchers. In particular, various aspects of these equations have been explored, including inverse problems [11, 12], as well as control and stabilization [9, 10, 13].

§ 1. Formulation of the problem

In the present paper, in a rectangular domain $\Omega = \{(x,t)\colon 0 < x < 1, 0 < t < T\}$, we consider the following degenerate equation of high even order

$$\frac{\partial^2 u}{\partial t^2} + (-1)^n \frac{\partial^n}{\partial x^n}\left(x^\alpha \frac{\partial^n u}{\partial x^n}\right) = f(x,t), \tag{1.1}$$

where $u = u(x,t)$ is an unknown function, $f(x,t)$ is a given sufficiently smooth function, and $\alpha \in \mathbb{R}$ is such that $1 < \alpha < n$, $n \in \mathbb{N}$, $n \geq 2$, and α is not an integer, that is

$$\alpha = \alpha_0 + \varepsilon, \quad \alpha_0 = [\alpha], \quad 0 < \varepsilon < 1,$$

where $[\alpha]$ denotes the integer part of α.

Our aim is to study the existence, uniqueness and continuous dependence of the solution on the given data for the following initial-boundary value problem.

Problem 1. Find a function $u(x,t)$ with the following properties:

1) u_t, $(\partial^j/\partial x^j)u$, $(\partial^j/\partial x^j)[x^\alpha(\partial^n/\partial x^n)u] \in C(\overline{\Omega})$, $j = \overline{0, n-1}$; $(\partial^n/\partial x^n)[x^\alpha(\partial^n/\partial x^n)u]$, $u_{tt} \in C(\Omega)$;

2) it satisfies the equation (1.1) in the domain Ω;

3) it satisfies the following initial conditions

$$u(x,0) = \varphi_1(x), \quad x \in [0,1], \quad u_t(x,0) = \varphi_2(x), \quad x \in [0,1], \tag{1.2}$$

and the boundary conditions

$$\begin{cases} \left.\dfrac{\partial^j}{\partial x^j} u(x,t)\right|_{x=0} = 0, & j = \overline{0,\, n-\alpha_0-1}, \\ \left|\lim_{x\to 0} \dfrac{\partial^j u(x,t)}{\partial x^j}\right| < \infty, & j = \overline{n-\alpha_0,\, n-1}, \quad t \in [0,T]; \\ \left.\dfrac{\partial^j}{\partial x^j} u(x,t)\right|_{x=1} = 0, & j = \overline{n-\alpha_0,\, n-1}, \\ \left.\dfrac{\partial^j}{\partial x^j}\left(x^\alpha \dfrac{\partial^n u(x,t)}{\partial x^n}\right)\right|_{x=1} = 0, & j = \overline{\alpha_0,\, n-1}, \quad t \in [0,T], \end{cases} \tag{1.3}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are sufficiently smooth given functions.

We now provide a brief overview of known results related to problems involving higher-order partial differential equations. In [14], in a rectangular domain of the plane an initial boundary value problem were formulated and studied for the following degenerate higher-order equation

$$B^t_{\gamma-1/2} u + (-1)^k \frac{\partial^k}{\partial x^k}\left(x^\alpha \frac{\partial^k u}{\partial x^k}\right) = f(x,t),$$

where $B^t_{\gamma-1/2} = \dfrac{\partial^2}{\partial t^2} + \dfrac{2\gamma}{t}\dfrac{\partial}{\partial t}$ is the Bessel operator, $0 \leqslant \alpha < 1$, $0 \leqslant \gamma < 1/2$, $k \in \mathbb{N}$. And in [15], for this equation with $\alpha = 0$ boundary value problems with non-local conditions were formulated and studied.

Also, in [16], for the following equation

$$\frac{\partial^{2n}}{\partial x^{2n}}\left(x^{\alpha}\frac{\partial^{2n}u}{\partial x^{2n}}\right)+u_{tt}+\frac{2\gamma}{t}u_t+bu=f(x,t),\quad 0<\alpha<1,$$

an initial boundary value problem involving the values of even-order partial derivatives of the unknown function with respect to spatial variable were studied by applying Fourier's method. Moreover, in the work [17], it was considered higher-even order degenerate partial differential equation with the following form of boundary conditions

$$\left.\frac{\partial^j u}{\partial x^j}\right|_{x=0}=0,\quad \left.\frac{\partial^j u}{\partial x^j}\right|_{x=1}=0,\quad j=\overline{0,k-1}.$$

Also, papers [18,19] are devoted to study the existence and uniqueness of the initial boundary value problems for weakly degenerate partial differential equations.

It should be noted that the above-mentioned works on degenerate partial differential equations have focused on weakly degenerate equations. Problems involving strongly degenerate equations have remained insufficiently studied (see [20]).

The novelty of our approach lies in formulating an initial boundary value problem for a higher-order partial differential equation in the case where the degeneracy degree of the equation may reach half the order of the considered equation.

§ 2. Study of the spectral problem

When the Fourier method is formally applied to the problem, the following equation arises:

$$M[v]\equiv(-1)^n\big(x^{\alpha}v^{(n)}(x)\big)^{(n)}=\lambda v(x),\quad 0<x<1. \tag{2.1}$$

Let us define the domain of the operator M in (2.1), denoted by $D(M)$, as the set of functions satisfying the following conditions:

$$\begin{cases} v^{(j)}(x),\quad \big(x^{\alpha}v^{(n)}(x)\big)^{(j)}\in C[0,1], & j=\overline{0,\,n-1};\\ v^{(j)}(0)=0, & j=\overline{0,\,n-\alpha_0-1},\\ \Big|\lim\limits_{x\to 0}v^{(j)}(x)\Big|<\infty, & j=\overline{n-\alpha_0,\,n-1};\\ v^{(j)}(1)=0, & j=\overline{n-\alpha_0,\,n-1},\\ \left.\big(x^{\alpha}v^{(n)}(x)\big)^{(j)}\right|_{x=1}=0, & j=\overline{\alpha_0,\,n-1}. \end{cases} \tag{2.2}$$

Before moving to the proof of the existence of the eigenvalues and eigenfunctions of the problem (2.1), (2.2), we will prove the following auxiliary lemma.

Lemma 2.1. *Let $\alpha_0=[\alpha]$ and let*

$$g(x)=x^{\alpha}v^{(n)}(x).$$

Assume that

$$v^{(j)}(x),\ g^{(j)}(x)\in C[0,1],\qquad j=0,1,\dots,n-1,$$

and

$$\Big|\lim_{x\to 0}v^{(j)}(x)\Big|<\infty,\qquad j=n-\alpha_0,\dots,n-1.$$

Then,

$$g^{(j)}(0)=0,\qquad j=0,1,\dots,\alpha_0-1.$$

Equivalently,

$$\left.\big(x^{\alpha}v^{(n)}(x)\big)^{(j)}\right|_{x=0}=0,\qquad j=0,1,\dots,\alpha_0-1.$$

P r o o f. We argue by contradiction. Suppose that there exists an integer $j \in \{0, 1, \ldots, \alpha_0 - 1\}$ such that $g^{(j)}(0) \neq 0$, and put

$$m = \min\bigl\{j \in \{0, 1, \ldots, \alpha_0 - 1\} : g^{(j)}(0) \neq 0\bigr\}.$$

Then,

$$g^{(j)}(0) = 0, \quad 0 \le j < m, \quad \text{and} \quad g^{(m)}(0) \neq 0.$$

By Taylor's formula,

$$g(x) = \frac{g^{(m)}(0)}{m!}\, x^m + o(x^m), \quad x \to 0,$$

from which

$$\lim_{x \to 0} \frac{g(x)}{x^m} = \frac{g^{(m)}(0)}{m!} = A \neq 0.$$

Choosing $\varepsilon = \dfrac{|A|}{2}$, by the definition of the limit, there is a $\delta > 0$ such that

$$\left| \frac{g(x)}{x^m} - A \right| < \frac{|A|}{2}, \quad 0 < x < \delta. \tag{2.3}$$

Hence, by the reverse triangle inequality,

$$\left| \frac{g(x)}{x^m} \right| \ge |A| - \left| \frac{g(x)}{x^m} - A \right| > \frac{|A|}{2},$$

that is,

$$|g(x)| \ge \frac{|g^{(m)}(0)|}{2\, m!}\, x^m, \quad 0 < x < \delta. \tag{2.4}$$

Moreover, the inequality (2.3) shows that $g(x)/x^m$ has the same sign as A on $(0, \delta)$; since $x^m > 0$ for $x > 0$, the function $g(x)$ has a constant sign on $(0, \delta)$, and therefore

$$v^{(n)}(x) = \frac{g(x)}{x^\alpha}$$

also has a constant sign on $(0, \delta)$.

Put

$$p = \alpha_0 - m.$$

Since $m \le \alpha_0 - 1$, we have $p \ge 1$. Writing $\alpha = \alpha_0 + \varepsilon_0$ with $0 < \varepsilon_0 < 1$, we have $m - \alpha = -p - \varepsilon_0$, so that, by (2.4),

$$|v^{(n)}(x)| = \frac{|g(x)|}{x^\alpha} \ge C_1\, x^{m-\alpha} = C_1\, x^{-p-\varepsilon_0}, \quad 0 < x < \delta, \tag{2.5}$$

for a suitable constant $C_1 > 0$, and $v^{(n)}(x)$ has a constant sign on $(0, \delta)$.

Fix $x \in (0, \delta)$ and take $0 < \eta < x$. Applying Taylor's formula with the integral form of the remainder to the function $v^{(n-p)}$ on the interval $[\eta, x]$ (equivalently, integrating the Newton–Leibniz identity p times), we obtain

$$v^{(n-p)}(x) = \sum_{k=0}^{p-1} \frac{v^{(n-p+k)}(\eta)}{k!}\, (x - \eta)^k + \frac{1}{(p-1)!} \int_\eta^x (x - t)^{p-1} v^{(n)}(t)\, dt. \tag{2.6}$$

The derivatives included in the sum have orders $n-p+k$ for $k=0,1,\dots,p-1$, i. e., they range over

$$n-p,\ n-p+1,\ \dots,\ n-1.$$

Since $n-p=n-\alpha_0+m$ and $0\le m\le \alpha_0-1$, these orders satisfy

$$n-\alpha_0\le n-p+k\le n-1,\qquad k=0,1,\dots,p-1,$$

which is exactly the range for which the hypotheses of the lemma guarantee that the limits $\lim\limits_{\eta\to+0} v^{(n-p+k)}(\eta)$ exist and are finite. Consequently, for the fixed x, every term of the sum in (2.6) tends to a finite limit as $\eta\to+0$, while $v^{(n-p)}(x)$ is a fixed finite number. Therefore the remainder

$$R(\eta)=\frac{1}{(p-1)!}\int_\eta^x (x-t)^{p-1}v^{(n)}(t)\,dt=v^{(n-p)}(x)-\sum_{k=0}^{p-1}\frac{v^{(n-p+k)}(\eta)}{k!}(x-\eta)^k \tag{2.7}$$

remains bounded as $\eta\to+0$.

On the other hand, since $v^{(n)}$ has a constant sign on $(0,\delta)$, we may pass to absolute values under the integral and use (2.5):

$$|R(\eta)|=\frac{1}{(p-1)!}\int_\eta^x (x-t)^{p-1}\,|v^{(n)}(t)|\,dt\ge\frac{C_1}{(p-1)!}\int_\eta^x (x-t)^{p-1}\,t^{-p-\varepsilon_0}\,dt. \tag{2.8}$$

Since we are interested in the limit $\eta\to+0$, we may assume $0<\eta<x/2$. Keeping only the subinterval $[\eta,x/2]$, on which $x-t\ge x/2$, we obtain

$$\int_\eta^x (x-t)^{p-1}t^{-p-\varepsilon_0}\,dt\ge\left(\frac{x}{2}\right)^{p-1}\int_\eta^{x/2} t^{-p-\varepsilon_0}\,dt=\left(\frac{x}{2}\right)^{p-1}\frac{\eta^{\,1-p-\varepsilon_0}-(x/2)^{\,1-p-\varepsilon_0}}{p+\varepsilon_0-1}. \tag{2.9}$$

Since $p\ge1$ and $\varepsilon_0>0$, we have $p+\varepsilon_0-1>0$, hence, $1-p-\varepsilon_0<0$ and

$$\eta^{\,1-p-\varepsilon_0}\longrightarrow+\infty\quad(\eta\to+0).$$

Combining (2.8)–(2.9), we conclude that

$$|R(\eta)|\longrightarrow+\infty\quad(\eta\to+0),$$

which contradicts the boundedness of $R(\eta)$ established in (2.7).

This contradiction shows that the assumption $g^{(m)}(0)\ne0$ is untenable, so $g^{(m)}(0)=0$. As the choice of $m\in\{0,1,\dots,\alpha_0-1\}$ was arbitrary, we conclude that

$$g^{(j)}(0)=0,\quad j=0,1,\dots,\alpha_0-1,$$

which completes the proof. □

Now, we will define the sign of λ in (2.1). To do this, we assume that there exist eigenfunctions and eigenvalues of the spectral problem (2.1), (2.2). Under this assumption multiplying equation (2.1) by $v(x)$ and integrating over $(0,1)$, and applying integration by parts n times yields

$$\lambda\int_0^1 v^2(x)\,dx=\sum_{k=0}^{n-1}(-1)^k\Big[v^{(k)}(x)\big(x^\alpha v^{(n)}(x)\big)^{(n-1-k)}\Big]_0^1+\int_0^1 x^\alpha\big(v^{(n)}(x)\big)^2\,dx.$$

We show that all boundary terms vanish. For $x = 1$, consider the boundary term

$$v^{(k)}(1)\big(x^\alpha v^{(n)}(x)\big)^{(n-1-k)}\Big|_{x=1}, \quad k = 0, 1, \dots, n-1.$$

If

$$k \geq n - \alpha_0,$$

then, by the boundary conditions,

$$v^{(k)}(1) = 0.$$

On the other hand, if

$$k \leq n - \alpha_0 - 1,$$

then

$$n - 1 - k \geq \alpha_0,$$

and therefore

$$\big(x^\alpha v^{(n)}(x)\big)^{(n-1-k)}\Big|_{x=1} = 0.$$

Hence, for every $k = 0, 1, \dots, n-1$, at least one factor in the product

$$v^{(k)}(x)\big(x^\alpha v^{(n)}(x)\big)^{(n-1-k)}\Big|_{x=1}$$

vanishes. Consequently,

$$v^{(k)}(x)\big(x^\alpha v^{(n)}(x)\big)^{(n-1-k)}\Big|_{x=1} = 0, \quad k = 0, 1, \dots, n-1.$$

Therefore, all boundary terms at $x = 1$ vanish.

At $x = 0$, if $k \leq n - \alpha_0 - 1$, then

$$v^{(k)}(0) = 0$$

by condition (2.2). On the other hand, if $k = n - \alpha_0, \dots, n-1$, then $n - 1 - k \leq \alpha_0 - 1$, and therefore, by Lemma 2.1,

$$\big(x^\alpha v^{(n)}(x)\big)^{(n-1-k)}\Big|_{x=0} = 0.$$

Consequently,

$$\Big[v^{(k)}(x)\big(x^\alpha v^{(n)}(x)\big)^{(n-1-k)}\Big]_0^1 = 0, \quad k = 0, \dots, n-1.$$

Thus,

$$\lambda \int_0^1 v^2(x)\,dx = \int_0^1 x^{-\alpha}\big[x^\alpha v^{(n)}(x)\big]^2\,dx. \tag{2.10}$$

From (2.10), it follows that $\lambda \geq 0$. Assume that $\lambda = 0$. Then, from equation (2.10), we have $v^{(n)}(x) = 0$, $0 < x < 1$. Hence, by virtue of the conditions $v^{(j)}(0) = 0$, $j = \overline{0, n - \alpha_0 - 1}$, $v^{(j)}(1) = 0$, $j = \overline{n - \alpha_0, n-1}$, we conclude that $v(x) \equiv 0$, $0 \leqslant x \leqslant 1$. Consequently, the problem (2.1), (2.2) can have non-trivial solutions only with $\lambda > 0$.

Assuming $\lambda > 0$, the existence of eigenvalues of the problem (2.1), (2.2) will be proved by the method of the Green's function. The Green's function $G(x, s)$ of this problem has the following properties (see [21, § 3, p. 38]):

1) the functions $(\partial^j/\partial x^j)G(x, s)$, $j = \overline{0, n-1}$, and $(\partial^j/\partial x^j)\big[x^\alpha(\partial^n/\partial x^n)G(x, s)\big]$, $j = \overline{0, n-2}$, are continuous for all $x, s \in [0, 1]$;

2) in each of the intervals $[0, s)$ and $(s, 1]$ there exists a continuous derivative of the function $(\partial^{n-1}/\partial x^{n-1})\big[x^\alpha(\partial^n/\partial x^n)G(x,s)\big]$, and at $x = s$ it has a jump:

$$(\partial^{n-1}/\partial x^{n-1})\big[x^\alpha(\partial^n/\partial x^n)G(x,s)\big]\Big|_{x=s-0}^{x=s+0} = (-1)^n;$$

3) the function $G(x, s)$, as a function of the argument x, on each of the intervals $(0, s)$ and $(s, 1)$, satisfies the following homogeneous equation:

$$MG(x,s) = 0, \quad x \in (0,s) \cup (s,1);$$

4) for $s \in (0, 1)$, with respect to x, the function $G(x, s)$ satisfies the following boundary conditions

$$\left.\frac{\partial^j G(x,s)}{\partial x^j}\right|_{x=0} = 0, \quad j = \overline{0, n-\alpha_0-1}, \qquad \left|\lim_{x\to 0}\frac{\partial^j G(x,s)}{\partial x^j}\right| < \infty, \quad j = \overline{n-\alpha_0, n-1}; \tag{2.11}$$

$$\left.\frac{\partial^j G(x,s)}{\partial x^j}\right|_{x=1} = 0, \quad j = \overline{n-\alpha_0, n-1}, \quad \left.\frac{\partial^j}{\partial x^j}\left(x^\alpha\frac{\partial^n G(x,s)}{\partial x^n}\right)\right|_{x=1} = 0, \quad j = \overline{\alpha_0, n-1}. \tag{2.12}$$

Now, we will construct the Green's function of the problem.

Using the form of the general solution of the equation $MG(x, s) = 0$, we will seek the function $G(x, s)$ in the following form in the intervals $(0, s)$ and $(s, 1)$:

$$G(x,s) = \begin{cases} \displaystyle\sum_{j=1}^{n} a_j \frac{x^{2n-\alpha-j}}{(n-j)!(n-\alpha+1-j)_n} + \sum_{j=1}^{n} a_{n+j}\frac{x^{n-j}}{(n-j)!}, & 0 \leqslant x \leqslant s, \\ \displaystyle\sum_{j=1}^{n} b_j \frac{x^{2n-\alpha-j}}{(n-j)!(n-\alpha+1-j)_n} + \sum_{j=1}^{n} b_{n+j}\frac{x^{n-j}}{(n-j)!}, & s \leqslant x \leqslant 1, \end{cases} \tag{2.13}$$

where a_j and $b_j, j = \overline{1, 2n}$, are unknown functions of the variable s, and $(z)_n = z(z + 1) \cdot \cdot (z + 2) \dots (z + n - 1)$ is the Pochhammer symbol. By obeying the function (2.13) to the properties 1) and 2) of the Green function, we obtain the following system of equations with respect to $b_j - a_j$, $j = \overline{1, 2n}$:

$$\begin{cases} b_1 - a_1 = (-1)^n, \\ \displaystyle\sum_{j=1}^{m_1}\frac{s^{m_1-j}}{(m_1-j)!}(b_j - a_j) = 0, \quad m_1 = \overline{2,n}, \\ \displaystyle\sum_{j=1}^{n}\frac{s^{n-\alpha+m_2-j}(b_j-a_j)}{(n-j)!(n-\alpha-j+1)_{m_2}} + \sum_{j=1}^{m_2}\frac{s^{m_2-j}(b_{n+j}-a_{n+j})}{(m_2-j)!} = 0, \quad m_2 = \overline{1,n}. \end{cases}$$

From this system, we find the following relations for $b_j - a_j$ and $b_{n+j} - a_{n+j}$, which are necessary for the construction of the Green's function [16]:

$$b_j - a_j = \frac{(-1)^{n+j-1}s^{j-1}}{(j-1)!}, \quad b_{n+j} - a_{n+j} = \frac{(-1)^{j-1}s^{n-1-\alpha+j}}{(j-1)!(j-\alpha)_n}, \quad j = \overline{1,n} \tag{2.14}$$

Then, substituting (2.13) into the conditions (2.11), we successively obtain

$$\begin{aligned} a_{2n} = a_{2n-1} = a_{2n-2} = \dots = a_{n+\alpha_0+2} = a_{n+\alpha_0+1} = 0, \\ a_n = a_{n-1} = a_{n-2} = \dots = a_{n-\alpha_0+2} = a_{n-\alpha_0+1} = 0. \end{aligned} \tag{2.15}$$

By virtue of these equations, from (2.14) it follows that

$$\begin{aligned} b_{n+j} &= (-1)^{j-1}\frac{s^{n-1-\alpha+j}}{(j-1)!(j-\alpha)_n}, \quad j=\overline{\alpha_0+1,n}, \\ b_j &= (-1)^{n+j-1}\frac{s^{j-1}}{(j-1)!}, \quad j=\overline{n-\alpha_0+1,n}. \end{aligned} \tag{2.16}$$

Now, substituting (2.13) into the second part of the conditions (2.12), we find

$$b_1 = b_2 = \ldots = b_{n-\alpha_0-1} = b_{n-\alpha_0} = 0. \tag{2.17}$$

Then, from (2.14), it follows that

$$a_j = \frac{(-1)^{n+j}s^{j-1}}{(j-1)!}, \quad j=\overline{1,n-\alpha_0}. \tag{2.18}$$

Next, substituting (2.13) into the first part of the conditions (2.12), we obtain

$$\sum_{j=n-\alpha_0+1}^{n}\frac{b_j}{(n-j)!(n-\alpha+1-j)_m} + \sum_{j=1}^{m}\frac{b_{n+j}}{(m-j)!} = 0, \quad m=\overline{1,\alpha_0}. \tag{2.19}$$

Taking into account that b_j, $j=\overline{n-\alpha_0+1,n}$, are known quantities (2.16), from (2.19), we find b_{n+j}, $j=\overline{1,\alpha_0}$:

$$\begin{aligned} b_{n+1} &= -\sum_{j=n-\alpha_0+1}^{n}\frac{(-1)^{n+j-1}s^{j-1}}{(j-1)!(n-j)!(n-\alpha+1-j)}, \\ b_{n+m} &= -\sum_{j=1}^{m-1}\frac{b_{n+j}}{(m-j)!} - \sum_{j=n-\alpha_0+1}^{n}\frac{(-1)^{n+j-1}s^{j-1}}{(j-1)!(n-j)!(n-\alpha+1-j)_m}, \quad m=\overline{2,\alpha_0}. \end{aligned} \tag{2.20}$$

Substituting (2.20) into (2.14), we obtain

$$a_{n+j} = b_{n+j} - (-1)^{j-1}\frac{s^{n-1-\alpha+j}}{(j-1)!(j-\alpha)_n}, \quad j=\overline{1,\alpha_0}. \tag{2.21}$$

Substituting (2.15), (2.16), (2.17), (2.18), (2.20), (2.21) into (2.13), we find the Green function in the following form

$$G(x,s) = \begin{cases} \displaystyle\sum_{j=1}^{n-\alpha_0}\frac{(-1)^{n+j}s^{j-1}x^{2n-\alpha-j}}{(j-1)!(n-j)!(n-\alpha+1-j)_n} \\ \displaystyle\quad + \sum_{j=1}^{\alpha_0}\left(b_{n+j} - \frac{(-1)^{j-1}s^{n-1-\alpha+j}}{(j-1)!(j-\alpha)_n}\right)\frac{x^{n-j}}{(n-j)!}, \quad 0\le x\le s, \\ \displaystyle\sum_{j=n-\alpha_0+1}^{n}\frac{(-1)^{n+j-1}s^{j-1}x^{2n-\alpha-j}}{(j-1)!(n-j)!(n-\alpha+1-j)_n} \\ \displaystyle\quad - \sum_{j=n-\alpha_0+1}^{n}\frac{(-1)^{n+j-1}s^{j-1}}{(j-1)!(n-j)!(n-\alpha+1-j)}\frac{x^{n-1}}{(n-1)!} \\ \displaystyle\quad + \sum_{m=2}^{\alpha_0}\left(-\sum_{j=1}^{m-1}\frac{b_{n+j}}{(m-j)!} - \sum_{j=n-\alpha_0+1}^{n}\frac{(-1)^{n+j-1}s^{j-1}}{(j-1)!(n-j)!(n-\alpha+1-j)_m}\right) \\ \displaystyle\quad \times\frac{x^{n-m}}{(n-m)!} + \sum_{j=\alpha_0+1}^{n}\frac{(-1)^{j-1}s^{n-\alpha-1+j}x^{n-j}}{(j-1)!(n-j)!(j-\alpha)_n}, \quad s\le x\le 1, \end{cases} \tag{2.22}$$

where b_{n+j}, $j=\overline{1,\alpha_0}$, are defined by (2.20).

From (2.22), when $n=2$, we obtain

$$G(x,s)=\begin{cases}-\dfrac{x^{3-\alpha}}{(2-\alpha)_2}+\frac{xs}{1-\alpha}-\dfrac{xs^{2-\alpha}}{(1-\alpha)_2}, & 0\leqslant x\leqslant s,\\ -\dfrac{s^{3-\alpha}}{(2-\alpha)_2}+\frac{sx}{1-\alpha}-\dfrac{sx^{2-\alpha}}{(1-\alpha)_2}, & s\leqslant x\leqslant 1.\end{cases}$$

Moreover, for $n=3$ and $\alpha_0=1$, from (2.22), it follows that

$$G(x,s)=\begin{cases}\dfrac{x^{5-\alpha}}{2!(3-\alpha)_3}-\dfrac{x^{4-\alpha}s}{(2-\alpha)_3}-\dfrac{x^2s^2}{2!2!(1-\alpha)}-\dfrac{x^2s^{3-\alpha}}{2!(1-\alpha)_3}, & 0\leqslant x\leqslant s,\\ \dfrac{s^{5-\alpha}}{2!(3-\alpha)_3}-\dfrac{s^{4-\alpha}x}{(2-\alpha)_3}-\dfrac{s^2x^2}{2!2!(1-\alpha)}-\dfrac{s^2x^{3-\alpha}}{2!(1-\alpha)_3}, & s\leqslant x\leqslant 1.\end{cases}$$

One can easily show that for $n=3$ and $\alpha_0=2$, the Green's function has the form:

$$G(x,s)=\begin{cases}\dfrac{x^{5-\alpha}}{2!(3-\alpha)_3}+\dfrac{x^2s}{2!(2-\alpha)}-\dfrac{x^2s^2}{2!2!(1-\alpha)}-\dfrac{x^2s^{3-\alpha}}{2!(1-\alpha)_3}\\ \qquad -\dfrac{xs}{3-\alpha}+\dfrac{xs^2}{2!(2-\alpha)}+\dfrac{xs^{4-\alpha}}{(2-\alpha)_3}, & 0\leqslant x\leqslant s,\\ \dfrac{s^{5-\alpha}}{2!(3-\alpha)_3}+\dfrac{s^2x}{2!(2-\alpha)}-\dfrac{s^2x^2}{2!2!(1-\alpha)}-\dfrac{s^2x^{3-\alpha}}{2!(1-\alpha)_3}\\ \qquad -\dfrac{sx}{3-\alpha}+\dfrac{sx^2}{2!(2-\alpha)}+\dfrac{sx^{4-\alpha}}{(2-\alpha)_3}, & s\leqslant x\leqslant 1.\end{cases}$$

Consequently, the Green's function that satisfies conditions 1)–4) exists and is unique. Now, we will prove that it is symmetric with respect to its arguments. It is difficult to prove this fact with the help of formula (2.15). So, to do this, we will prove that the operator M is self-adjoint. Then, based on the general theory, the Green function will be symmetric with respect to its arguments. We note that the operator M is symmetric if it satisfies the following equality

$$(Mv,w)=(v,Mw),\quad \forall\, v,w\in D(M).$$

We equip $L_2(0,1)$ with the inner product

$$(v,w)=\int_0^1 v(x)w(x)\,dx.$$

Then,

$$(Mv,w)=(-1)^n\int_0^1\left(x^\alpha v^{(n)}(x)\right)^{(n)}w(x)\,dx.$$

Performing integration by parts $2n$ times in the last equality yields the following result:

$$\begin{aligned}(Mv,w)&=(-1)^n\left\{\sum_{k=0}^{n-1}(-1)^k\left[w^{(k)}(x)\left(x^\alpha v^{(n)}(x)\right)^{(n-1-k)}\right]\Big|_0^1+(-1)^n\int_0^1 x^\alpha w^{(n)}(x)v^{(n)}(x)\,dx\right\}\\ &=(-1)^{n+k}\sum_{k=0}^{n-1}\left[w^{(k)}(x)\left(x^\alpha v^{(n)}(x)\right)^{(n-1-k)}\right]\Big|_0^1+\int_0^1 x^\alpha w^{(n)}(x)v^{(n)}(x)\,dx\\ &=\sum_{k=0}^{n-1}(-1)^{n+k}\left[w^{(k)}(x)\left(x^\alpha v^{(n)}(x)\right)^{(n-1-k)}\right]\Big|_0^1+\sum_{k=0}^{n-1}(-1)^k\left[\left(x^\alpha w^{(n)}(x)\right)^{(k)}v^{(n-1-k)}(x)\right]\Big|_0^1\\ &\quad+(-1)^n\int_0^1\left(x^\alpha w^{(n)}(x)\right)^{(n)}v(x)\,dx.\end{aligned}$$

Since $v, w \in D(M)$ in the resulting equality, we arrive at the following:

$$(Mv, w) = (-1)^n \int_0^1 \left(x^\alpha w^{(n)}(x)\right)^{(n)} v(x)\, dx = (v, Mw).$$

Therefore, the operator M is self-adjoint in $L_2(0,1)$.

Thus, the spectral problem (2.1), (2.2) is self-adjoint, hence, the Green's function is symmetric with respect to its arguments.

The boundary value problem (2.1), (2.2) is equivalent to the following homogeneous Fredholm integral equation of the second kind:

$$v(x) = \lambda \int_0^1 G(x,s)v(s)\, ds. \tag{2.23}$$

The proof follows the same line of reasoning as that of Theorem 2 in [21, § 3, pp. 40–42]. Although the boundary conditions considered there differ from (2.2), the argument remains valid in the present case, since it uses only Properties 1)–4) of the Green function established above. Repeating the proof of Theorem 2 with the Green function constructed here, one verifies that the function

$$v(x) = \int_0^1 G(x,s)f(s)\, ds$$

satisfies

$$M[v] = f$$

together with the boundary conditions (2.2). Therefore, setting

$$f(x) = \lambda v(x),$$

we arrive at the homogeneous Fredholm integral equation (2.23).

This equivalence is established because $G(x,s)$ is the Green's function of the differential operator M under the specified boundary conditions, which allows the differential operator to be represented as an inverse integral operator. Since the kernel $G(x,s)$ is real, symmetric (as shown above) and continuous, by the theory of symmetric integral equations the corresponding integral operator possesses a countable set of characteristic numbers $|\lambda_1| \leqslant |\lambda_2| \leqslant \ldots \leqslant |\lambda_k| \leqslant \ldots$, (see [22, § 26, pp. 140–145]), and, by the Hilbert–Schmidt theorem (see [22, § 27, Theorem 1, p. 150]), the corresponding system of eigenfunctions is complete in $L_2(0,1)$. Recall that, as shown above, every eigenvalue satisfies $\lambda > 0$; hence the eigenvalues can be arranged as

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots \leqslant \lambda_k < \ldots, \quad \lambda_k \to +\infty,$$

and the corresponding system of eigenfunctions $v_1(x), v_2(x), v_3(x), \ldots, v_k(x), \ldots$ forms an orthonormal system in space $L_2(0,1)$ [22].

In addition, it is not difficult to verify that the system of functions $s^{\alpha/2} v_k^{(n)}(s)/\sqrt{\lambda_k}$, $k = 1, 2, \ldots$, also forms an orthonormal system in $L_2(0,1)$.

The following lemma demonstrates which type of functions can be expanded into series by the system of eigenfunctions of the problem (2.1), (2.2).

Lemma 2.2. *Let the function $g(x)$ satisfy the conditions (2.2) and let $Mg(x) \in C(0,1) \cap L_2(0,1)$. Then, the function $g(x)$ can be expanded on a segment $[0,1]$ into an absolutely and uniformly converging series by the system of eigenfunctions of the problem (2.1), (2.2).*

P r o o f. First, we will show the validity of the following equality under the assumptions of Lemma 2.2:

$$\int_0^1 G(x,s)Mg(s)\,ds = (-1)^n \int_0^1 G(x,s)\left[s^\alpha g^{(n)}(s)\right]^{(n)} ds = g(x).$$

Indeed, considering the properties of the functions $G(x,s)$ and $g(x)$, we have

$$(-1)^n \int_0^1 G(x,s)\left[s^\alpha g^{(n)}(s)\right]^{(n)} ds$$
$$= (-1)^n \left\{ \sum_{k=0}^{n-1} (-1)^k \left[G_s^{(k)}(x,s)\left[s^\alpha g^{(n)}(s)\right]^{(n-1-k)} \right]\Big|_0^1 + (-1)^n \int_0^1 s^\alpha G_s^{(n)}(x,s) g^{(n)}(s)\,ds \right\}$$
$$= \sum_{k=0}^{n-1} (-1)^{n+k} \left[G_s^{(k)}(x,s)\left[s^\alpha g^{(n)}(s)\right]^{(n-1-k)} \right]\Big|_0^1 + \int_0^1 s^\alpha G_s^{(n)}(x,s) g^{(n)}(s)\,ds$$
$$= \sum_{k=0}^{n-1} (-1)^{n+k} \left[G_s^{(k)}(x,s)\left[s^\alpha g^{(n)}(s)\right]^{(n-1-k)} \right]\Big|_0^1 + \sum_{k=0}^{n-1} (-1)^k \left[\left[s^\alpha G_s^{(n)}(x,s)\right]^{(k)} g^{(n-1-k)}(s) \right]\Big|_0^1$$
$$+ (-1)^n \int_0^1 \left[s^\alpha G_s^{(n)}(x,s)\right]^{(n)} g(s)\,ds.$$

By the properties of the Green function, we have

$$(-1)^n \int_0^1 G(x,s)\left[s^\alpha g^{(n)}(s)\right]^{(n)} ds = (-1)^{n-1} \left[g(s)\left(s^\alpha G_s^{(n)}(x,s)\right)^{(n-1)} \right]\Big|_{s=0}^{s=x-0}$$
$$- (-1)^{n-1} \left[g(s)\left(s^\alpha G_s^{(n)}(x,s)\right)^{(n-1)} \right]\Big|_{s=x+0}^{s=1} + (-1)^n \int_0^1 g(s)\left(s^\alpha G_s^{(n)}(x,s)\right)^{(n)} ds = g(x),$$

since the following equality holds true by virtue of the second property of the Green's function:

$$\left[\left(s^\alpha G_s^{(n)}(x,s)\right)^{(n-1)} \right]\Big|_{s=x-0}^{s=x+0} = (-1)^n.$$

Consequently, $g(x)$ is a function that can be represented through the kernel $G(x,s)$. Moreover, one can easily show that the validity of the following inequality:

$$\int_0^1 G^2(x,s)\,ds \le C_2 = \text{const} < +\infty.$$

Then, by the Hilbert–Schmidt theorem (see [22, § 27, Theorem 1, p. 150]), the function $g(x)$ on the interval $[0,1]$ is expanded into a uniformly convergent Fourier series in terms of the system of eigenfunctions $\{v_k(x)\}_{k=1}^{+\infty}$. Lemma 2.2 has been proved. □

Now, we will prove the convergence of certain series related to the eigenvalues and eigenfunctions of the spectral problem (2.1), (2.2).

Lemma 2.3. *The following series converge uniformly on a segment* $[0,1]$*:*

$$\sum_{k=1}^{+\infty} \frac{\left[v_k^{(j)}(x)\right]^2}{\lambda_k}, \qquad \sum_{k=1}^{+\infty} \frac{\left(\left[x^\alpha v_k^{(n)}(x)\right]^{(j)}\right)^2}{\lambda_k^2}, \qquad j = \overline{0, n-1}. \tag{2.24}$$

P r o o f. Considering the equation (2.1) and the properties $G(x,s)$ of the function, from (2.23) taking $v(x) \equiv v_k(x)$, we obtain

$$v_k^{(j)}(x) = \lambda_k \int_0^1 \frac{\partial^j}{\partial x^j} G(x,s) v_k(s)\, ds = (-1)^n \int_0^1 \left[s^\alpha v_k^{(n)}(s)\right]^{(n)} \frac{\partial^j}{\partial x^j} G(x,s)\, ds, \quad j = \overline{0, n-1}.$$

Hence, applying the rule of integration by parts n times, and then taking the conditions (2.2) into account, we have

$$v_k^{(j)}(x) = \int_0^1 s^\alpha v_k^{(n)}(s) \frac{\partial^{n+j}}{\partial x^j \partial s^n} G(x,s)\, ds, \quad j = \overline{0, n-1}.$$

Hence, by virtue of $\lambda_k > 0$, it follows the equality

$$\frac{v_k^{(j)}(x)}{\sqrt{\lambda_k}} = \int_0^1 \left(s^{\alpha/2} \frac{\partial^{n+j}}{\partial x^j \partial s^n} G(x,s)\right)\left(\frac{s^{\alpha/2} v_k^{(n)}(s)}{\sqrt{\lambda_k}}\right) ds, \quad j = \overline{0, n-1}. \tag{2.25}$$

From (2.25), it follows that $v_k^{(j)}(x)/\sqrt{\lambda_k}$ is the Fourier coefficient of the function $s^{\alpha/2} \dfrac{\partial^{n+j}}{\partial x^j \partial s^n} G(x,s)$ by the orthonormal system of the functions $\left\{s^{\alpha/2} v_k^{(n)}(s)/\sqrt{\lambda_k}\right\}_{k=1}^{+\infty}$. Therefore, according to the Bessel inequality (see: [22, pp. 23–24]), we have

$$\begin{aligned} \sum_{k=1}^{+\infty} \frac{\left[v_k^{(j)}(x)\right]^2}{\lambda_k} &\leqslant \int_0^1 s^\alpha \left[\frac{\partial^{n+j}}{\partial x^j \partial s^n} G(x,s)\right]^2 ds \\ &\leqslant \int_0^1 s^{-\alpha} \left[\frac{\partial^j}{\partial x^j}\left(s^\alpha \frac{\partial^n}{\partial s^n} G(x,s)\right)\right]^2 ds, \quad j = \overline{0, n-1}. \end{aligned} \tag{2.26}$$

By virtue of $s^\alpha \dfrac{\partial^n G(x,s)}{\partial s^n} = s^{\alpha_0} O(1)$, the integral in the right-hand side of inequality (2.26) is uniformly bounded for $j = \overline{0, n-1}$, from which it follows that the first series in (2.24) converge uniformly.

The convergence of the remained series can be proved similarly.

Lemma 2.3 has been proved. □

Lemma 2.4. *Let the following conditions be fulfilled:*

$$g^{(j)}(x) \in C[0,1], \quad j = \overline{0, n-1}, \qquad x^{\alpha/2} g^{(n)}(x) \in C(0,1) \cap L_2(0,1),$$

$$g^{(j)}(0) = 0, \quad j = \overline{0, n-\alpha_0-1}, \qquad g^{(j)}(1) = 0, \quad j = \overline{n-\alpha_0, n-1}.$$

Then, the following inequality holds true:

$$\sum_{k=1}^{+\infty} \lambda_k g_k^2 \leqslant \int_0^1 x^\alpha \left[g^{(n)}(x)\right]^2 dx. \tag{2.27}$$

In particular, the series on the left side converges, where $g_k = \int_0^1 g(x) v_k(x)\, dx$, $k \in \mathbb{N}$.

P r o o f. Using the equation (2.1), we have

$$\lambda_k^{1/2} g_k = \lambda_k^{1/2} \int_0^1 g(x) v_k(x)\, dx = (-1)^n \lambda_k^{-1/2} \int_0^1 g(x) \left[x^\alpha v_k^{(n)}(x)\right]^{(n)} dx.$$

From this equation, applying the rule of integration by parts times n and taking into account the properties of the functions $g(x)$ and $v_k(x)$, we get

$$\lambda_k^{1/2} g_k = \int_0^1 \{x^{\alpha/2} g^{(n)}(x)\}\{\lambda_k^{-1/2} x^{\alpha/2} v_k^{(n)}(x)\}\, dx.$$

Hence, it follows that $\lambda_k^{1/2} g_k$ is the Fourier coefficient of the function $x^{\alpha/2} g^{(n)}(x)$ by the orthonormal system $\left\{x^{\alpha/2} v^{(n)}(x)/\sqrt{\lambda_k}\right\}_{k=1}^{+\infty}$. Then, applying the Bessel inequality (see: [22, pp. 23–24]), we have the validity of the inequality (2.27).

Lemma 2.4 has been proved. □

Lemma 2.5. *Let the function $g(x)$ satisfy the condition* (2.2) *and $Mg(x) \in C(0,1) \cap L_2(0,1)$. Then, the following inequality holds true*

$$\sum_{k=1}^{+\infty} \lambda_k^2 g_k^2 \leqslant \int_0^1 \big[Mg(x)\big]^2\, dx. \tag{2.28}$$

P r o o f. Using the formula for g_k and the equation (2.1), we have

$$\lambda_k g_k = \lambda_k \int_0^1 g(x) v_k(x)\, dx = (-1)^n \int_0^1 g(x) \big[x^\alpha v_k^{(n)}(x)\big]^{(n)}\, dx.$$

Hence, applying the rule of integration by parts $2n$ times to the integral on the right-hand side of the latter equality, and considering the properties of the functions $g(x)$ and $v_k(x)$, we obtain

$$\lambda_k g_k = (-1)^n \int_0^1 \big[x^\alpha g^{(n)}(x)\big]^{(n)} v_k(x)\, dx = \int_0^1 \big[Mg(x)\big] v_k(x)\, dx.$$

From the last, it follows that $\lambda_k g_k$ is the Fourier coefficient of the function $Mg(x)$ by the orthonormal system of functions $\{v_k(x)\}_{k=1}^{+\infty}$. Then, based on Bessel inequality (see: [22, pp. 23–24]), it follows the inequality (2.28).

The proof of Lemma 2.5 is complete. □

Similarly, one can prove the following lemma.

Lemma 2.6. *Let the function $g(x)$ satisfy the conditions* (2.2) *and $\big[Mg(x)\big]^{(j)} \in C[0,1]$, $j = \overline{0, n-1}$, $x^{\alpha/2}\big[Mg(x)\big]^{(n)} \in C(0,1) \cap L_2(0,1)$, $\big[Mg(x)\big]^{(j)}\big|_{x=0} = 0$, $j = \overline{0, n-\alpha_0-1}$, $\big[Mg(x)\big]^{(j)}\big|_{x=1} = 0$, $j = \overline{n-\alpha_0, n-1}$.*

Then, the following inequality holds true:

$$\sum_{k=1}^{+\infty} \lambda_k^3 g_k^2 \leqslant \int_0^1 x^\alpha \big\{\big[Mg(x)\big]^{(n)}\big\}^2\, dx.$$

§ 3. Existence, uniqueness and continuous dependence of the solution on the given data

In this section, we establish the existence, uniqueness, and the continuous dependence of the solution of Problem 1 on the given data. We will seek a solution to the problem in the following form

$$u(x,t) = \sum_{k=1}^{+\infty} u_k(t) v_k(x), \tag{3.1}$$

where $v_k(x)$, $k \in \mathbb{N}$, are the eigenfunctions of the problem (2.1), (2.2), and $u_k(t)$, $k \in \mathbb{N}$, are unknown functions to be defined.

Substituting (3.1) into equation (1.1) and into the initial condition (1.2), we have the following problem with respect to $u_k(t)$:

$$u_k''(t) + \lambda_k u_k(t) = f_k(t), \quad t \in (0,T), \quad k \in \mathbb{N},$$
$$u_k(0) = \varphi_{1k}, \quad u_k'(0) = \varphi_{2k},$$

where

$$\varphi_{jk} = \int_0^1 \varphi_j(x) v_k(x)\,dx, \quad j = \overline{1,2}; \qquad f_k(t) = \int_0^1 f(x,t) v_k(x)\,dx, \quad k \in \mathbb{N}.$$

It is known that the solution of the last problem exists, unique and is determined by the following formula:

$$u_k(t) = \varphi_{1k} \cos\Big(t\sqrt{\lambda_k}\Big) + \varphi_{2k}\lambda_k^{-1/2} \sin\Big(t\sqrt{\lambda_k}\Big) + \lambda_k^{-1/2} \int_0^t f_k(\tau) \sin\Big[(t-\tau)\sqrt{\lambda_k}\Big]\,d\tau, \quad 0 \leqslant t \leqslant T. \tag{3.2}$$

Hence, the following estimate easily follows

$$|u_k(t)| \leqslant |\varphi_{1k}| + \frac{1}{\sqrt{\lambda_k}}|\varphi_{2k}| + \frac{\sqrt{T}}{\sqrt{\lambda_k}}\sqrt{\int_0^T f_k^2(\tau)\,d\tau}, \quad 0 \leqslant t \leqslant T. \tag{3.3}$$

Theorem 3.1. *Assume that the function $\varphi_1(x)$ satisfies the conditions of Lemma 2.6, the function $\varphi_2(x)$ satisfies the conditions of Lemma 2.5, and the function $f(x,t)$ satisfies the conditions of Lemma 2.5 on the argument x uniformly over t. Then the series (3.1), whose coefficients are defined by (3.2), will be a solution to Problem 1.*

P r o o f. To do this, it is necessary to prove the uniform convergence of the series (3.1) and the following series, formally obtained from (3.1) in $\overline{\Omega}$:

$$u_t(x,t) = \sum_{k=1}^{+\infty} u_k'(t) v_k(x), \quad \frac{\partial^j u(x,t)}{\partial x^j} = \sum_{k=1}^{+\infty} u_k(t) v_k^{(j)}(x), \quad j = \overline{1, n-1},$$
$$\frac{\partial^j}{\partial x^j}\left(x^\alpha \frac{\partial^n u(x,t)}{\partial x^n}\right) = \sum_{k=1}^{+\infty} u_k(t)\big(x^\alpha v_k^{(n)}(x)\big)^{(j)}, \quad j = \overline{0, n-1},$$

and uniform convergence in any compact set $\Omega_0 \subset \Omega$ of the following series:

$$\frac{\partial^n}{\partial x^n}\left(x^\alpha \frac{\partial^n u(x,t)}{\partial x^n}\right) = \sum_{k=1}^{+\infty} u_k(t)\big(x^\alpha v_k^{(n)}(x)\big)^{(n)}, \tag{3.4}$$

$$u_{tt}(x,t) = \sum_{k=1}^{+\infty} u_k''(t) v_k(x). \tag{3.5}$$

Consider the series (3.1). By virtue of (3.3) of (3.1), for any $(x,t) \in \overline{\Omega}$, we have

$$|u(x,t)| \leqslant \sum_{k=1}^{+\infty} |u_k(t)||v_k(x)| \leqslant \sum_{k=1}^{+\infty} \frac{|v_k(x)|}{\sqrt{\lambda_k}}\left(\sqrt{\lambda_k}|\varphi_{1k}| + |\varphi_{2k}| + \sqrt{T}\sqrt{\int_0^T f_k^2(\tau)\,d\tau}\right).$$

Hence, applying the Cauchy–Schwarz inequality, we obtain

$$|u(x,t)| \leqslant \sqrt{\sum_{k=1}^{+\infty} \frac{v_k^2(x)}{\lambda_k}}\left(\sqrt{\sum_{k=1}^{+\infty} \lambda_k \varphi_{1k}^2} + \sqrt{\sum_{k=1}^{+\infty} \varphi_{2k}^2} + \sqrt{T}\sqrt{\int_0^T \sum_{k=1}^{+\infty} \big[f_k(\tau)\big]^2\,d\tau}\right). \tag{3.6}$$

The series on the right sides of this inequality, by virtue of the condition of Theorem 3.1, according to Lemmas 2.3 and 2.4, converge uniformly. Therefore, the series standing on the left side, that is the series (3.1), converges uniformly in $\overline{\Omega}$.

Now, consider the series (3.4). By virtue of the equation (2.1), in any compact Ω_0, the series in (3.4) can be written as

$$\sum_{k=1}^{+\infty}(-1)^n \lambda_k u_k(t) v_k(x). \tag{3.7}$$

In order to prove the uniform convergence of the series (3.7), according to (3.3), it is sufficient to prove the absolute and uniform convergence of series

$$\begin{gathered}\sum_{k=1}^{+\infty}(-1)^n \lambda_k \varphi_{1k} v_k(x), \quad \sum_{k=1}^{+\infty}(-1)^n \sqrt{\lambda_k}\varphi_{2k} v_k(x), \\ \sum_{k=1}^{+\infty}(-1)^n \sqrt{\lambda_k}\sqrt{T}\sqrt{\int_0^T \left[f_k(\tau)\right]^2 d\tau\, v_k(x)}.\end{gathered} \tag{3.8}$$

In Ω_0 for each of these series, applying the Cauchy–Schwarz inequality, we have

$$\left|\sum_{k=1}^{+\infty}(-1)^n \lambda_k \varphi_{1k} v_k(x)\right| \leqslant \sum_{k=1}^{+\infty}\left|\sqrt{\lambda_k^3}\varphi_{1k}\frac{v_k(x)}{\sqrt{\lambda_k}}\right| \leqslant \left[\sum_{k=1}^{+\infty}\lambda_k^3\varphi_{1k}^2 \sum_{k=1}^{\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\left|\sum_{k=1}^{+\infty}(-1)^n \sqrt{\lambda_k}\varphi_{2k} v_k(x)\right| \leqslant \sum_{k=1}^{+\infty}\left|\lambda_k\varphi_{2k}\frac{v_k(x)}{\sqrt{\lambda_k}}\right| \leqslant \left[\sum_{k=1}^{+\infty}\lambda_k^2\varphi_{2k}^2 \sum_{k=1}^{\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\left|\sum_{k=1}^{+\infty}(-1)^n \sqrt{\lambda_k}\sqrt{T}\sqrt{\int_0^T \left[f_k(\tau)\right]^2 d\tau\, v_k(x)}\right| \leqslant \sum_{k=1}^{+\infty}\left|\sqrt{T}\sqrt{\lambda_k^2\int_0^T \left[f_k(\tau)\right]^2 d\tau}\,\frac{v_k(x)}{\sqrt{\lambda_k}}\right|$$

$$\leqslant \sqrt{T}\left[\int_0^T \sum_{k=1}^{+\infty}\lambda_k^2\left[f_k(\tau)\right]^2 d\tau \sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2}.$$

The series standing on the right-hand sides of these inequalities, by virtue of the condition of Theorem 3.1, according to Lemmas 2.4, 2.5 and 2.6, converge uniformly. Then the series standing on the left sides, that is series (3.8), converge absolutely and uniformly in Ω_0. Therefore, the series (3.7), hence, the series in (3.4), converge uniformly in the compact Ω_0. The uniform convergence of the series (3.5) in Ω_0 follows from the convergence of the series (3.4) and the validity of equation (1.1).

Similarly, the uniform convergence of the rest of the series can be proved.

Theorem 3.1 has been proved. □

Now, we will prove the uniqueness of the solution to Problem 1.

Theorem 3.2. *Problem 1 cannot have more than one solution.*

P r o o f. Suppose that there exist two solutions $u_1(x,t)$ and $u_2(x,t)$ of Problem 1. Let us denote by $u(x,t)$ their difference. Then, the function $u(x,t)$ satisfies equation (1.1) with $f(x,t) \equiv 0$, and the conditions (1.2) and (1.3) with $\varphi_1(x) \equiv \varphi_2(x) \equiv 0$.

Let $T_0 \in (0,T]$ and $\Omega_1 = \{(x,t)\colon 0 < x < 1,\ 0 < t < T_0\}$. Obviously, $\overline{\Omega}_1 \subset \overline{\Omega}$. We introduce the following function:

$$\omega(x,t) = -\int_t^{T_0} u(x,\xi)\, d\xi, \quad (x,t) \in \overline{\Omega}_1.$$

It has the following properties:

1) ω_t, ω_{tt}, $\dfrac{\partial^j \omega}{\partial x^j}$, $\dfrac{\partial^j}{\partial x^j}\left(x^\alpha \dfrac{\partial^n \omega}{\partial x^n}\right) \in C\left(\overline{\Omega}_1\right)$, $j = \overline{0, n-1}$;

2) it satisfies the conditions (1.3) for $t \in [0, T_0]$.

Consider equation (1.1) with $f(x,t) \equiv 0$, multiplying it by the function $\omega(x,t)$ and integrating the resulting equation over the domain Ω_1, we have

$$\int_{\Omega_1} \omega(x,t)\left\{u_{tt}(x,t) + (-1)^n \frac{\partial^n}{\partial x^n}\left[x^\alpha \frac{\partial^n u(x,t)}{\partial x^n}\right]\right\} dt\, dx = 0.$$

We rewrite this equality in the following form

$$\int_0^{T_0} dt \int_0^1 (-1)^n \omega(x,t) \frac{\partial^n}{\partial x^n}\left[x^\alpha \frac{\partial^n u(x,t)}{\partial x^n}\right] dx + \int_0^1 dx \int_0^{T_0} \omega(x,t) u_{tt}(x,t)\, dt = 0.$$

Hence, applying the rule of integration by parts, we obtain the following equality

$$\int_0^{T_0} (-1)^n \left[\omega(x,t) \frac{\partial^{n-1}}{\partial x^{n-1}}\left(x^\alpha \frac{\partial^n u(x,t)}{\partial x^n}\right) - \frac{\partial \omega(x,t)}{\partial x} \frac{\partial^{n-2}}{\partial x^{n-2}}\left(x^\alpha \frac{\partial^n u(x,t)}{\partial x^n}\right) + \ldots \right.$$
$$\left. + (-1)^{n-1} \frac{\partial^{n-1}\omega(x,t)}{\partial x^{n-1}}\left(x^\alpha \frac{\partial^n u(x,t)}{\partial x^n}\right)\right]_{x=0}^{x=1} dt + \int_0^{T_0} dt \int_0^1 x^\alpha \frac{\partial^n \omega(x,t)}{\partial x^n} \frac{\partial^n u(x,t)}{\partial x^n}\, dx$$
$$+ \int_0^1 \left[\omega(x,t) \frac{\partial u(x,t)}{\partial t}\Big|_{t=0}^{t=T_0} - \int_0^{T_0} \frac{\partial \omega(x,t)}{\partial t} \frac{\partial u(x,t)}{\partial t}\right] dx = 0,$$

from which, by virtue of the properties of the function $\omega(x,t)$ and $u(x,t)$, it follows that

$$\int_0^{T_0} dt \int_0^1 x^\alpha \frac{\partial^n \omega(x,t)}{\partial x^n} \frac{\partial^n u(x,t)}{\partial x^n}\, dx - \int_0^1 dx \int_0^{T_0} \frac{\partial \omega(x,t)}{\partial t} \frac{\partial u(x,t)}{\partial t}\, dt = 0.$$

Taking into account the equalities $u = \dfrac{\partial \omega}{\partial t}$ and $\dfrac{\partial^n u}{\partial x^n} = \dfrac{\partial^{n+1} \omega}{\partial x^n \partial t}$, we have

$$\int_0^1 x^\alpha\, dx \int_0^{T_0} \frac{\partial^n \omega(x,t)}{\partial x^n} \frac{\partial^{n+1} \omega(x,t)}{\partial x^n \partial t}\, dt - \int_0^1 dx \int_0^{T_0} u(x,t) \frac{\partial u(x,t)}{\partial t}\, dt = 0.$$

Further, taking into account the following equalities

$$u(x,t) \frac{\partial u(x,t)}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t}\left[u(x,t)\right]^2, \qquad \frac{\partial^n \omega(x,t)}{\partial x^n} \frac{\partial^{n+1} \omega(x,t)}{\partial x^n \partial t} = \frac{1}{2} \frac{\partial}{\partial t}\left[\frac{\partial^n \omega(x,t)}{\partial x^n}\right]^2,$$

and applying the rule of integration by parts with respect to the variable t, and taking into account that $\omega(x, T_0) = 0$, $u(x, 0) = 0$, we have

$$\int_0^1 u^2(x, T_0)\, dx + \int_0^1 x^\alpha \left[\frac{\partial^n \omega(x,t)}{\partial x^n}\right]_{t=0}^2 dx = 0.$$

From the latter equality, it follows that $u(x, T_0) \equiv 0$, $x \in [0,1]$. Since T_0 is an arbitrary number from $[0, T]$, then $u(x,t) \equiv 0$, $(x,t) \in \overline{\Omega}$. Thus, $u_1(x,t) \equiv u_2(x,t)$, $(x,t) \in \overline{\Omega}$ and the uniqueness follows. Theorem 3.2 has been proved. □

Now, we will prove certain estimates for the solution to Problem 1.

Theorem 3.3. *Let the functions $\varphi_1(x), \varphi_2(x)$ and $f(x,t)$ satisfy the conditions of Theorem 3.1. Then, the following estimates are valid:*

$$\|u(\cdot,t)\|^2_{L_2(0,1)} \leqslant C_3\big[\|\varphi_1\|^2_{L_2(0,1)} + \|\varphi_2\|^2_{L_2(0,1)} + \|f(\cdot,t)\|^2_{L_2(\Omega)}\big], \tag{3.9}$$

$$\|u(x,t)\|_{C(\overline{\Omega})} \leqslant C_4\Big[\big\|\varphi_1^{(n)}\big\|_{L_{2,r}(0,1)} + \|\varphi_2\|_{L_2(0,1)} + \sqrt{T}\|f(\cdot,t)\|_{L_2(\Omega)}\Big], \tag{3.10}$$

where $\|\varphi_1\|_{L_{2,r}(0,1)} = \left[\int_0^1 x^\alpha \big[\varphi_1(x)\big]^2 dx\right]^{1/2}$, $r = r(x) = x^\alpha$, C_3 *and* C_4 *are* $C_3 = 3C_5$, $C_5 = \max(1, 1/\lambda_1, T/\lambda_1)$, $C_4 = \sup_{[0,1]} \sqrt{\sum\limits_{k=1}^{+\infty} v_k^2(x)/\lambda_k}$.

P r o o f. Since the system $\{v_k(x)\}_{k=1}^{+\infty}$ is an orthonormal system, then using inequality (3.3), from (3.2), we obtain

$$\begin{aligned}\|u(\cdot,t)\|^2_{L_2(0,1)} &= \sum_{k=1}^{+\infty} u_k^2(t) \leqslant \sum_{k=1}^{+\infty}\left[|\varphi_{1k}| + \frac{1}{\sqrt{\lambda_k}}|\varphi_{2k}| + \frac{\sqrt{T}}{\sqrt{\lambda_k}}\|f_k(t)\|_{L_2(0,T)}\right]^2 \\ &\leqslant 3\sum_{k=1}^{+\infty}\left[\varphi_{1k}^2 + \frac{1}{\lambda_k}\varphi_{2k}^2 + \frac{T}{\lambda_k}\|f_k(t)\|^2_{L_2(0,T)}\right] \\ &\leqslant 3\sum_{k=1}^{+\infty}\left[\varphi_{1k}^2 + \frac{1}{\lambda_1}\varphi_{2k}^2 + \frac{T}{\lambda_1}\|f_k(t)\|^2_{L_2(0,T)}\right].\end{aligned}$$

Hence, applying the Bessel inequality (see [22, pp. 23–24]), we have

$$\|u(\cdot,t)\|^2_{L_2(0,1)} \leqslant C_3\left(\|\varphi_1\|^2_{L_2(0,1)} + \|\varphi_2\|^2_{L_2(0,1)} + \sum_{k=1}^{+\infty}\|f_k(t)\|^2_{L_2(0,T)}\right). \tag{3.11}$$

Taking into account the easily verifiable equality

$$\|f(\cdot,t)\|^2_{L_2(\Omega)} = \sum_{k=1}^{+\infty}\|f_k(t)\|^2_{L_2(0,T)},$$

from (3.11), we get the estimate (3.9).

Taking into account the statements of Lemmas 2.3 and 2.4 from (3.6), we obtain

$$\|u(x,t)\|_{C(\overline{\Omega})} \leqslant C_4\left\{\sqrt{\int_0^1 x^\alpha\big[\varphi_1^{(n)}(x)\big]^2 dx} + \|\varphi_2(x)\|_{L_2(0,1)} + \sqrt{T}\sqrt{\sum_{k=1}^{+\infty}\|f_k(t)\|^2_{L_2(0,T)}}\right\}.$$

Hence, by virtue of the introduced designations, follows the estimate (3.10).

Theorem 3.3 has been proved. □

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Корректность начально-краевой задачи для вырожденного дифференциального уравнения в частных производных чётного высокого порядка

Ключевые слова: вырожденное уравнение, краевая задача с начальным условием, метод разделения переменных, спектральная задача, метод функции Грина, интегральное уравнение, ряд Фурье.

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В настоящей работе сформулирована и исследована краевая задача с начальным условием для вырожденного дифференциального уравнения в частных производных чётного высокого порядка в прямоугольной области. С помощью метода разделения переменных получена спектральная задача для обыкновенного дифференциального уравнения. Путем построения функции Грина эта спектральная задача эквивалентно сведена к интегральному уравнению Фредгольма второго рода с симметрическим ядром. На основе теории интегральных уравнений с симметрическим ядром доказано существование собственных значений и системы собственных функций спектральной задачи. Кроме того, с помощью полученного интегрального уравнения и теоремы Мерсера доказана равномерная сходимость некоторых билинейных рядов, зависящих от найденных собственных функций. Решение рассматриваемой задачи представлено в виде суммы ряда Фурье по системе собственных функций спектральной задачи. С помощью метода энергетических интегралов доказана единственность решения задачи. Для демонстрации непрерывной зависимости решения от заданных данных получены соответствующие оценки.

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