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THE EXISTENCE OF COINCIDENCE POINTS IN A SUBSET ON b-METRIC SPACE

In this paper, we extend some results of coincidence points for mappings (ψ, φ) acting from a b-metric space (X, ρ_X) to a space Y equipped with a distance d satisfying only the axiom of identity. Here, the mapping ψ is α-covering while the mapping φ is β-Lipschitz, $\alpha > \beta \geq 0$, only on a given subset $V \subset X$. An α-covering set ($\alpha > 0$) of the mapping ψ on V is a set of (x, y) such that there exists $u \in V$ with $\psi(u) = y$, $\rho_X(x, u) \leq \alpha^{-1} d(\psi(x), y)$, and a β-Lipschitz set ($\beta \geq 0$) of the mapping φ on V is a set of (x, y) such that for every $u \in V$, $\varphi(u) = y \ \Rightarrow \ d(y, \varphi(x)) \leq \beta \rho_X(u, x)$. Also, we study the stability of the sequence of coincidence points ξ_n for mappings ψ_n, φ_n, when we have some convergence to ψ and φ respectively, when $n \to \infty$.

Keywords: coincidence points, metric space, b-metric space, covering mapping, Lipschitz mapping.

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Introduction

The coincidence point theorem established by Arutyunov [1] has received considerable attention across various branches of mathematics, including integral equations [2, 3], differential equations and control problems [4–6]. In response to this widespread interest, numerous studies have been undertaken to extend this framework. Initially, the baseline established in [1] considered two mappings defined between regular metric spaces. Subsequent works [7–9] explored scenarios where the two mappings are defined from one (q_1, q_2)-quasimetric space to another. Later, this was further generalized in [10] for mappings defined from a b-metric space to a space equipped with a distance satisfying only the identity condition. Finally, a similar relaxation was applied in [11], where the mappings operate from a (q_1, q_2)-quasimetric space to a space equipped with a distance satisfying only the identity condition. In all of these studies, one of the mappings is a graph-closed and α-covering and the other one is a β-Lipschitz in the whole space $(X \to Y)$ with $(\alpha > \beta)$.

Nevertheless, these extensions have primarily addressed enlargements of the metric spaces, without relaxing the constraints imposed on the mappings themselves except in [12, 13], where the conditions on the mappings were weakened so that the required properties were verified only on a subset of the metric space. In this context, the present work aims to further generalize this latter result [12, 13] to a more generalized metric spaces (in our paper it's the b-metric space), which likewise encompasses and extends the work presented in [10].

§ 1. Main concept

Let us give some definitions necessary for our research.

Definition 1.1. Let X be a nonempty set. Let $s \geq 1$. The mapping $\rho_X \colon X \times X \to \mathbb{R}_+$ is defined by the following way. Let the following relations are satisfied for all $x, y, z \in X$:

(1) $\rho_X(x, y) = 0 \Leftrightarrow x = y$ (identity);

(2) $\rho_X(x, y) = \rho_X(y, x)$ (symmetry);

(3) $\rho_X(x, z) \leq s(\rho_X(x, y) + \rho_X(y, z))$ (the s-relaxed triangle inequality).

Then, the distance ρ_X is called b-metric, and the pair (X, ρ_X) is called a b-metric space.

Remark 1.1. if $s = 1$, then (X, ρ_X) is a standard metric space.

Definition 1.2. Let (X, ρ_X) be a b-metric space. For $x_0 \in X$ and $r > 0$, the closed ball centered at x_0 with radius r is defined by

$$\overline{B}(x_0, r) := \{x \in X \colon \rho_X(x_0, x) \leq r\}.$$

Remark 1.2 (see [14]). Unlike the metric case, the closed ball $\overline{B}(x_0, r)$ in a b-metric space is not necessarily closed with respect to convergence. Indeed, if $\{x_n\} \subset \overline{B}(x_0, r)$ and $x_n \to x$ in (X, ρ_X), then, using the s-relaxed triangle inequality, we have

$$\rho_X(x_0, x) \leq s\big(\rho_X(x_0, x_n) + \rho_X(x_n, x)\big).$$

Passing to the limit as $n \to \infty$, we obtain

$$\rho_X(x_0, x) \leq sr.$$

Hence, in general,

$$x \in \overline{B}(x_0, sr), \quad \text{but is not necessarily} \quad x \in \overline{B}(x_0, r).$$

Let (X, ρ_X) be a b-metric space. We give the definitions of Cauchy sequence, convergent sequence and complete b-metric space (for example, see [10]).

Definition 1.3. Let (X, ρ_X) be a b-metric space.

(i) A sequence $\{x_n\}$ in X is called *Cauchy* (or *fundamental*) if

$$\rho_X(x_n, x_m) \longrightarrow 0 \ \text{ as } \ n, m \to +\infty.$$

(ii) A sequence $\{x_n\}$ in X is said to *converge* to a point $x \in X$ if

$$\rho_X(x_n, x) \longrightarrow 0 \ \text{ as } \ n \to +\infty;$$

in this case we write

$$x_n \longrightarrow x \ \text{ or } \ \lim_{n \to +\infty} x_n = x.$$

(iii) The b-metric space (X, ρ_X) is called *complete* if every Cauchy sequence in X converges to a point of X.

Definition 1.4. Let Y be a non-empty set, and let

$$d \colon Y \times Y \to \mathbb{R}_+$$

be a function satisfying only the *identity axiom*:

$$\forall y_1, y_2 \in Y \quad d(y_1, y_2) = 0 \ \Leftrightarrow \ y_1 = y_2.$$

The pair (Y, d) is called a *generalized metric space*.

In Y, we define the notion of convergence of a sequence $\{y_i\} \subset Y$ to an element $y \in Y$ by

$$d(y_i, y) \to 0 \ \text{ as } \ i \to \infty.$$

We denote the set of all limits of $\{y_i\}$ by $\operatorname{Lim} y_i = \{y \in Y \mid y_i \to y\}$.

Remark 1.3.

1. Let $\{y_i\} \subset Y$, the limits of sequences may *not be unique*, and even if $d(y_i, y) \to 0$ this does not mean that $d(y, y_i) \not\to 0$, as $i \to +\infty$.

2. If d additionally satisfies *symmetry* and the *triangle inequality*, then (Y, d) reduces to a standard metric space.

From [1], we extend some definitions for mappings defined from a b-metric space to a space equipped with a distance that only satisfy the first axiom of metric.

Definition 1.5. Let $\alpha > 0$. The mapping $\psi\colon X \to Y$ is called α-covering if the following relation is satisfied:

$$\forall x_0 \in X \;\; \forall y \in Y \;\; \exists x \in X\colon \psi(x) = y,$$

and

$$\rho_X(x, x_0) \leq \frac{1}{\alpha} d\big(\psi(x), \psi(x_0)\big).$$

Remark 1.4. We note from Definition 1.5 that every single-valued α-covering mapping is surjective, but not necessarily injective. (As an example, the mapping $\psi\colon \mathbb{R} \to [0, \infty)$ defined by $\psi(x) = |x|$ is a 1-covering mapping. It is surjective but not injective.) If, in addition, the mapping is injective, then its inverse exists and is α^{-1}-Lipschitz. In contrast, for a multivalued α-covering mapping, its inverse is automatically α^{-1}-Lipschitz mapping.

Definition 1.6. Let $\beta \geq 0$. The mapping $\varphi\colon X \to Y$ is called β-Lipschitz if the following relation is satisfied:

$$\forall x, x_0 \in X \;\; d\big(\varphi(x), \varphi(x_0)\big) \leq \beta \rho_X(x, x_0).$$

A mapping $\psi\colon X \to Y$ is called *continuous* at $x \in X$ if for every sequence $\{x_i\} \subset X$ with $x_i \to x$, we have $\psi(x_i) \to \psi(x)$, i. e., $\psi(x) \in \operatorname{Lim} \psi(x_i)$. A mapping continuous at every point of $V \subset X$ is called continuous on V, and in the case $V = X$ such mapping is called continuous. Note that uniqueness of the limit $\psi(x)$ in $\{\psi(x_i)\}$ is not required for continuity at x (see Example 1.1 from [12]). From [9], we say that a mapping $\psi\colon X \to Y$ is called *graph-closed* at a point $x \in X$, if for every sequence $\{x_i\} \subset X$ with $x_i \to x$, whenever the sequence $\{\psi(x_i)\}$ converges to some $y \in Y$, then necessarily

$$y = \psi(x).$$

A mapping is graph-closed on every point of $V \subset X$ is called graph-closed on V, and in the case $V = X$ such mapping is called graph-closed. Thus, uniqueness of the limit $\{\psi(x_i)\}$ is not necessary for graph-closedness. Moreover, continuity at x does not imply its graph-closedness at x (see Example 1.1 from [13]).

From [13], we give the definition of the weakened graph-closedness property, which is satisfied by all continuous mappings from X to Y.

Definition 1.7. A mapping $\psi\colon X \to Y$ is called *d-closed* at a point $x \in X$ if, for any sequence $\{x_i\} \subset X$ such that $x_i \to x$ and $\operatorname{Lim} \psi(x_i) \neq \varnothing$, the inclusion $\psi(x) \in \operatorname{Lim} \psi(x_i)$ holds. A mapping that is d-closed at every point of a set $V \subset X$ is called d-closed on V, and, in the case $V = X$, such mapping is called *d-closed*.

From [10], we formulate the conditions for the existence of a coincidence point for two mappings defined from a b-metric space to a space equipped by a generalized distance.

Firstly, we recall that a point $\xi \in X$ is called a coincidence point for the mappings $\psi\colon X \to Y$ and $\varphi\colon X \to Y$ if the following relation is satisfied:

$$\psi(\xi) = \varphi(\xi). \tag{1.1}$$

For any $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$, we denote by $R(\theta, n)$ the sum of n terms of a geometric series $\sum_{i=0}^{n-1} \theta^i$, thus,

$$R(\theta, n) = \frac{1-\theta^n}{1-\theta}.$$

We assume that $R(\theta, 0) = 0$.

Let the real numbers $\alpha > \beta \geq 0$ be given, and we assume that $\beta^0 = 1$ when $\beta = 0$. Let

$$m_0 = \min\{j \in \mathbb{N} \mid s\beta^j < \alpha^j\},$$

and also we denote

$$\tilde{R}(j) = R\left(s\frac{\beta}{\alpha}, j-1\right) + s^{j-2}\frac{\beta^{j-1}}{\alpha^{j-1}}, \quad j \in \mathbb{N}, \quad \tilde{R}(0) = 0.$$

Theorem 1.1. *Let a b-metric space (X, ρ_X) be complete. Let the following conditions hold: mapping $\psi\colon X \to Y$ is α-covering and graph-closed, and mapping $\varphi\colon X \to Y$ is β-Lipschitz with $\alpha > \beta$.*

Then, for any $x_0 \in X$, there exists a coincidence point ξ for ψ and φ such that

$$\rho_X(x_0, \xi) \leq \frac{s^3\alpha^{m_0-1}R(s\beta/\alpha, m_0-1) + s^2(s\beta)^{m_0-1}}{\alpha^{m_0} - s\beta^{m_0}} d\big(\varphi(x_0), \psi(x_0)\big). \tag{1.2}$$

Remark 1.5. The statement (1.2) can also be written in another form

$$\begin{aligned}\rho_X(x_0, \xi) &\leq \frac{s^3\tilde{R}(m_0)}{1 - s\frac{\beta^{m_0}}{\alpha^{m_0}}}\alpha^{-1} d\big(\varphi(x_0), \psi(x_0)\big) \\ &= \frac{s^3\alpha^{m_0-1}R(s\beta/\alpha, m_0-1) + s^2(s\beta)^{m_0-1}}{\alpha^{m_0} - s\beta^{m_0}} d\big(\varphi(x_0), \psi(x_0)\big).\end{aligned}$$

§ 2. Main results

In this section, we will give some results of coincidence point. Firstly, we give a result about the existence of the coincidence point ξ for two mappings ψ, φ defined from a b-metric space (X, ρ_X) to a space (Y, d) (d is a mapping satisfying only the first condition of a metric). One of these mappings is α-covering and the other one is β-Lipschitz on a subset $V \subset X$. The second result is about the stability of the sequence of coincidence points ξ_n for mappings ψ_n, φ_n, when we have some convergence to ψ and φ respectively, as $n \to \infty$.

From [3, 15] we formulate the generalizations of Definitions 1.2 and 1.3 proposed in [16]. Let a set $V \subset X$ be given. For mappings $\psi\colon X \to Y$ and $\varphi\colon X \to Y$, we define the following two sets:

$$\mathrm{Cov}_\alpha[\psi; V] := \left\{ (x, y) \in X \times Y \;\middle|\; \begin{array}{l} d\big(y, \psi(x)\big) < +\infty \;\Rightarrow\; \exists u \in V \text{ such that} \\ \psi(u) = y \text{ and } \rho_X(u, x) \leq \dfrac{1}{\alpha} d\big(y, \psi(x)\big) \end{array} \right\},$$

$$\mathrm{Lip}_\beta[\varphi; V] := \big\{(x, y) \in X \times Y \;\big|\; \forall u \in V,\ \varphi(u) = y \implies d\big(y, \varphi(x)\big) \leq \beta\, \rho_X(u, x)\big\}.$$

The first of these sets is called the *set of α-covering* of the mapping ψ relative to the set V, and the second is called the *set of β-Lipschitzness* of the mapping φ relative to V. Obviously, the relation

$$\mathrm{Cov}_\alpha[\psi; X] = X \times Y$$

means that the mapping ψ is α-covering, and the relation

$$\mathrm{Lip}_\beta[\varphi; X] = X \times Y$$

means that the mapping φ is β-Lipschitz. Note that the sets of covering and Lipschitzness monotonically depend on the set $V \subset X$ by inclusion, namely, the following relations hold:

$$V \subset \widetilde{V} \subset X \ \Rightarrow\ \mathrm{Cov}_\alpha[\psi; V] \subset \mathrm{Cov}_\alpha[\psi; \widetilde{V}], \qquad \mathrm{Lip}_\beta[\varphi; V] \supset \mathrm{Lip}_\beta[\varphi; \widetilde{V}].$$

2.1. Coincidence point in a local set from *b*-metric space

Our main result, stated in the following theorem, deals with the existence of a coincidence point for two mappings ψ, φ defined from b-metric space to a space equipped by a distance satisfying only the first condition of a metric.

Theorem 2.1. *Let (X, ρ_X) be a complete b-metric space, $\alpha > \beta \geq 0$, and let $x_0 \in X$ satisfy $d\big(\psi(x_0), \varphi(x_0)\big) < \infty$. We set*

$$V = \bar{B}(x_0, L),$$
$$L = \frac{s^3\alpha^{m_0-1}R(s\beta/\alpha, m_0-1) + s^2(s\beta)^{m_0-1}}{\alpha^{m_0} - s\beta^{m_0}} d\big(\psi(x_0), \varphi(x_0)\big).$$

Assume that, for every $x \in V$, $(x, \psi(x)) \in \mathrm{Lip}_\beta[\varphi; V]$ and $(x, \varphi(x)) \in \mathrm{Cov}_\alpha[\psi; X]$; the mapping ψ is graph-closed and the mapping φ is continuous on V.

Then, there exists a solution of (1.1) *in V.*

P r o o f. Let $V \subset X$. We will prove that there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subset X$ satisfying the following relation:

$$x_i \in V, \quad \psi(x_i) = \varphi(x_{i-1}),\ i = 1, 2, \dots, \ \text{ and } \ \rho_X(x_{i-1}, x_i) \leq \frac{\beta}{\alpha}\rho_X(x_{i-2}, x_{i-1}),\ i \geq 2. \quad (2.1)$$

The main idea of this proof is to show that all elements of the sequence $\{x_i\}$ belong to the closed ball $V = \bar{B}(x_0, L)$, thereby preserving the Cauchy convergence property. To achieve this, we give a partition of the set

$$H = \{x_0, x_1, x_2, \dots\}$$

that representing the elements of the sequence $\{x_i\}$, into finite subsets of the form $B_1 = \{x_0, x_1, \dots, x_{m_0}\}$, $B_2 = \{x_{m_0}, x_{m_0+1}, \dots, x_{2m_0}\}, \dots, B_n = \{x_{(n-1)m_0}, \dots, x_{nm_0}\}$. So, first of all, we prove that $H = \bigcup_{n\geq 1} B_n$. Then, we apply backward mathematical induction to prove that every element of each subset (B_n) lies inside the ball.

1. We show that $H = \bigcup_{n\geq 1} B_n$.

Let $\alpha > \beta \geq 0$ with $s > 1$ be fixed, and define $m_0 = \min\{j \in \mathbb{N}\colon s\beta^j < \alpha^j\}$. Since $\alpha/\beta > 1$, so $(\alpha/\beta)^j \to \infty$ as $j \to \infty$, hence, such m_0 exists. Clearly, $m_0 \neq 0$ because for $j = 0$, $s\beta^0 = s > \alpha^0 = 1$. Thus, $m_0 \geq 1$.

Let $H = \{x_0, x_1, x_2, \dots\}$, and define for each $n \in \mathbb{N}$, $B_n = \{x_{(n-1)m_0}, x_{(n-1)m_0+1}, \dots, x_{nm_0}\}$. Each B_n is nonempty since it contains exactly $m_0 + 1$ consecutive elements of X.

For any $x_k \in H$, $k \in \mathbb{N}$, there exists a unique n such that $(n-1)m_0 < k \leq nm_0$, meaning that k lies between two consecutive multiples of m_0. Hence, $x_k \in B_n$, and, therefore, $H \subset \bigcup_{n\geq 1} B_n$. Also, it's clear that $\bigcup_{n\geq 1} B_n \subset H$ holds. Thus, we have $\bigcup_{n\geq 1} B_n = H$.

2. We denote $M = d\big(\psi(x_0), \varphi(x_0)\big)$. Now, we will show that $\bigcup_{n\geq 1} B_n \subset V$.

Base step $B_1 \subset V$:

Let x_0 satisfy the hypothesis of the theorem and $y_0 = \varphi(x_0)$. Since $(x_0, y_0) \in \mathrm{Cov}_\alpha[\psi; X]$, there exists $x_1 \in X$ such that

$$\psi(x_1) = \varphi(x_0), \quad \rho_X(x_0, x_1) \leq \alpha^{-1} d\big(\psi(x_0), \varphi(x_0)\big). \tag{2.2}$$

Firstly, we observe that

$$0 \leq 1 - s\frac{\beta^{m_0}}{\alpha^{m_0}} \leq 1.$$

Consequently,

$$1 \leq \frac{1}{1 - s\dfrac{\beta^{m_0}}{\alpha^{m_0}}} \ \Rightarrow \ \alpha^{-1} \leq \frac{\alpha^{-1}}{1 - s\dfrac{\beta^{m_0}}{\alpha^{m_0}}}.$$

Moreover, by hypothesis, we have $1 \leq s$ and $1 \leq s^3\tilde{R}(m_0)$.
It follows that

$$\alpha^{-1} d\big(\psi(x_0), \varphi(x_0)\big) \leq \frac{s^3\tilde{R}(m_0)}{1 - s\frac{\beta^{m_0}}{\alpha^{m_0}}} \alpha^{-1} d\big(\psi(x_0), \varphi(x_0)\big).$$

Therefore, we deduce $\alpha^{-1}M \leq \ L$.

From (2.2), we get $\rho_X(x_0, x_1) \leq L$, so $x_1 \in V$. Let $y_1 = \varphi(x_1)$. Since $(x_1, y_1) \in \mathrm{Lip}_\beta[\varphi; V]$, we have

$$d(y_0, y_1) = d\big(\varphi(x_0), \varphi(x_1)\big) \leq \beta\rho_X(x_0, x_1). \tag{2.3}$$

Case $m_0 = 1$. This case is well verified in x_1. Thus, our base step ends here. Otherwise, we have: since $(x_1, y_1) \in \mathrm{Cov}_\alpha[\psi; X]$, there exists $x_2 \in X$ such that

$$\psi(x_2) = \varphi(x_1), \quad \rho_X(x_1, x_2) \leq \alpha^{-1} d\big(\psi(x_1), \varphi(x_1)\big).$$

From (2.3), we deduce that $\rho_X(x_1, x_2) \leq \frac{\beta}{\alpha}\rho_X(x_0, x_1)$. From (2.2),

$$\rho_X(x_0, x_2) \leq s\big(\rho_X(x_0, x_1) + \rho_X(x_1, x_2)\big) \leq s\left(1 + \frac{\beta}{\alpha}\right)\rho_X(x_0, x_1) \leq \alpha^{-1}s\left(1 + \frac{\beta}{\alpha}\right)M.$$

Case $m_0 = 2$. Recall that

$$L = \frac{s^3\alpha^{m_0-1}R\big(\frac{s\beta}{\alpha}, m_0 - 1\big) + s^2(s\beta)^{m_0-1}}{\alpha^{m_0} - s\beta^{m_0}} M.$$

For $m_0 = 2$, we have $m_0 - 1 = 1$ and

$$R\left(\frac{s\beta}{\alpha}, 1\right) = 1.$$

Hence,

$$L = \frac{s^3\alpha + s^3\beta}{\alpha^2 - s\beta^2} M = \frac{s^3(\alpha + \beta)}{\alpha^2 - s\beta^2} M.$$

Thus,

$$\rho_X(x_0, x_2) \leq \alpha^{-1}s\left(1 + \frac{\beta}{\alpha}\right)M = \frac{s(\alpha + \beta)}{\alpha^2} M \leq \frac{s(\alpha + \beta)}{\alpha^2 - s\beta^2} M \leq \frac{s^3(\alpha + \beta)}{\alpha^2 - s\beta^2} M = L.$$

Case $m_0 > 2$.

$$\rho_X(x_0,x_2) \le \alpha^{-1}s\left(1+\frac{\beta}{\alpha}\right)M \le \alpha^{-1}s\left(1+\frac{s\beta}{\alpha}\right)M \le \alpha^{-1}sR\left(\frac{s\beta}{\alpha},2\right)M$$
$$\le \alpha^{-1}sR\left(\frac{s\beta}{\alpha},m_0\right)M \le \frac{s^3\tilde{R}(m_0)}{1-s\frac{\beta^{m_0}}{\alpha^{m_0}}}\alpha^{-1}M = L,$$

so, $x_2 \in V$. Thus, (2.1) holds for $i=1,2$.

We suppose that (2.1) holds for i and $i+1$, where $0 \le i, i+1 \le m_0$. Since $(x_i,y_i) \in$ $\in \mathrm{Cov}_\alpha[\psi;X]$, there exists $x_{i+1} \in X$ such that

$$\psi(x_{i+1}) = \varphi(x_i), \quad \rho_X(x_i,x_{i+1}) \le \alpha^{-1}d\big(\psi(x_i),\varphi(x_i)\big). \tag{2.4}$$

Hence,

$$\rho_X(x_i,x_{i+1}) \le \alpha^{-1}d\big(\varphi(x_{i-1}),\varphi(x_i)\big) \le \frac{\beta}{\alpha}\rho_X(x_{i-1},x_i). \tag{2.5}$$

From (2.1) and (2.5), we get

$$\rho_X(x_i,x_{i+1}) \le \alpha^{-1}\left(\frac{\beta}{\alpha}\right)^i d\big(\psi(x_0),\varphi(x_0)\big). \tag{2.6}$$

From [10] (or by using (2.6)), we have

$$\begin{aligned}
\rho_X(x_i,x_{i+j}) &\le s\big(\rho_X(x_i,x_{i+1})+\rho_X(x_{i+1},x_{i+j})\big)\\
&\le s\,\rho_X(x_i,x_{i+1}) + s^2\big(\rho_X(x_{i+1},x_{i+2})+\rho_X(x_{i+2},x_{i+j})\big)\\
&\le s\,\rho_X(x_i,x_{i+1}) + s^2\rho_X(x_{i+1},x_{i+2}) + \cdots + s^{j-1}\rho_X(x_{i+j-2},x_{i+j-1})\\
&\quad + s^{j-1}\rho_X(x_{i+j-1},x_{i+j})\\
&\le \left(s\frac{\beta^i}{\alpha^{i+1}} + s^2\frac{\beta^{i+1}}{\alpha^{i+2}} + \cdots + s^{j-1}\frac{\beta^{i+j-2}}{\alpha^{i+j-1}} + s^j\frac{\beta^{i+j-1}}{\alpha^{i+j}}\right)M\\
&= s\frac{\beta^i}{\alpha^{i+1}}\,M\,\widetilde{R}(j),
\end{aligned} \tag{2.7}$$

where

$$\tilde{R}(j) = R\left(s\frac{\beta}{\alpha},j-1\right) + s^{j-2}\frac{\beta^{j-1}}{\alpha^{j-1}}, \quad j \in \mathbb{N}, \quad \tilde{R}(0)=0.$$

From (2.6) and (2.7), we obtain

$$\rho_X(x_0,x_{i+1}) \le s\alpha^{-1}M\tilde{R}(i+1) \le s\alpha^{-1}M\tilde{R}(m_0) \le \frac{s^3\tilde{R}(m_0)}{1-s\frac{\beta^{m_0}}{\alpha^{m_0}}}\alpha^{-1}M \le L.$$

So, $(x_{i+1},y_{i+1}) \in \mathrm{Cov}_\alpha[\psi;X]$ and (2.1) is verified.

Thus, we have established that property (2.1) holds for every x_i, $0 \le i \le m_0$. This completes the base step of the induction, namely, the proof that $B_1 \subset V$. We now proceed to the induction step.

Inductive step: Suppose that $\bigcup_{1\le j\le n} B_j \subset V$. We must now prove that $B_{n+1} \subset V$. By the induction hypothesis, we have $x_{nm_0} \in B_n \cap B_{n+1}$ and $(x_{nm_0},y_{nm_0}) \in \mathrm{Cov}_\alpha[\psi;X]$. Consequently, for every index i satisfying $nm_0 \le i$ and $i+1 \le (n+1)m_0$, if $(x_i,y_i) \in \mathrm{Cov}_\alpha[\psi;X]$, then there exists an element $x_{i+1} \in X$ such that property (2.4) holds. Moreover, relations (2.1), (2.5), and (2.6) remain valid and by applying (2.7), we obtain

$$\begin{aligned}
\rho_X(x_{nm_0}, x_{i+1}) &\le s\big(\rho_X(x_{nm_0}, x_{nm_0+1}) + \rho_X(x_{nm_0+1}, x_{i+1})\big)\\
&\le s\,\rho_X(x_{nm_0}, x_{nm_0+1}) + s^2\big(\rho_X(x_{nm_0+1}, x_{nm_0+2}) + \rho_X(x_{nm_0+2}, x_{i+1})\big)\\
&\le s\,\rho_X(x_{nm_0}, x_{nm_0+1}) + s^2\rho_X(x_{nm_0+1}, x_{nm_0+2}) + \cdots\\
&\qquad + s^{i-nm_0}\rho_X(x_{i-1}, x_i) + s^{i+1-nm_0}\rho_X(x_i, x_{i+1})\\
&\le \left(s\frac{\beta^{nm_0}}{\alpha^{nm_0+1}} + s^2\frac{\beta^{nm_0+1}}{\alpha^{nm_0+2}} + \cdots + s^{i-nm_0}\frac{\beta^{i-1}}{\alpha^{i}} + s^{i+1-nm_0}\frac{\beta^{i}}{\alpha^{i+1}}\right)M\\
&= s\frac{\beta^{nm_0}}{\alpha^{nm_0+1}}\, M\,\widetilde{R}\big((i+1) - nm_0\big).
\end{aligned}$$

From [10], for any non negative integers i, k and $r = \left[\frac{k}{m_0}\right]$, we have

$$\begin{aligned}
\rho_X(x_i, x_{i+k}) &\le s\big(\rho_X(x_i, x_{i+m_0}) + \rho_X(x_{i+m_0}, x_{i+k})\big)\\
&\le s\,\rho_X(x_i, x_{i+m_0}) + s^2\big(\rho_X(x_{i+m_0}, x_{i+2m_0}) + \rho_X(x_{i+2m_0}, x_{i+k})\big)\\
&\le s\,\rho_X(x_i, x_{i+m_0}) + s^2\rho_X(x_{i+m_0}, x_{i+2m_0}) + \cdots + s^r\rho_X(x_{i+(r-1)m_0}, x_{i+rm_0})\\
&\quad + s^{r+1}\rho_X(x_{i+rm_0}, x_{i+k})\\
&\le s^2\frac{\beta^{i}}{\alpha^{i+1}}M\widetilde{R}(m_0) + s^3\frac{\beta^{i+m_0}}{\alpha^{i+m_0+1}}M\widetilde{R}(m_0) + \cdots\\
&\quad + s^{r+1}\frac{\beta^{i+(r-1)m_0}}{\alpha^{i+(r-1)m_0+1}}M\widetilde{R}(m_0) + s^{r+1}\frac{\beta^{i+rm_0}}{\alpha^{i+rm_0+1}}M\widetilde{R}(k-rm_0)\\
&\le s^2\frac{\beta^{i}}{\alpha^{i+1}}M\widetilde{R}(m_0)R\left(\frac{s\beta^{m_0}}{\alpha^{m_0}}, r\right) + s^{r+1}\frac{\beta^{i+rm_0}}{\alpha^{i+rm_0+1}}M\widetilde{R}(k-rm_0).
\end{aligned} \tag{2.8}$$

Case: $i+1 < (n+1)m_0$.
We have $r = \left\lfloor\frac{i+1}{m_0}\right\rfloor = n$, and using estimates (2.7) and (2.8), we obtain

$$\begin{aligned}
\rho_X(x_0, x_{i+1}) &\le s\big(\rho_X(x_0, x_{m_0}) + \rho_X(x_{m_0}, x_{i+1})\big)\\
&\le s\rho_X(x_0, x_{m_0}) + s^2\big(\rho_X(x_{m_0}, x_{2m_0}) + \rho_X(x_{2m_0}, x_{i+1})\big)\\
&\ \ \vdots\\
&\le s\rho_X(x_0, x_{m_0}) + s^2\rho_X(x_{m_0}, x_{2m_0}) + \cdots + s^n\rho_X(x_{(n-1)m_0}, x_{nm_0})\\
&\qquad + s^n\rho_X(x_{nm_0}, x_{i+1})\\
&\le s^2\frac{1}{\alpha}M\tilde{R}(m_0) + s^3\frac{\beta^{m_0}}{\alpha^{m_0+1}}M\tilde{R}(m_0) + \cdots + s^{r+1}\frac{\beta^{(n-1)m_0}}{\alpha^{(n-1)m_0+1}}M\tilde{R}(m_0)\\
&\qquad + s^{r+1}\frac{\beta^{nm_0}}{\alpha^{nm_0+1}}M\tilde{R}\big((i+1) - nm_0\big)\\
&\le s^2\frac{1}{\alpha}M\tilde{R}(m_0) + s^3\frac{\beta^{m_0}}{\alpha^{m_0+1}}M\tilde{R}(m_0) + \cdots + s^{n+1}\frac{\beta^{(n-1)m_0}}{\alpha^{(n-1)m_0+1}}M\tilde{R}(m_0)\\
&\qquad + s^{n+1}\frac{\beta^{nm_0}}{\alpha^{nm_0+1}}M\tilde{R}\big((i+1) - nm_0\big)\\
&\le s^2\frac{1}{\alpha}M\tilde{R}(m_0)R\left(s\frac{\beta^{m_0}}{\alpha^{m_0}}, n\right) + s^{n+1}\frac{\beta^{nm_0}}{\alpha^{nm_0+1}}M\tilde{R}\big((i+1) - nm_0\big),
\end{aligned} \tag{2.9}$$

and also since

$$0 \le (i+1) - nm_0 \le m_0 \ \text{ and } \ s^{n+1} \le s^{n+2},$$

we obtain

$$\begin{aligned}\rho_X(x_0,x_{i+1}) &\leq s^2\frac{1}{\alpha}M\tilde{R}(m_0)R\bigg(s\frac{\beta^{m_0}}{\alpha^{m_0}},n\bigg)+s^{n+1}\frac{\beta^{nm_0}}{\alpha^{nm_0+1}}M\tilde{R}\big((i+1)-nm_0\big).\\ &\leq s^2\frac{1}{\alpha}M\tilde{R}(m_0)R\bigg(s\frac{\beta^{m_0}}{\alpha^{m_0}},n\bigg)+s^{n+2}\frac{\beta^{nm_0}}{\alpha^{nm_0+1}}M\tilde{R}(m_0).\\ &\leq s^2\frac{1}{\alpha}M\tilde{R}(m_0)R\bigg(s\frac{\beta^{m_0}}{\alpha^{m_0}},n+1\bigg)\leq L,\end{aligned}$$

Case: $i+1=(n+1)m_0$.
We have $r=n+1$, and, from (2.7), we obtain:

$$\begin{aligned}\rho_X(x_{nm_0},x_{(n+1)m_0}) &\leq s\big(\rho_X(x_{nm_0},x_{nm_0+1})+\rho_X(x_{nm_0+1},x_{(n+1)m_0})\big)\\ &\leq s\,\rho_X(x_{nm_0},x_{nm_0+1})+s^2\big(\rho_X(x_{nm_0+1},x_{nm_0+2})+\rho_X(x_{nm_0+2},x_{(n+1)m_0})\big)\\ &\leq s\,\rho_X(x_{nm_0},x_{nm_0+1})+s^2\rho_X(x_{nm_0+1},x_{nm_0+2})+\cdots\\ &\quad+s^{(n+1)m_0-nm_0-1}\rho_X(x_{(n+1)m_0-1},x_{(n+1)m_0})\\ &\leq\bigg(s\frac{\beta^{nm_0}}{\alpha^{nm_0+1}}+s^2\frac{\beta^{nm_0+1}}{\alpha^{nm_0+2}}+\cdots+s^{(n+1)m_0-nm_0-1}\frac{\beta^{(n+1)m_0-1}}{\alpha^{(n+1)m_0}}\bigg)M\\ &=s\frac{\beta^{nm_0}}{\alpha^{nm_0+1}}\,M\,\widetilde{R}\big((n+1)m_0-nm_0\big)=s\frac{\beta^{nm_0}}{\alpha^{nm_0+1}}\,M\,\widetilde{R}(m_0).\end{aligned}$$

From (2.8), we find

$$\begin{aligned}\rho_X(x_0,x_{(n+1)m_0}) &\leq s^2\frac{1}{\alpha}M\tilde{R}(m_0)R\bigg(s\frac{\beta^{m_0}}{\alpha^{m_0}},n+1\bigg)+s^{n+2}\frac{\beta^{(n+1)m_0}}{\alpha^{(n+1)m_0+1}}M\tilde{R}(0)\\ &\leq s^2\frac{1}{\alpha}M\tilde{R}(m_0)R\bigg(s\frac{\beta^{m_0}}{\alpha^{m_0}},n+1\bigg)\leq L,\end{aligned}$$

so, $(x_{i+1},y_{i+1})\in \mathrm{Cov}_\alpha[\psi;X]$ and (2.1) is verified. Thus, $B_{n+1}\subset V$, and from this we find that for all $n\geq 1$ $\bigcup_{1\leq j\leq n}B_j\subset V$. From (2.9), it follows that $\{x_i\}$ is a Cauchy sequence, and as X is a complete space, so $\lim_{i\to+\infty}x_i=\xi$.

Now, it remains to prove that $\xi\in V$, and for that we have:

$$\begin{aligned}\rho_X(x_0,\xi) &\leq s\rho_X(x_0,x_n)+s\rho_X(x_n,\xi)\leq \lim_{n\to+\infty}\big(s\rho_X(x_0,x_n)+s\rho_X(x_n,\xi)\big)\\ &\leq \lim_{n\to+\infty}\bigg(s^3\frac{1}{\alpha}M\tilde{R}(m_0)R\bigg(s\frac{\beta^{m_0}}{\alpha^{m_0}},n+1\bigg)+s\rho_X(x_n,\xi)\bigg)\\ &\leq \lim_{n\to+\infty}s^3\frac{1}{\alpha}M\tilde{R}(m_0)R\bigg(s\frac{\beta^{m_0}}{\alpha^{m_0}},n+1\bigg)+0\leq L.\end{aligned}$$

That proves $\xi=\lim x_i\in V$. By the continuity of φ on V, we have $\varphi(x_i)\to\varphi(\xi)$. By using $\psi(x_{i+1})=\varphi(x_i)$ and the graph-closedness of ψ on V, we have $\psi(\xi)=\varphi(\xi)$. $\square$

Remark 2.1.

1. As in [13], the graph-closedness assumption on ψ may be replaced by the requirement that ψ be d-closed and that the following condition be satisfied:

$$\forall\{x_i\}\subset V\ \forall x\in V\ x_i\to x\ \text{ and }\ \mathrm{Lim}\,\psi(x_i)=\mathrm{Lim}\,\varphi(x_i)\neq\emptyset\implies\psi(x)=\varphi(x). \tag{2.10}$$

2. In analogous, coincidence point theorems for mappings in metric spaces, the continuity of φ is not assumed, since it follows from the Lipschitz condition. However, in Theorem 2.1 we employ a weakened Lipschitz assumption, namely, $((x, \psi(x)) \in \mathrm{Lip}_\beta[\varphi; V]$ for all $x \in V)$ which does not guarantee the continuity of φ. Consequently, the continuity of φ must be explicitly imposed here.

3. Theorem 2.1 still valid if X and Y are b-metric spaces.

Example 2.1. Let $X = Y = \mathbb{R}$ endowed with the quadratic complete b-metric space ($s = 2$) equipped by the distance

$$d(x, y) = (x - y)^2, \quad x, y \in \mathbb{R}.$$

We define two mappings

$$g(x) = \begin{cases} e^x - 1, & x < 0, \\ -e^{-x} + 1, & x \geq 0, \end{cases} \qquad f(x) = \begin{cases} 1 + 3x + \sin(x), & x \in [-\pi, \pi], \\ 0, & x \notin [-\pi, \pi]. \end{cases}$$

Properties of g (Lipschitz via Mean Value Theorem)

For every $x \in \mathbb{R}$ the derivative of g satisfies $g'(x) = e^{-|x|}$, so $0 < g'(x) \leq 1$. Consequently, for every $x, y \in \mathbb{R}$ there exists ξ between x and y with

$$|g(x) - g(y)| = |g'(\xi)|\, |x - y| \leq |x - y|.$$

Thus, g is 1-Lipschitz in the usual metric and therefore also 1-Lipschitz for the quadratic b-metric, i. e., $d\big(g(x), g(y)\big) = (g(x) - g(y))^2 \leq (x - y)^2 = d(x, y)$.

Properties of f on $[-\pi, \pi]$

On $[-\pi, \pi]$, the mapping f equals $x \mapsto 1 + 3x + \sin(x)$, hence, $\min_{x \in [-\pi, \pi]} f'(x) = 2$ and $f'(x) > 0$ for all $x \in [-\pi, \pi]$. Therefore, for any $x, y \in [-\pi, \pi]$, $|f(x) - f(y)| \geq 2|x - y|$. Squaring both sides, $d\big(f(x), f(y)\big) = \big(f(x) - f(y)\big)^2 \geq 4(x - y)^2 = 4\, d(x, y)$. So, inside $[-\pi, \pi]$ we have an expansion inequality $d\big(f(x), f(y)\big) \geq \alpha\, d(x, y)$ with $\alpha = 4$.

Application to a coincidence point theorem

Let (X, d) be a complete b-metric space, where $X = \mathbb{R}$ and $d(x, y) = (x - y)^2$, $x, y \in \mathbb{R}$. It is well known that the quadratic distance and so d is a b-metric with coefficient $s = 2$, that is,

$$d(x, z) \leq 2\big(d(x, y) + d(y, z)\big) \qquad \forall x, y, z \in X.$$

And as (Y, d) is a b-metric space, so, the classical coincidence stated in [1, 12, 13, 17] cannot be applied, since they require both spaces to be standard metric spaces.

Moreover, the mappings considered in [7, 10, 11] assume the β-Lipschitz and α-covering properties on the whole space. In our case, the mapping $f\colon \mathbb{R} \to \mathbb{R}$ is not α-covering on $\mathbb{R}$, since it is not surjective. Therefore, the previous results are not applicable.

Now, we apply our main theorem. We choose the parameters

$$s = 2, \quad m_0 = 1, \quad \alpha = 4, \quad \beta = 1, \quad x_0 = 0.$$

Using the expression of the constant L, we compute explicitly

$$L = \frac{s^3 \alpha^{m_0 - 1} R\big(\frac{s\beta}{\alpha}, m_0 - 1\big) + s^2 (s\beta)^{m_0 - 1}}{\alpha^{m_0} - s\beta^{m_0}}\, d\big(f(x_0), g(x_0)\big) = \frac{0 + 4}{2} \cdot 1 = 2.$$

Since the metric is given by $d(x, y) = |x - y|^2$, we have $d(0, x) = |x|^2$. Choosing $L = 2$, the corresponding closed ball centered at the origin is

$$V = \{x \in \mathbb{R} \colon d(0, x) \le L\} = \{x \in \mathbb{R} \colon |x|^2 \le 2\} = [-\sqrt{2}, \sqrt{2}].$$

By direct computation, we have

$$g(V) \subset f(V),$$

which implies that, for every $x \in V$, the pair $(x, g(x))$ belongs to the covering set $\mathrm{Cov}_4[f, R]$.

Furthermore, for every $x \in V$, the pair $(x, f(x))$ belongs to $\mathrm{Lip}_1[g, V]$. The mapping f is graph-closed on V and the mapping g is continuous on V.

Consequently, all the assumptions of our theorem are satisfied, and we conclude that there exists a point $x^* \in V$ such that

$$f(x^*) = g(x^*),$$

i. e., f and g admit a coincidence point in V.

2.2. Numerical coincidence points (computed)

Solving $f(x) = g(x)$ numerically yields the approximate solution $x \approx -0.31977$.

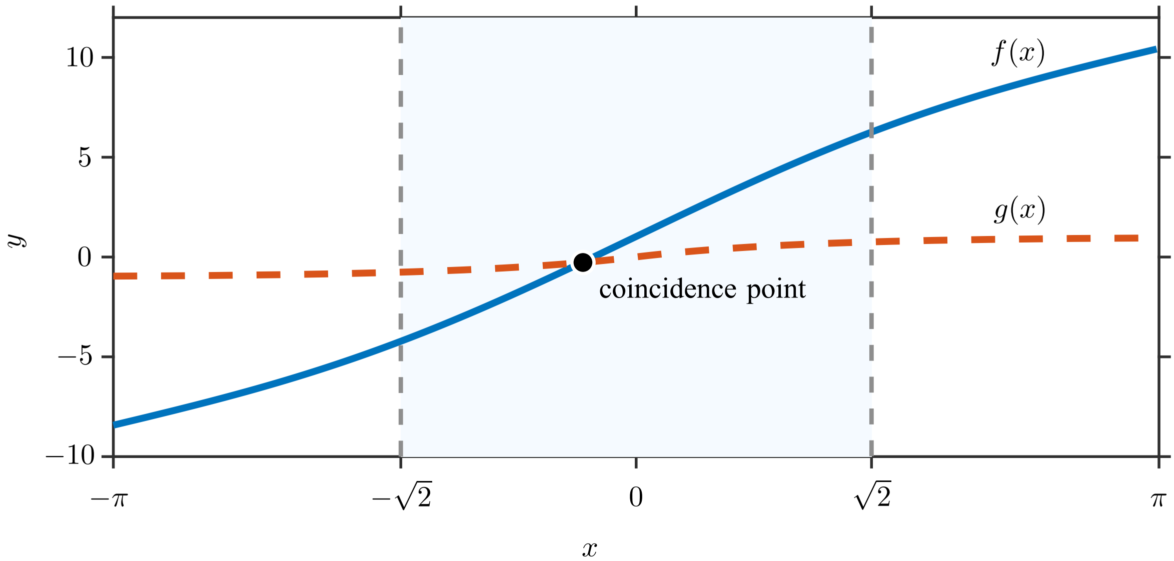

Fig. 1. Graphs of the mappings f and g and their coincidence point $x^* \approx -0.31977$. The dotted vertical lines correspond to the interval $V = [-\sqrt{2}, \sqrt{2}]$

§ 3. The stability of coincidence point

Let for all $n \in \mathbb{N}$ mappings $\psi_n, \varphi_n \colon X \to Y$ be given. In this section, we will study the stability of coincidence point for two mappings ψ_n, φ_n, $n \in \mathbb{N}$, when we have some convergence to ψ, φ, respectively. Therefore, we will prove the convergence of ξ_n to ξ, where ξ_n is the coincidence point for ψ_n, φ_n and ξ is the coincidence point for $\psi, \varphi \colon X \to Y$.

Recall that a coincidence point of the mappings $\psi_n, \varphi_n \colon X \to Y$ is the solution of the equation

$$\psi_n(x) = \varphi_n(x). \tag{3.1}$$

Let (X, ρ_X) be a b-metric space, and let (Y, d) be a set endowed with a distance satisfying only the identity axiom (as in the previous section). Let $\psi, \varphi \colon X \to Y$ be two mappings admitting a coincidence point $\xi \in X$, i. e., $\psi(\xi) = \varphi(\xi)$.

Let $(\psi_n)_{n\in\mathbb{N}}$ and $(\varphi_n)_{n\in\mathbb{N}}$ be sequences of mappings from $X \to Y$.

Theorem 3.1. *Let (X, ρ_X) be a complete b-metric space. Assume that there exist sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ such that $0 \le \beta_n < \alpha_n$ for all $n \in \mathbb{N}$.*

For each $n \in \mathbb{N}$, we define

$$L_n = \frac{s^3 \alpha_n^{m_0-1} R\big(\frac{s\beta_n}{\alpha_n}, m_0 - 1\big) + s^2 (s\beta_n)^{m_0-1}}{\alpha_n^{m_0} - s\beta_n^{m_0}} \, d\big(\psi_n(\xi), \varphi_n(\xi)\big),$$

and introduce the closed ball

$$V_n := \overline{B}(\xi, L_n).$$

Assume that, for every $n \in \mathbb{N}$, the following conditions are satisfied:

(1) *for all $x \in V_n$, $(x, \psi_n(x)) \in \mathrm{Lip}_{\beta_n}[\varphi_n; V_n]$, and $(x, \varphi_n(x)) \in \mathrm{Cov}_{\alpha_n}[\psi_n; X]$;*

(2) *on V_n, ψ_n is graph-closed and φ_n is continuous;*

(3) *$L_n \longrightarrow 0$ as $n \to +\infty$.*

Then, for each $n \in \mathbb{N}$, there exists a coincidence point $\xi_n \in X$ of the mappings ψ_n and φ_n, that is, such that $\xi_n \longrightarrow \xi$ as $n \to +\infty$ in (X, ρ_X).

P r o o f. For every $n \in \mathbb{N}$, the conditions of Theorem 2.1 are satisfied for the mappings ψ_n, φ_n, with $x_0 = \xi$. Consequently, for each n, there exists a solution ξ_n for the equation (3.1) such that $\rho_X(\xi, \xi_n) \le L_n$. Due to the convergence $L_n \to 0$, when $n \to \infty$, we obtain $\xi_n \to \xi$ when $n \to \infty$. □

As in Theorem 2.1, the graph-closedness assumption on ψ may be replaced by the requirement that ψ be d-closed and that the condition (2.10) be satisfied.

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Существование точек совпадения в подмножестве на b-метрическом пространстве

Ключевые слова: точки совпадения, метрическое пространство, b-метрическое пространство, накрывающее отображение, липшицевое отображение.

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В данной статье мы расширяем некоторые результаты о точках совпадения для отображений (ψ, φ), действующих из b-метрического пространства (X, ρ_X) в пространство Y, с расстоянием d, удовлетворяющим лишь аксиоме тождества. Здесь отображение ψ является α-накрывающим, а отображение φ является β-липшицевым $(\alpha > \beta \geqslant 0)$ только на данном подмножестве $V \subset X$. α-накрывающее множество $(\alpha > 0)$ для отображения ψ на V — это множество пар (x, y) таких, что существует $u \in V$ с $\psi(u) = y$, $\rho_X(x, u) \leqslant \alpha^{-1} d(\psi(x), y)$, а β-липшицево множество $(\beta \geqslant 0)$ для отображения φ на V — это множество пар (x, y) таких, что для любого $u \in V$, если $\varphi(u) = y$, то $d(y, \varphi(x)) \leqslant \beta \rho_X(u, x)$. Кроме того, в работе исследуется устойчивость последовательности точек совпадения $\{\xi_n\}$ для ψ_n и φ_n в предположении, что при $n \to \infty$ имеет место сходимость $\psi_n \to \psi$ и $\varphi_n \to \varphi$ соответственно.

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